Lecture Notes on
Cut and Identity Elimination

15-836: Substructural Logics
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Lecture 3
September 5, 2023

1 Introduction

As we have seen in the last lecture, in order to capture the meaning of implication we needed a hypothetical judgment with a single conclusion C

$$\begin{array}{c} A_1 \;\ldots\; A_n \\ \vdots \\ C \end{array}$$

where the hypotheses $A_1 \;\ldots\; A_n$ are interpreted according to the structural properties under consideration (a set for structural logic, a multiset for linear logic, and a sequence for ordered logic). We wrote this as $\Gamma \vdash A$ (structural), $\Delta \vdash A$ (linear) and $\Omega \vdash A$ (ordered) and wrote some inference rules for the logical connectives.

But how do we know the logical rules are correct? A standard approach due to Tarski [1931] provides mathematical models for the language of logical formulas and thereby gives external notions of soundness and completeness for a set of rules. This is in the tradition of what I called the "descriptive" approach to logic and a worthwhile enterprise. While this is fine as a way to analyze logic from a mathematical point of view, we do have to accept mathematics to start with as the basis for the external semantics, so it has its limitations from the foundational point of view.

In the current section of the course we are more interested in what I called the *creative* use of logic, where the rules are justified internally and therefore themselves create a computational universe. This is often called a *proof-theoretic semantics* because truth is justified by proofs and their structure. There is a long philosophical tradition for such an understanding of logic, perhaps starting with Gentzen [1935] and worked out further by Dummett [1991] and others. The connection between such an approach to logic and computation was noted by Curry [1934] and

Howard [1969]. A seminal paper merging these threads by Martin-Löf [1983] also highlights the difference between propositions (such as $A \supset B$) and judgments (such as "A *is true*" or "A *is false*"). This analysis comes into play here since we differentiate between "A *is a hypothesis*" and "A *is a conclusion*".

In a hypothetical judgment, we can fundamentally either work forward from the hypotheses or backwards from the conclusion we are trying to prove. Because the word "conclusion" is somewhat overloaded (also meaning the judgment "concluded" by an inference rule) we write a *sequent* as $\Omega \vdash A$ and refer to Ω as the *antecedents* and A as the *succedent*. This notion of a sequent (for structural logic) originated in Gentzen [1935]; the version of ordered logic is due to Lambek [1958]. The rules that work forward from the antecedents are called *left rules*, because they apply to propositions on the left of the turnstile '$\vdash$'. The rules that work backward from the succedent are called *right rules* because they apply to propositions on the right of the turnstile. These rules should fundamentally in balance in the following ways:

- If we have an antecedent A we should be able to conclude the succedent A. This corresponds to closing the gap betwee hypotheses and conclusion, if we think in two dimensions.
- If we have proved a succedent A we should be able to assume it as a hypothesis. This corresponds to justifying a hypothesis by a proof, so it no longer is a needed hypothesis.

When taken as rules in ordered logic, these take the following forms

$$\frac{}{A \vdash A}\ \mathsf{id}_A \qquad\qquad \frac{\Omega \vdash A \quad \Omega_L\ A\ \Omega_R \vdash C}{\Omega_L\ \Omega\ \Omega_R \vdash C}\ \mathsf{cut}_A$$

We are careful here about the order about the antecedents because, well, we are reasoning in ordered logic.

While these rules are certainly sound and useful, they should also be somehow redundant. For example, there should be sufficiently strong left rules so we can extract the component from a compound antecedent A to prove the succedent A. Conversely, when an antecedent A is used in a proof of the succedent C, we should be able to just use the proof of A wherever we use the hypothesis A. This is the essence of the properties of *identity elimination* and *cut elimination* we tackle in this lecture. They express a form of *harmony* between the left and right rules for a connectives, a property usually shown for *natural deduction* [Gentzen, 1935, Prawitz, 1965, Dummett, 1991] rather than the sequent calculus. Since natural deduction for substructural logics is somewhat delicate, we express and prove the corresponding properties on the sequent calculus—incidentally the path also taken by Gentzen.

2 Right Rules Meeting Left Rules

Rather than presenting fully formed proofs of harmony between the right and left rules, we proceed in small steps, eventually building up enough knowledge to then state the desired theorems and assemble the pieces into their proofs. We proceed connective by connective, which has the added advantage of arriving at our understanding in a modular way. When adding connectives, we (mostly) have to consider the new cases. Some of this reviews the material from the last lecture in a new light.

We foreshadow the idea that we end up with the sequent calculus *without* the rule of cut, and a form of identity that is limited to atomic propositions. In this calculus, cut and general identity should be *admissible*, that is, every instance of these rules should be valid. We used dashed lines to indicate admissible rules, so we are ultimately trying to justify

$$\frac{}{A \vdash A}\,\mathsf{id}_A \qquad \frac{\Omega \vdash A \quad \Omega_L\ A\ \Omega_R \vdash C}{\Omega_L\ \Omega\ \Omega_R \vdash C}\,\mathsf{cut}_A$$

2.1 Ordered Conjunction $A \bullet B$

Ordered conjunction $A \bullet B$ (pronounced A *fuse* B) internalizes the operation that concatenates two ordered states. Therefore we may think of the left rule as *defining* the connective.

$$\frac{\Omega_1\ A\ B\ \Omega_2 \vdash C}{\Omega_1\ (A \bullet B)\ \Omega_2 \vdash C}\,\bullet L$$

We claimed that the corresponding right rule should split the ordered antecedents somewhere devoting the first portion Ω_1 to proving A and the second portion Ω_2 to proving B.

$$\frac{\Omega_1 \vdash A \quad \Omega_2 \vdash B}{\Omega_1\ \Omega_2 \vdash A \bullet B}\,\bullet R$$

First, we want to check that the identity at $A \bullet B$ can be reduced to the identity at A and B.

$$\frac{}{A \bullet B \vdash A \bullet B}\,\mathsf{id}_{A \bullet B} \quad \longrightarrow_E \quad \frac{\dfrac{\dfrac{}{A \vdash A}\,\mathsf{id}_A \quad \dfrac{}{B \vdash B}\,\mathsf{id}_B}{A\ B \vdash A \bullet B}\,\bullet R}{A \bullet B \vdash A \bullet B}\,\bullet L$$

Notice that the first step (thinking bottom-up as we should) in the proof is forced: trying to use the $\bullet R$ rule will fail since there is only a single antecedent which we cannot split into two yet. The good news is that the left/right split worked out in this case.

Second, we should check if we can reduce a cut of proposition $A \bullet B$ into cuts at propositions A and B while preserving the conclusion. The critical we examine here is if a right rule for a connectives meets a corresponding left rule.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Omega_1 \vdash A\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \Omega_2 \vdash B\end{array}}{\Omega_1\ \Omega_2 \vdash A \bullet B}\ \bullet R \quad \dfrac{\begin{array}{c}\mathcal{E}'\\ \Omega_L\ A\ B\ \Omega_R \vdash C\end{array}}{\Omega_L\ (A \bullet B)\ \Omega_R \vdash C}\ \bullet L}{\Omega_L\ \Omega_1\ \Omega_2\ \Omega_R \vdash C}\ \mathsf{cut}_{A \bullet B}$$

We have given names to the derivations of the premises of $\bullet R$ and $\bullet L$ so we can refer to them after the transformation. We observe we can appeal to cut on A between $\mathcal{D}_1$ and $\mathcal{E}'$, and then again on B between $\mathcal{D}_2$ and the result. This yields the following:

$$\longrightarrow_R \quad \dfrac{\begin{array}{c}\mathcal{D}_2\\ \Omega_2 \vdash B\end{array} \quad \dfrac{\begin{array}{c}\mathcal{D}_1\\ \Omega_1 \vdash A\end{array} \quad \begin{array}{c}\mathcal{E}'\\ \Omega_L\ A\ B\ \Omega_R \vdash C\end{array}}{\Omega_L\ \Omega_1\ B\ \Omega_R \vdash C}\ \mathsf{cut}_A}{\Omega_L\ \Omega_1\ \Omega_2\ \Omega_R \vdash C}\ \mathsf{cut}_B$$

As we can see, everything works out in both cases.

But what goes wrong if we had the incorrect right rule, swapping the antecedents?

$$\dfrac{\Omega_2 \vdash A \quad \Omega_1 \vdash B}{\Omega_1\ \Omega_2 \vdash A \bullet B}\ \bullet R?$$

First, we notice that the identity is no longer admissible in the calculus without cut. Here is a counterexample with atomic propositions P and Q.

$$\begin{array}{c}\vdots\\ P \bullet Q \vdash P \bullet Q\end{array}$$

Starting with $\bullet R?$ won't work, since there is only a single antecedent, which would either have to go to the first or second premise, while the other is empty. Proceeding with $\bullet L$:

$$\dfrac{\begin{array}{c}\vdots\\ P\ Q \vdash P \bullet Q\end{array}}{P \bullet Q \vdash P \bullet Q}\ \bullet L$$

Now there are only one rule and three possible pairs of premises with this conclusion, depending on how the antecedents are split.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ P\ Q \vdash P\end{array} \quad \begin{array}{c}\vdots\\ \cdot \vdash Q\end{array}}{P\ Q \vdash P \bullet Q}\ \bullet R?}{P \bullet Q \vdash P \bullet Q}\ \bullet L \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\\ Q \vdash P\end{array} \quad \begin{array}{c}\vdots\\ P \vdash Q\end{array}}{P\ Q \vdash P \bullet Q}\ \bullet R?}{P \bullet Q \vdash P \bullet Q}\ \bullet L \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \cdot \vdash P\end{array} \quad \begin{array}{c}\vdots\\ P\ Q \vdash Q\end{array}}{P\ Q \vdash P \bullet Q}\ \bullet R?}{P \bullet Q \vdash P \bullet Q}\ \bullet L$$

No rules apply to any of these premises. Since we have explored all possibilities, the identity is not admissible for the incorrect right rules.

To make matters worse, cut would also not be admissible! You could try the reduction and see that any natural attempt would change the order of Ω_1 and Ω_2 in the conclusion.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Omega_2 \vdash A\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \Omega_1 \vdash B\end{array}}{\Omega_1\ \Omega_2 \vdash A \bullet B}\ {\bullet}R? \qquad \dfrac{\begin{array}{c}\mathcal{E}'\\ \Omega_L\ A\ B\ \Omega_R \vdash C\end{array}}{\Omega_L\ (A \bullet B)\ \Omega_R \vdash C}\ {\bullet}L}{\Omega_L\ \Omega_1\ \Omega_2\ \Omega_R \vdash C}\ \mathsf{cut}_{A \bullet B}$$

$$\longrightarrow_R ? \qquad \dfrac{\begin{array}{c}\mathcal{D}_2\\ \Omega_1 \vdash B\end{array} \quad \dfrac{\begin{array}{c}\mathcal{D}_1\\ \Omega_2 \vdash A\end{array} \quad \begin{array}{c}\mathcal{E}'\\ \Omega_L\ A\ B\ \Omega_R \vdash C\end{array}}{\Omega_L\ \Omega_2\ B\ \Omega_R \vdash C}\ \mathsf{cut}_A}{\Omega_L\ \Omega_2\ \Omega_1\ \Omega_R \vdash C}\ \mathsf{cut}_B$$

Of course, this is not a *refutation* of cut elimination. It only shows that a particular way to attempt to prove the admissibility of cut does not work. But we can fashion this failed proof attempt into a counterexample. Let's pick $\Omega_1 = B$ and $\Omega_2 = A$. Then the figure can become

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\ \mathsf{id}_A \quad \dfrac{}{B \vdash B}\ \mathsf{id}_B}{B\ A \vdash A \bullet B}\ {\bullet}R? \qquad \dfrac{\begin{array}{c}\mathcal{E}'\\ \Omega_L\ A\ B\ \Omega_R \vdash C\end{array}}{\Omega_L\ (A \bullet B)\ \Omega_R \vdash C}\ {\bullet}L}{\Omega_L\ B\ A\ \Omega_R \vdash C}\ \mathsf{cut}_{A \bullet B}$$

Now we see that if cut (and identity) were admissible, the rule of *exchange* between two ordered(!) antecedents would also be admissible. In other words, our ordered logic would become linear!

If cut and identity were primitive rules (our starting point) then we could even derive exchange, and ordered logic would collapse to linear logic—order wouldn't mean anything. Clearly, the ${\bullet}R?$ rule would be wrong.

2.2 Left Implication $A \setminus B$

In ordered logic, the usual implication $A \supset B$ splits into two different connectives, $A \setminus B$ (pronounced A under B) and $B \mathbin{/} A$ (pronounced B over A), depending on whether A is added to the left end or the right and of the antecedents. In this case, we view the *right* rule as a definition of the connective.

$$\dfrac{A\ \Omega \vdash B}{\Omega \vdash A \setminus B}\ \setminus R$$

What should the matching left rule be? Starting the proof of the identity gives some intuition.

$$\dfrac{\begin{matrix}\vdots\\ A\ (A \setminus B) \vdash B\end{matrix}}{A \setminus B \vdash A \setminus B}\ \setminus R$$

It looks as if the proof of A needs to come from the left of the implication. So let's hypothesize:

$$\dfrac{\Omega_A \vdash A \quad \Omega_L\ B\ \Omega_R \vdash C}{\Omega_L\ \Omega_A\ (A \setminus B)\ \Omega_R \vdash C}\ \setminus L$$

We first check identity expansion.

$$\dfrac{}{A \setminus B \vdash A \setminus B}\ \mathsf{id}_{A \setminus B} \quad \longrightarrow_E \quad \dfrac{\dfrac{\dfrac{}{A \vdash A}\ \mathsf{id}_A \quad \dfrac{}{B \vdash B}\ \mathsf{id}_B}{A\ (A \setminus B) \vdash B}\ \setminus L}{A \setminus B \vdash A \setminus B}\ \setminus R$$

Next, the cut reduction when the right rule meets the left rule.

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}'\\ A\ \Omega \vdash B\end{matrix}}{\Omega \vdash A \setminus B}\ \setminus R \quad \dfrac{\begin{matrix}\mathcal{E}_1\\ \Omega_A \vdash A\end{matrix} \quad \begin{matrix}\mathcal{E}_2\\ \Omega_L\ B\ \Omega_R \vdash C\end{matrix}}{\Omega_L\ \Omega_A\ (A \setminus B)\ \Omega_R \vdash C}\ \setminus L}{\Omega_L\ \Omega_A\ \Omega\ \Omega_R \vdash C}\ \mathsf{cut}_{A \setminus B}$$

$$\longrightarrow_R \quad \dfrac{\dfrac{\begin{matrix}\mathcal{E}_1\\ \Omega_A \vdash A\end{matrix} \quad \begin{matrix}\mathcal{D}'\\ A\ \Omega \vdash B\end{matrix}}{\Omega_A\ \Omega \vdash B}\ \mathsf{cut}_A \quad \begin{matrix}\mathcal{E}_2\\ \Omega_L\ B\ \Omega_R \vdash C\end{matrix}}{\Omega_L\ \Omega_A\ \Omega\ \Omega_R \vdash C}\ \mathsf{cut}_B$$

It works out! A significant observation here is that the result of the reduction is not unique. For example, we could have cut $\mathcal{D}'$ with $\mathcal{E}_2$ first, and then $\mathcal{E}_1$ with the end sequent.

3 Right Implication $B \,/\, A$

This is symmetric to the left implication, so we just show the rules. It is a recommended exercise to go through the cases of identity expansion and cut reduction.

$$\dfrac{\Omega\ A \vdash B}{\Omega \vdash B \,/\, A}\ /R \qquad \dfrac{\Omega_A \vdash A \quad \Omega_L\ B\ \Omega_R \vdash C}{\Omega_L\ (B \,/\, A)\ \Omega_A\ \Omega_R \vdash C}\ /L$$

4 Excursion: Parsing with the Lambek Calculus

A point in Lambek's original calculus [Lambek, 1958] is to model natural language parsing by (ordered) logical inference. For this purpose, we use the following specialized versions of the $\backslash L$ and $\backslash R$ rules:

$$\frac{\Omega_L\ B\ \Omega_R \vdash C}{\Omega_L\ A\ (A \backslash B)\ \Omega_R \vdash C}\ \backslash L^* \qquad \frac{\Omega_L\ B\ \Omega_R \vdash C}{\Omega_L\ (B \,/\, A)\ A\ \Omega_R \vdash C}\ /L^*$$

Their soundness is easy to see, but it turns out we can't quite replace the general rules with these specialized ones (see Tasks 8 and 9 of Assignment 2).

Then we assign a syntactic category to every word appearing in a sentence. We start with n for names such as *Alice* or *Bob*. We also have s for complete sentences. An intransitive verb such as *works* has category $n \backslash s$ which means that if we find a name to its left then their combination forms a sentence.

To start the inference process we annotate each word with its syntactic category, writing $(w : A)$. We would like to parse a sequence of words as a sentence, so the succedent of our sequent is s.

$$(\mathit{Alice} : n)\ (\mathit{works} : n \backslash s) \vdash s$$

In this case, we can complete it in just two steps. We just concatenate the two proof terms for the result of $\backslash L^*$.

$$\dfrac{\dfrac{}{(\mathit{Alice} \cdot \mathit{works}) : s \vdash s}\ \mathsf{id}}{(\mathit{Alice} : n)\ (\mathit{works} : n \backslash s) \vdash s}\ \backslash L^*$$

Note that the proof of s here represents a parse tree, in this case trivial. What about adjectives such as *poor*? If *poor* precedes a name, the phrase again functions as a name. So $(\mathit{poor} : n \,/\, n)$. We would then parse "*poor Alice works*" as

$$\dfrac{\dfrac{\dfrac{}{((\mathit{poor} \cdot \mathit{Alice}) \cdot \mathit{works} : s) \vdash s}\ \mathsf{id}}{(\mathit{poor} \cdot \mathit{Alice} : n)\ (\mathit{works} : n \backslash s) \vdash s}\ \backslash L^*}{(\mathit{poor} : n \,/\, n)\ (\mathit{Alice} : n)\ (\mathit{works} : n \backslash s) \vdash s}\ /L^*$$

A transitive verb such as *likes* has category $n \backslash (s \,/\, n)$: if there is a name to its left and to its right, then the result is a sentence. For example:

$$(\mathit{poor} : n \,/\, n)\ (\mathit{Alice} : n)\ (\mathit{likes} : n \backslash (s \,/\, n))\ (\mathit{Bob} : n) \vdash s$$

This example illustrates some nondeterminism in the parsing process. For example, $(\mathit{Alice} : n)$ is to the left of $(\mathit{likes} : n\backslash(s/n)$ so we could apply $\backslash L^*$, but eventually we would get stuck at

$$(\mathit{poor} : n \,/\, n)\ ((\mathit{Alice} \cdot \mathit{likes}) \cdot \mathit{Bob} : s) \vdash s$$

which we cannot complete. Instead, we need to group *poor* with *Alice* first so that $(\mathit{poor} \cdot \mathit{Alice} : n)$. Then we can arrive at

$$(((\mathit{poor} \cdot \mathit{Alice}) \cdot \mathit{likes}) \cdot \mathit{Bob} : s) \vdash s$$

Below is a small table with syntactic categories of other words.

Word	Type	Part of Speech
works	$n \setminus s$	intransitive verb
poor	$n \,/\, n$	adjective
here	$s \setminus s$	adverb
never	$(n \setminus s) \,/\, (n \setminus s)$	adverb
likes	$n \setminus (s \,/\, n)$	transitive verb
and	$s \setminus (s \,/\, s)$	conjunction
and	$s \setminus (n^* \,/\, s)$	conjunction

For the last example we use n^* for a plural name so that phrases like *Alice and Bob work* can be parsed correctly with $(\mathit{work} : n^* \setminus s)$. But "*and*" is more complicated because it is overloaded. For example, it would also have syntactic type $s\setminus(s/s)$ because it can conjoin two sentences into a longer sentence. This cannot be expressed in Lambek's original calculus, but it fits into the so-called *full Lambek calculus* which we have called *ordered logic*. We can write

$$\mathit{and} : (s \setminus (s \,/\, s)) \mathbin{\&} (s \setminus (n^* \,/\, s))$$

where $A \mathbin{\&} B$ is a new connective. If we have the antecedent $A \mathbin{\&} B$ we can choose between A and B, while $A \bullet B$ necessarily gives us both, next to each other. We give the rules below, in Section 6.

5 A Small Example

From the parsing intuition, we would expect $A \setminus (C \,/\, B)$ to be equivalent to $(A \setminus C) \,/\, B$—a form of associativity. Whether we prove A to the left or B to the right first should be irrelevant, since we need both before we obtain C.

Let's prove one direction.

$$\begin{array}{c} \vdots \\ A \setminus (C \,/\, B) \vdash (A \setminus C) \,/\, B \end{array}$$

It turns out that the right rules for the two forms of ordered implication are *invertible* in the sense that we can always apply them during bottom-up proof construction without ever considering alternatives.

It turns out the every connective is either invertible on the right or on the left. A quick test (although not a proof) to see which one, see which side the proof of the identity starts on. For B / A and $A \backslash B$ it starts with a right rule. So we can start as follows without having to think:

$$\dfrac{\dfrac{\vdots}{A\ (A \backslash (C / B))\ B \vdash C}}{\dfrac{(A \backslash (C / B))\ B \vdash A \backslash C}{A \backslash (C / B) \vdash (A \backslash C) / B}\ /R}\ \backslash R$$

At this point we can only apply $\backslash L$ because it is the only connective at the top level among all antecedents and the succedent.

$$\dfrac{\dfrac{\dfrac{\overline{A \vdash A}\ \mathsf{id} \quad \dfrac{\vdots}{(C / B)\ B \vdash C}}{A\ (A \backslash (C / B))\ B \vdash C}\ \backslash L}{(A \backslash (C / B))\ B \vdash A \backslash C}\ \backslash R}{A \backslash (C / B) \vdash (A \backslash C) / B}\ /R$$

The open subgoal now follows by $/L$ and two identities.

$$\dfrac{\dfrac{\dfrac{\overline{A \vdash A}\ \mathsf{id} \quad \dfrac{\overline{B \vdash B}\ \mathsf{id} \quad \overline{C \vdash C}\ \mathsf{id}}{(C / B)\ B \vdash C}\ /L}{A\ (A \backslash (C / B))\ B \vdash C}\ \backslash L}{(A \backslash (C / B))\ B \vdash A \backslash C}\ \backslash R}{A \backslash (C / B) \vdash (A \backslash C) / B}\ /R$$

The entailment in the other direction proceeds in a similar vein.

6 External Choice ($A \,\&\, B$)

The parsing example suggests the following left rules:

$$\dfrac{\Omega_L\ A\ \Omega_R \vdash C}{\Omega_L\ (A \,\&\, B)\ \Omega_R \vdash C}\ \&L_1 \qquad \dfrac{\Omega_L\ B\ \Omega_R \vdash C}{\Omega_L\ (A \,\&\, B)\ \Omega_R \vdash C}\ \&L_1$$

What is the corresponding right rule? Rather then splitting the antecedents as for $A \bullet B$, we propagate all of them to both premises.

$$\dfrac{\Omega \vdash A \quad \Omega \vdash B}{\Omega \vdash A \,\&\, B}\ \&R$$

At first glance it may seem that this violates the principle that every hypothesis is used exactly once. But any *use* of an antecedent $A \mathbin{\&} B$ will pick either just A or just B. This is reflected in the two local reductions.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Omega \vdash A\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \Omega \vdash B\end{array}}{\Omega \vdash A \mathbin{\&} B}\,\&R \quad \dfrac{\begin{array}{c}\mathcal{E}'\\ \Omega_L\ A\ \Omega_R \vdash C\end{array}}{\Omega_L\ (A \mathbin{\&} B)\ \Omega_R \vdash C}\,\&L_1}{\Omega_L\ \Omega\ \Omega_R \vdash C}\,\mathsf{cut}_{A\&B}$$

$$\longrightarrow_R \quad \dfrac{\begin{array}{c}\mathcal{D}_1\\ \Omega \vdash A\end{array} \quad \begin{array}{c}\mathcal{E}'\\ \Omega_L\ A\ \Omega_R \vdash C\end{array}}{\Omega_L\ \Omega\ \Omega_R \vdash C}\,\mathsf{cut}_A$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Omega \vdash A\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \Omega \vdash B\end{array}}{\Omega \vdash A \mathbin{\&} B}\,\&R \quad \dfrac{\begin{array}{c}\mathcal{E}'\\ \Omega_L\ B\ \Omega_R \vdash C\end{array}}{\Omega_L\ (A \mathbin{\&} B)\ \Omega_R \vdash C}\,\&L_2}{\Omega_L\ \Omega\ \Omega_R \vdash C}\,\mathsf{cut}_{A\&B}$$

$$\longrightarrow_R \quad \dfrac{\begin{array}{c}\mathcal{D}_2\\ \Omega \vdash B\end{array} \quad \begin{array}{c}\mathcal{E}'\\ \Omega_L\ B\ \Omega_R \vdash C\end{array}}{\Omega_L\ \Omega\ \Omega_R \vdash C}\,\mathsf{cut}_B$$

We see both premises of $\&R$ are necessary, because we don't know which left rule ($\&L_1$ or $\&L_2$) it might meet. Also, if we had split the antecedents, then we wouldn't have enough of them in one branch or the other, or both. This would lend itself to a counterexample.

The identity is straightforward, and we must start with the right rule. This means the right rule for $A \mathbin{\&} B$ is invertible.

$$\dfrac{}{A \mathbin{\&} B \vdash A \mathbin{\&} B}\,\mathsf{id}_{A\&B} \quad \longrightarrow_E \quad \dfrac{\dfrac{\dfrac{}{A \vdash A}\,\mathsf{id}_A}{A \mathbin{\&} B \vdash A}\,\&L_1 \quad \dfrac{\dfrac{}{B \vdash B}\,\mathsf{id}_B}{A \mathbin{\&} B \vdash B}\,\&L_2}{A \mathbin{\&} B \vdash A \mathbin{\&} B}\,\&R$$

Here we notice that both left rules are necessary. If we just had one (say, $\wedge L_1$) we wouldn't be able to complete the identity expansion because $A \mathbin{\&} B \vdash B$ would not have a proof.

7 The Empty State (1)[1]

[1] covered in Lecture 4, but for continuity included here

We can also internalize the empty state as the proposition $\mathbf{1}$. As a left rule, the proposition simply disappears. As a right rule, the state must be empty. We can think of this as the unit for *fuse* in the sense that $A \bullet \mathbf{1} \dashv\vdash A \dashv\vdash \mathbf{1} \bullet A$.

$$\frac{}{\cdot \vdash \mathbf{1}}\ \mathbf{1}R \qquad \frac{\Omega_L\ \Omega_R \vdash C}{\Omega_L\ (\mathbf{1})\ \Omega_R \vdash C}\ \mathbf{1}L$$

Identity expansion and cut reduction are immediate, since there are no subformulas of $\mathbf{1}$.

$$\frac{\dfrac{}{\cdot \vdash \mathbf{1}}\ \mathbf{1}R \quad \dfrac{\begin{matrix}\mathcal{E}' \\ \Omega_L\ \Omega_R \vdash C\end{matrix}}{\Omega_L\ (\mathbf{1})\ \Omega_R \vdash C}\ \mathbf{1}L}{\Omega_L\ \Omega_R \vdash C}\ \mathsf{cut}_{\mathbf{1}} \quad \longrightarrow_R \quad \begin{matrix}\mathcal{E}' \\ \Omega_L\ \Omega_R \vdash C\end{matrix}$$

$$\frac{}{\mathbf{1} \vdash \mathbf{1}}\ \mathsf{id}_{\mathbf{1}} \quad \longrightarrow_E \quad \frac{\dfrac{}{\cdot \vdash \mathbf{1}}\ \mathbf{1}R}{\mathbf{1} \vdash \mathbf{1}}\ \mathbf{1}L$$

8 Disjunction ($A \oplus B$)[2]

Disjunction (also called *internal choice*) changes remarkably little from structural to linear to ordered logic. We may think of it as being defined by the following two right rules:

$$\frac{\Omega \vdash A}{\Omega \vdash A \oplus B}\ \oplus R_1 \qquad \frac{\Omega \vdash B}{\Omega \vdash A \oplus B}\ \oplus R_2$$

The situation is almost entirely symmetric to the one for *external choice* ($A \mathbin{\&} B$). If $A \oplus B$ is among our antecedents, we do not now whether A or B will be true, but we know only one of the two rules will be applied.

$$\frac{\Omega_L\ A\ \Omega_R \vdash C \quad \Omega_L\ B\ \Omega_R \vdash C}{\Omega_L\ (A \oplus B)\ \Omega_R \vdash C}\ \oplus L$$

You can now easily convince yourself that identity expansion and cut reduction are possible.

9 Truth ($\top$)

Truth is the unit of external choice in the sense that $A \mathbin{\&} \top \dashv\vdash A \dashv\vdash \top \mathbin{\&} A$. Because external choice has two left rules, its nullary version will have none. Conversely,

[2] covered in Lecture 4, but for continuity included here

the right rule for external choice has two premises, so the right rule for its nullary version has none.

$$\frac{}{\Omega \vdash \top}\ \top R \qquad\qquad \text{no } \top L \text{ rules}$$

Because there is no left rule, the right rule cannot meet any left rule and there is no cut reduction. But there is a simple identity expansion:

$$\frac{}{\top \vdash \top}\ \mathsf{id}_\top \quad \longrightarrow_E \quad \frac{}{\top \vdash \top}\ \top R$$

10 Falsehood (0)

Falsehood is the unit of internal choice in the sense that $\mathbf{0} \oplus A \dashv\vdash A \dashv\vdash A \oplus \mathbf{0}$. Its properties are symmetric to that of truth ($\top$).

$$\text{no } \mathbf{0}R \text{ rules} \qquad\qquad \frac{}{\Omega_L\ (\mathbf{0})\ \Omega_R \vdash C}\ \mathbf{0}L$$

Because there are no right rules, there cannot be a case where a right rule meets a left rule. But an identity expansion is possible.

$$\frac{}{\mathbf{0} \vdash \mathbf{0}}\ \mathsf{id}_{\mathbf{0}} \quad \longrightarrow_E \quad \frac{}{\mathbf{0} \vdash \mathbf{0}}\ \mathbf{0}L$$

This concludes the introduction of the connectives; see the summary in Figure 1.

11 Admissibility of Identity, as a Theorem

We can put together all the cases for identity expansions in the following theorem.

Theorem 1 (Admissibility of Identity) *In the system with identity restricted to atomic propositions, the rule*

$$\frac{}{A \vdash A}\ \mathsf{id}_A$$

is admissible for every A.

Proof: By induction on the structure of A. Many of the individual cases were presented in lecture; the others are analogous. □

12 Admissibility of Cut, as a Theorem

The cut reductions we have presented so far only cover the case where the cut combines a right rule for a connective with its left rule. There are two other classes of cases: if one of the premises is the identity (whether atomic or not), and when the last inference on one or both sides is *not* on the cut formula.

Cuts and identity cancel each other. There are only two cases; we show one.

$$\dfrac{\dfrac{}{A \vdash A}\,\mathsf{id}_A \quad \begin{matrix}\mathcal{E}' \\ \Omega_L\ A\ \Omega_R \vdash C\end{matrix}}{\Omega_L\ A\ \Omega_R \vdash C}\,\mathsf{cut}_A \quad \longrightarrow_R \quad \begin{matrix}\mathcal{E}' \\ \Omega_L\ A\ \Omega_R \vdash C\end{matrix}$$

Now we consider a case of $\bullet L$ in $\mathcal{D}$, where $\mathcal{E}$ remains completely arbirary.

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}' \\ \Omega_1\ B_1\ B_2\ \Omega_2 \vdash A\end{matrix}}{\Omega_1\ (B_1 \bullet B_2)\ \Omega_2 \vdash A}\,\bullet L \quad \begin{matrix}\mathcal{E} \\ \Omega_L\ A\ \Omega_R \vdash C\end{matrix}}{\Omega_L\ \Omega_1\ (B_1 \bullet B_2)\ \Omega_2\ \Omega_R \vdash C}\,\mathsf{cut}_A$$

Since $\mathcal{D}'$ still has succedent A, we can now cut it with $\mathcal{E}$ and then apply $\bullet L$ *afterwards*.

$$\longrightarrow_R \quad \dfrac{\dfrac{\begin{matrix}\mathcal{D}' \\ \Omega_1\ B_1\ B_2\ \Omega_2 \vdash A\end{matrix} \quad \begin{matrix}\mathcal{E} \\ \Omega_L\ A\ \Omega_R \vdash C\end{matrix}}{\Omega_L\ \Omega_1\ B_1\ B_2\ \Omega_2\ \Omega_R \vdash C}\,\mathsf{cut}_A}{\Omega_L\ \Omega_1\ (B_1 \bullet B_2)\ \Omega_2\ \Omega_R \vdash C}\,\bullet L$$

In essence, we are "pushing the cut up", past the preceding inference. There is some nondeterminism here. For example, if $\mathcal{E}$ ends in a right rule, or a left rule on a proposition that is not A, then we cut push it up into the second premise as well.

The immediate concern should be that we have *not reduced the structure of the cut formula*: it is still A! However, we reduced the cut from one on $\mathcal{D}$ (the first premise of the cut) to $\mathcal{D}'$, a subproof.

All other cases of this kind often called *commutative cases* proceed in an analogous manner. So we summarize: we reduce a cut on

$$\dfrac{\begin{matrix}\mathcal{D} \\ \Omega \vdash A\end{matrix} \quad \begin{matrix}\mathcal{E} \\ \Omega_L\ A\ \Omega_R\end{matrix}}{\Omega_L\ \Omega\ \Omega_R \vdash C}\,\mathsf{cut}_A$$

either

1. to cuts on a subformulas of A, or

2. to cuts on the same formula A but subderivations of $\mathcal{D}$ or $\mathcal{E}$.

Such reductions must always terminate, either because we reach derivations without subderivations, or because we reach formulas without subformula. Formally, it is a well-founded induction on a *lexicographic ordering*, first on A and then on $\mathcal{D}$ and $\mathcal{E}$. This is also called a *nested induction*. We summarize in the following theorem.

Theorem 2 (Admissibility of Cut) *In the inference system without cut (where the identity may be general or restricted to atoms), the rule*

$$\dfrac{\begin{array}{cc} \mathcal{D} & \mathcal{E} \\ \Omega \vdash A & \Omega_L\ A\ \Omega_R \end{array}}{\Omega_L\ \Omega\ \Omega_R \vdash C}\ \mathsf{cut}_A$$

is admissible.

Proof: By nested induction, first on the structure of A, and second on the structures of $\mathcal{D}$ and $\mathcal{E}$. We have the following classes of cases:

Principal Cases: $\mathcal{D}$ ends in a right rule inferring A and $\mathcal{E}$ ends in a left rule inferring A. We have shown several of these cases and we appeal to the induction hypothesis on subformulas on A. If A has no subformulas we have a base case (as for $\mathbf{1}$).

Identity Cases: Either $\mathcal{D}$ or $\mathcal{E}$ is an identity. Then we directly reduce to the other derivation.

Commuting Cases: Either $\mathcal{D}$ or $\mathcal{E}$ ends in an inference on a formula other than A. In this case we appeal to the induction hypothesis, possible in more than one way, on the same A and a subderivation on $\mathcal{D}$ or $\mathcal{E}$, and the reapply the inference.

□

From the admissibility of identity and cut we obtain the following straightforward corollaries that follow by straightforward induction over the structure of the given derivation.

Corollary 3 (Identity Elimination) *Given an arbitrary sequent derivation with uses of the identity (with or without cut). Then we can eliminate all uses of the identity except on atomic propositions P, obtaining a derivation with or without cut, respectively.*

Proof: By induction on the structure of the given derivation. We appeal to the induction hypothesis and reapply the rule, except for identity when we appeal to the admissibility of identity. □

Corollary 4 (Cut Elimination) *Given an arbitary sequent derivation with uses of cut (with or without a general identity). Then we can eliminate all uses of cut, obtaining a derivation with or without general identity, respectively.*

Proof: By induction on the structure of the given derivation. We appeal to the induction hypothesis and reapply the rule, except for cut when we appeal to the admissibility of cut. □

13 Summary

The rules for the sequent calculus, marking cut and identity as admissible, are summarized in Figure 1.

The admissibility of cut and identity in the sequent calculus without these rules (except for identity on atomic formulas) is the key property to ensure we have a proof-theoretic semantics for a logic. Cut-free proofs in particular always only refer to subformulas of the original goal sequents, so any semantic content is internal to what we are trying prove.

While we have shown some details for ordered logic, similar (and slightly more complicated) arguments apply to linear and structural logics and, eventually, to logics mixing these. This is the blueprint, and future proofs of identity and cut elimination will often be discussed via the differences from the present approach.

In a few lectures from now we will see that sequent proofs correspond to programs, and principal cut reductions play a fundamental role in interpreting the dynamics of programs. Identity expansions and commuting reductions correspond to equality between programs and are therefore slightly less significant.

$$\frac{}{A \vdash A}\ \mathsf{id}_A \qquad \frac{}{P \vdash P}\ \mathsf{id}^*_P \qquad \frac{\Omega \vdash A \quad \Omega_L\ A\ \Omega_R \vdash C}{\Omega_L\ \Omega\ \Omega_R \vdash C}\ \mathsf{cut}_A$$

$$\frac{\Omega_1 \vdash A \quad \Omega_2 \vdash B}{\Omega_1\ \Omega_2 \vdash A \bullet B}\ \bullet R \qquad \frac{\Omega_1\ A\ B\ \Omega_2 \vdash C}{\Omega_1\ (A \bullet B)\ \Omega_2 \vdash C}\ \bullet L$$

$$\frac{A\ \Omega \vdash B}{\Omega \vdash A \setminus B}\ \setminus R \qquad \frac{\Omega_A \vdash A \quad \Omega_L\ B\ \Omega_R \vdash C}{\Omega_L\ \Omega_A\ (A \setminus B)\ \Omega_R \vdash C}\ \setminus L$$

$$\frac{\Omega\ A \vdash B}{\Omega \vdash B \,/\, A}\ /R \qquad \frac{\Omega_A \vdash A \quad \Omega_L\ B\ \Omega_R \vdash C}{\Omega_L\ (B \,/\, A)\ \Omega_A\ \Omega_R \vdash C}\ /L$$

$$\frac{\Omega \vdash A \quad \Omega \vdash B}{\Omega \vdash A \mathbin{\&} B}\ \& R \qquad \frac{\Omega_L\ A\ \Omega_R \vdash C}{\Omega_L\ (A \mathbin{\&} B)\ \Omega_R \vdash C}\ \& L_1 \qquad \frac{\Omega_L\ B\ \Omega_R \vdash C}{\Omega_L\ (A \mathbin{\&} B)\ \Omega_R \vdash C}\ \& L_1$$

$$\frac{\Omega \vdash A}{\Omega \vdash A \oplus B}\ \oplus R_1 \qquad \frac{\Omega \vdash B}{\Omega \vdash A \oplus B}\ \oplus R_2 \qquad \frac{\Omega_L\ A\ \Omega_R \vdash C \quad \Omega_L\ B\ \Omega_R \vdash C}{\Omega_L\ (A \oplus B)\ \Omega_R \vdash C}\ \oplus L$$

$$\frac{}{\cdot \vdash \mathbf{1}}\ \mathbf{1}R \qquad \frac{\Omega_L\ \Omega_R \vdash C}{\Omega_L\ (\mathbf{1})\ \Omega_R \vdash C}\ \mathbf{1}L$$

$$\frac{}{\Omega \vdash \top}\ \top R \qquad \text{no } \top L \text{ rules}$$

$$\text{no } \mathbf{0}R \text{ rules} \qquad \frac{}{\Omega_L\ (\mathbf{0})\ \Omega_R \vdash C}\ \mathbf{0}L$$

Figure 1: Ordered Sequent Calculus

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