Lecture Notes on Resource Semantics

15-816: Linear Logic
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In this lecture we explore a new presentation of linear logic, one where resources are explicitly tracked in the judgments. It is a new form of semantics, given by intuitionistic means, while generally semantic investigations take a classical point of view even if the studied subject is intuitionistic.

In the present lecture we change the judgments, but we try to disturb the nature of proofs as little as possible. In the following lecture, we will take a less restrictive view, which leads to new ways to reason linearly.

One of the important reasons to investigate a resource semantics is that it allows us to express new properties and relations, beyond what is possible in linear logic itself. Further materials and properties on resource semantics are given by Reed [Ree09].

1 Resource-Aware Judgments

In order to give a Kripke-like resource semantics we label all the resources with unique labels representing that resource. In the succedent we record all the resources that may be used, which may be a subset of the resources listed in the antecedent. So a sequent has the form

$$A_1@\alpha_1, \ldots, A_n@\alpha_n \vdash C@p$$

where p is formed from $\alpha_1, \ldots, \alpha_n$ with a binary resource combination $*$. In addition we have the empty resource label ϵ, which is the unit of $*$. Resource combination is associative and commutative, so we have the laws

$$\begin{array}{rcl} p * \epsilon & = & p \\ \epsilon * q & = & q \\ (p * q) * r & = & p * (q * r) \\ p * q & = & q * p \end{array}$$

We will apply these equations silently, just as we, for example, silently reorder hypotheses.

By labeling resources we recover the property of weakening.

> **Weakening:** If $\Gamma \vdash C@p$ then $\Gamma, A@\alpha \vdash C@p$. Here α must be new in order to maintain the invariant on seqents that all antecedents are labeled with distinct resource parameters.

Contraction cannot be quite formulated, since we cannot contract $A@\alpha, A@\beta$ to $A@\alpha * \beta$ because at least for the moment, hypotheses can only be labeled with resource parameters and not combinations of them.

The identity is quite straightforward:

$$\frac{}{\Gamma, A@\alpha \vdash A@\alpha}\ \mathsf{init}$$

Cut is a bit more complicated, because resource labels in succedents are more general then in antecedents. But we have already seen this in substitution principles with proof terms, so we imitate this solution, substitution resources p for resource parameter α:

$$\frac{\Gamma \vdash A@p \quad \Gamma, A@\alpha \vdash C@\alpha * q}{\Gamma \vdash C@p * q}\ \mathsf{cut}$$

We now revisit each of the connectives so far in turn, deriving the appropriate rules. The goal is to achieve an exact isomorphism between the linear logic inference rules based on hypothetical judgments and the inference rules based on resources. Hidden behind the isomorphism is the equational reasoning in the resource algebra.

Simultaneous Conjunction. The resources available to achieve the goals are split between the two premises. Previously, this was achieved by splitting the context itself. Note that we use Γ here to stand for a context in which all assumptions are labeled with unique resource parameters.

$$\frac{\Gamma \vdash A@p \quad \Gamma \vdash B@q}{\Gamma \vdash A \otimes B@p * q}\ \otimes R$$

In order to apply a left rule to a given assumption $A@\alpha$, the resource α must actually be available, which is recorded in the succedent. Upon application of the rule the resource is no longer available, but new resources may now be available (depending on the connective).

$$\frac{\Gamma, A \otimes B@\alpha, A@\beta, B@\gamma \vdash @p * \beta * \gamma}{\Gamma, A \otimes B@\alpha \vdash C@p * \alpha} \otimes L^{\beta,\gamma}$$

In this rule, α is consumed and new resources β and γ are introduced.

Linear Implication. The intuitions above give us enough information to write out these rules directly, modeling the linear sequent calculus.

$$\frac{\Gamma, A@\alpha \vdash B@p * \alpha}{\Gamma \vdash A \multimap B@p} \multimap R^{\alpha}$$

In the elimination rule we see again how a split between the antecedents is represented as a split between the resources.

$$\frac{\Gamma, A \multimap B@\alpha \vdash A@q \quad \Gamma, A \multimap B@\alpha, B@\beta \vdash C@p * \beta}{\Gamma, A \multimap B@\alpha \vdash C@p * q * \alpha} \multimap L^{\beta}$$

By strengthening, we can see that the antecedent $A \multimap B@\alpha$ can not be used in either premise.

Unit. Here, we just have to enforce the emptiness of the resources.

$$\frac{}{\Gamma \vdash \mathbf{1}@\epsilon} \mathbf{1}R \qquad \frac{\Gamma, \mathbf{1}@\alpha \vdash C@p}{\Gamma, \mathbf{1}@\alpha \vdash C@p * \alpha} \mathbf{1}L$$

In the $\mathbf{1}L$ rule we replace α by ϵ and then use its unit property to obtain p.

2 Exponentials

Our representation technique for the sequent calculus using explicit resources is already rich enough to handle persistence. We just allow assumptions $A@\epsilon$ to indicate $A\ pers$. Since resource-annotated hypothesis already allow weakening and contraction, no additional structural rules are required. In $!R$, both premise and conclusion may not use any resources, so they are at ϵ. In $!L$, the resource α labeling $!A$ is consumed (that is, replaced by ϵ, which is eliminated by the unit property of '$*$').

$$\frac{\Gamma \vdash A@\epsilon}{\Gamma \vdash !A@\epsilon}\ !R \qquad \frac{\Gamma, !A@\alpha, A@\epsilon \vdash C@p}{\Gamma, !A@\alpha \vdash C@p * \alpha}\ !L$$

If we want to maintain a bijection between sequent proofs in the two systems, we also need a special judgmental rule which creates a fresh resource α and $A@\alpha$ from $A@\epsilon$. This is justified by the resource semantics, but nevertheless somewhat unexpected. A different solution will be presented in the next lecture.

$$\frac{\Gamma, A@\epsilon, A@\alpha \vdash C@p * \alpha}{\Gamma, A@\epsilon \vdash C@p}\ \mathsf{copy}^{\alpha}$$

The resulting system is summarized in Figure 1.

An interesting aspect of this system is that we did not need to generalize the available judgments when we added the exponentials; the empty resource and the hypothetical judgment was sufficient. We did however, need a new form of cut because the previous version allowed only to cut a hypothesis $A@\alpha$ for a resource parameter α.

3 Correspondence

It is now easy to establish that the resource calculus is in bijective correspondence with the linear sequent calculus. Moroever, it satisfies the expected properties of cut and identity. Key is the crucial strengthening property.

For the remainder of this lecture we assume that a resource context has hypotheses of the form $A@\epsilon$ and $A@\alpha$, where all resource parameters α are distinct, and the succedent has the form $C@p$, where p is a product of distinct resource parameters. We write $\alpha \notin p$ if α does not occur in p. The equational theory for resources remains associativity and commutatitivity for '$*$' with unit ϵ.

Theorem 1 (Strengthening for Resource Semantics) *If $\Gamma, A@\alpha \vdash C@p$ and $\alpha \notin p$ then $\Gamma \vdash C@p$ with the same proof.*

Proof: By induction on the structure of the given proof. □

Theorem 2 (Identity) *In the system where identity is restricted to atomic propositions, general identity is admissible. That is, $A@\alpha \vdash A@\alpha$ for any proposition A.*

$$\frac{}{\Gamma, A@\alpha \vdash A@\alpha}\ \mathsf{init}$$

$$\frac{\Gamma \vdash A@p \quad \Gamma \vdash A@\alpha \vdash C@\alpha * q}{\Gamma \vdash C@p * q}\ \mathsf{cut} \qquad \frac{\Gamma \vdash A@\epsilon \quad \Gamma, A@\epsilon \vdash C@p}{\Gamma \vdash C@p}\ \mathsf{cut!}$$

$$\frac{\Gamma \vdash A@p \quad \Gamma \vdash B@q}{\Gamma \vdash A \otimes B@p * q}\ \otimes R$$

$$\frac{\Gamma, A \otimes B@\alpha, A@\beta, B@\gamma \vdash @p * \beta * \gamma}{\Gamma, A \otimes B@\alpha \vdash C@p * \alpha}\ \otimes L^{\beta,\gamma}$$

$$\frac{\Gamma, A@\alpha \vdash B@p * \alpha}{\Gamma \vdash A \multimap B@p}\ \multimap R^{\alpha}$$

$$\frac{\Gamma, A \multimap B@\alpha \vdash A@q \quad \Gamma, A \multimap B@\alpha, B@\beta \vdash C@p * \beta}{\Gamma, A \multimap B@\alpha \vdash C@p * q * \alpha}\ \multimap L^{\beta}$$

$$\frac{}{\Gamma \vdash \mathbf{1}@\epsilon}\ \mathbf{1}R \qquad \frac{\Gamma, \mathbf{1}@\alpha \vdash C@p}{\Gamma, \mathbf{1}@\alpha \vdash C@p * \alpha}\ \mathbf{1}L$$

$$\frac{\Gamma \vdash A@\epsilon}{\Gamma \vdash !A@\epsilon}\ !R \qquad \frac{\Gamma, !A@\alpha, A@\epsilon \vdash C@p}{\Gamma, !A@\alpha \vdash C@p * \alpha}\ !L$$

$$\frac{\Gamma, A@\epsilon, A@\alpha \vdash C@p * \alpha}{\Gamma, A@\epsilon \vdash C@p}\ \mathsf{copy}^{\alpha}$$

Figure 1: Resource Semantics for Multiplicative Exponential Linear Logic

Proof: By induction on the structure of A. □

Theorem 3 (Cut) *In the system without the* cut *and* cut! *rules, they are admissible. That is,*

(i) If $\Gamma \vdash A@q$ *and* $\Gamma, A@\alpha \vdash C@p * \alpha$ *then* $\Gamma \vdash C@p * q$*.*

(ii) If $\Gamma \vdash A@\epsilon$ *and* $\Gamma, A@\epsilon \vdash C@p$ *then* $\Gamma \vdash C@p$*.*

Proof: By nested induction, first on the cut formula A, then on the form of cut where $(i) < (ii)$, then on the structure of the proofs in the two premises (one must become smaller while the other remains the same). □

In order to formulate a correspondence theorem, we need to express relationships between assumptions. We write $(A_1, \ldots, A_n)@\epsilon = A_1@\epsilon, \ldots, A_n@\epsilon$ and $(A_1, \ldots, A_n)@\vec{\alpha} = A_1@\alpha_1, \ldots, A_n@\alpha_n$. Furthermore, we need to construct a pair of contexts $\Delta; \Gamma$ from given a resource context. For ease of definition, we do not require a separation of zones but write Ψ for a mixed context with assumptions $A\ pers$ and $A\ eph$ (expressing a truth antecedent which is linear).

$$
\begin{array}{lcl}
(\cdot)|_\epsilon & = & (\cdot) \\
(\Gamma, \Gamma')|_{p*q} & = & \Gamma|_p, \Gamma'|_q \\
(A@\alpha)|_\alpha & = & A\ eph \\
(A@\alpha)|_\epsilon & = & (\cdot) \\
(A@\epsilon)|_\epsilon & = & A\ pers
\end{array}
$$

Because of the equational theory, this definition has some nondeterminism. Under the general assumptions of this section, $\Gamma|_p$ will be defined and unique.

Theorem 4 (Correspondence)

(i) If $\Delta; \Gamma \vdash C$ *then* $\Delta@\epsilon, \Gamma@\vec{\alpha} \vdash C@\alpha_1 * \cdots * \alpha_n$*.*

(ii) If $\Gamma \vdash C@p$ *then* $\Gamma|_p \vdash C$*.*

Moreover, the correspondence between linear and resource proofs is a bijection.

Proof: By straightforward inductions, exploiting strengthening. □

References

[Ree09] Jason C. Reed. *A Hybrid Logical Framework*. PhD thesis, Carnegie Mellon University, September 2009. Available as Technical Report CMU-CS-09-155.