Lecture Notes on
Resource Management

15-816: Linear Logic
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Backward chaining, or other linear logic proof search procedures that work backwards from the conclusion, have to deal with the problem of *resource management*. The basic issue can be seen in the rule for $\otimes R$:

$$\frac{\Gamma \,;\, \Delta_1 \rightarrow [A] \quad \Gamma \,;\, \Delta_2 \rightarrow [B]}{\Gamma \,;\, \Delta_1, \Delta_2 \rightarrow [A \otimes B]} \otimes R$$

The persistent antecedents Γ are propagated to both subgoals, but the linear antecedents have to be split into Δ_1 and Δ_2. A priori, we do not know how to divide up the context, and guessing would be extremely inefficient. In Lecture 23 we will see a general way to address this through *resource constraints*. Here, we look for a simpler and more efficient solution that is appropriate in backward chaining. The difference is that backward chaining, since it forms the basis of a logic programming language, should proceed according to fixed search strategy. We have already mentioned that it selects subgoals in left-to-right order. We exploit this in the following manner: instead of splitting the resources, we propagate *all* resources to the first subgoal and keep track of which resources are consumed. We then propagate those that are *not* consumed to the second subgoal.

In the remainder of this lecture we will work at making this intuition precise. As we have done so far, we capture a particular search strategy by providing an inference system whose proofs follow the desired behavior.

Although research management need not be strictly tied to focusing, we will do so here. Other presentations have been given in the literature [HM94, CHP00]. Recall the grammar for the backward-chaining fragment of linear logic:

Clauses	D	$::=$	$P^- \mid G \multimap D \mid D_1 \mathbin{\&} D_2 \mid \top \mid \forall x.\, D$
Goals	G	$::=$	$P^+ \mid G_1 \otimes G_2 \mid \mathbf{1} \mid G_1 \oplus G_2 \mid \mathbf{0} \mid \exists x.\, G \mid !D \mid D$

1 Multiplicative Connectives

The multiplicative connectives $A \otimes B$, $\mathbf{1}$, $A \multimap B$ provide the motivation for resource management and are therefore also easily described. Our main jugments are

Right focus	$\Gamma \,;\, \Delta_I/\Delta_O \to [G]$
Left focus	$\Gamma \,;\, \Delta_I/\Delta_O \,;\, [D] \to P^-$
Inversion	$\Gamma \,;\, \Delta_I/\Delta_O \,;\, \Omega \to D$
Stable	$\Gamma \,;\, \Delta_I/\Delta_O \to P^-$

where

Inversion context	Ω	$::=$	$\cdot \mid G, \Omega$
Clauses	Δ	$::=$	$\cdot \mid \Delta, D$
Programs	Γ	$::=$	$\cdot \mid \Gamma, D$

Since this is a refinement of focusing, at most one formula in a sequent can be in focus. The *inversion context* Ω is *ordered*, in effect fixing the order in which the inversion steps are performed. We already know that the order does not matter, so this does not change provability.

The notation Δ_I/Δ_O means that we have *input resources* Δ_I and *output resources* Δ_O. It will always be the case that $\Delta_I \supseteq \Delta_O$, and the difference $\Delta_I - \Delta_O$ are the resources consumed in the proof of the judgment.

We start with the motivating example. We omit Γ in the rules since they are (boringly) propagated everywhere.

$$\frac{\Delta_I/\Delta_M \to [G_1] \quad \Delta_M/\Delta_O \to [G_2]}{\Delta_I/\Delta_O \to [G_1 \otimes G_2]} \otimes R$$

The strategy to always search for a proof of the first premise is baked into this rule, although not in any formal way. We can get a proof of the conclusion if we have proofs of both premises, but if our search is to respect the modes (Δ_I is input and Δ_O is output), the the first premise needs to be attacked first.

The left rule is invertible and does not affect the resources, except by removing the resource $G_1 \otimes G_2$ itself. We model this by separating the antecedents that are decomposed from those are are passed along.

$$\frac{\Delta_I/\Delta_O \; ; G_1, G_2, \Omega \rightarrow D'}{\Delta_I/\Delta_O \; ; G_1 \otimes G_2, \Omega \rightarrow D'} \otimes L$$

When we reach clauses (that is, negative propositions) in Δ, they are moved into Δ_I. As part of this move we label them with a new variable x. This is so we can ensure that this particular resource is indeed consumed in the subproof. We cannot pass x out of its proper scope.

$$\frac{\Delta_I, x{:}D/\Delta_O \; ; \Omega \rightarrow D' \quad (x{:}D \notin \Delta_O)}{\Delta_I/\Delta_O \; ; D, \Omega \rightarrow D'} \; \mathsf{deactivate}$$

The way assumptions are consumed is by focusing on them.

$$\frac{\Delta_I/\Delta_O \; ; [D] \rightarrow P^-}{\Delta_I, x{:}D/\Delta_O \rightarrow P^-} \; \mathsf{foc}L$$

At the identity rule, no assumption besides the one in focus is removed, so all of the input resources are returned as output resources:

$$\frac{}{\Delta_I/\Delta_I \; ; [P^-] \rightarrow P^-} \; \mathsf{id}_{P^-}$$

The rules for linear implication follow the dual pattern to that for multiplicative conjunction.

$$\frac{\Delta_I/\Delta_M \; ; [D] \rightarrow P^- \quad \Delta_M/\Delta_O \rightarrow [G]}{\Delta_I/\Delta_O \; ; [G \multimap D] \rightarrow P^-} \multimap L$$

The only consideration here is the order of the two subgoals. We first check to see if D eventually matches P^-, which is therefore our first premise, and then solve G only if the first premise succeeds and returns Δ_M as unused so far. For the right rule, we require the inversion context Ω to be empty. This is just part of our technique of removing nondeterminism from the inversion phase of focusing.

$$\frac{\Delta_I/\Delta_O \; ; G \rightarrow D}{\Delta_I/\Delta_O \; ; \cdot \rightarrow G \multimap D} \multimap R$$

Eventually, inversion finishes with an atomic conclusion and no positive antecedents, in which case we have reached a stable sequent.

$$\frac{\Delta_I/\Delta_O \to P^-}{\Delta_I/\Delta_O \,;\, \cdot \to P^-}\ \mathsf{stabilize}$$

The multiplicative unit ($\mathbf{1}$), meaning empty resources, follows a similar patter.

$$\frac{}{\Delta_I/\Delta_I \to [\mathbf{1}]}\ \mathbf{1}R \qquad \frac{\Delta_I/\Delta_O \,;\, \Omega \to D}{\Delta_I/\Delta_O \,;\, \mathbf{1}, \Omega \to D}\ \mathbf{1}L$$

2 Soundness and Completeness

As a restriction or modification of focusing, we have to show that the new rules are both *sound* (do not prove too much) and *complete* (do not prove too little).

We proceed by writing down the "obvious", to see if we can perhaps prove it easily. As usual, we ignore the persistent resources for now. We write $\Delta_I - \Delta_O$ for set difference, that is, the set of elements of Δ_I that are not in Δ_O. We have labeled each element in Δ_I with a unique label, so that these really are sets of $x{:}D$. We also always have the $\Delta_I \supseteq \Delta_O$, which can easily be shown by induction on the judgments. At the moment, we can only check the cases for the connectives we have already discussed, but we can take the proof structure as a guide for further rules.

Theorem 1 (Soundness of Resource Management, Version I)

(i) *If* $\Delta_I/\Delta_O \to [G]$ *then* $(\Delta_I - \Delta_O) \to [G]$

(ii) *If* $\Delta_I/\Delta_O \,;\, [D] \to P^-$ *then* $(\Delta_I - \Delta_O), [D] \to P^-$

(iii) *If* $\Delta_I/\Delta_O \,;\, \Omega \to D$ *then* $(\Delta_I - \Delta_O), \Omega \to D$

(iv) *If* $\Delta_I/\Delta_O \to P^-$ *then* $(\Delta_I - \Delta_O) \to P^-$

Proof: By a straightforward mutual induction over the given derivations. The crucial observation, for example in the case of $\otimes R$ is that $(\Delta_I - \Delta_O) = (\Delta_I - \Delta_M) \cup (\Delta_M - \Delta_O)$. □

Completeness is more difficult. This is usually the case when proposing a more deterministic or restrictive search strategy since we have to show that we don't miss anything provable.

Again, we start with the straightforward statement. We formulate it only for stable sequents; the modification for the other forms of focusing sequents should be clear: in

Theorem 2 (Completeness of Resource Management, Version I)

(i) *If* $\Delta \rightarrow [G]$ *then* $\Delta/\cdot \rightarrow [G]$.

(ii) *If* $\Delta, [D] \rightarrow P^-$ *then* $\Delta/\cdot\, ; [D] \rightarrow P^-$.

(iii) *If* $\Delta, \Omega \rightarrow D$ *then* $\Delta/\cdot\, ; \Omega \rightarrow D$.

(iv) *If* $\Delta \rightarrow P^-$ *then* $\Delta/\cdot \rightarrow P^-$.

Proof: For *(iii)*, we have to show that the order of the resources in Ω does not matter, a property we leave as a later exercise. The remainder of the proof is just concerned with the input/output interpretation of linear resources.

At first, this does not look very promising, since we only appear to construct sequents with empty output, while we know that more complicated ones arise. Trying an induction over the structure of the given derivation, and examining a critical case:

Case:

$$\frac{\Delta_1 \rightarrow [G_1] \quad \Delta_2 \rightarrow [G_2]}{\Delta_1, \Delta_2 \rightarrow [G_1 \otimes G_2]} \otimes R$$

where $\Delta = (\Delta_1, \Delta_2)$. Applying the induction hypothesis and attempting to reassemble a proof, we get

$$\frac{\begin{array}{cc}\text{i.h.}(i) & \text{i.h.}(i) \\ \Delta_1/\cdot \rightarrow [G_1] & \Delta_2/\cdot \rightarrow [G_2]\end{array}}{\Delta_1, \Delta_2/\cdot \rightarrow [G_1 \otimes G_2]} \otimes R?$$

We see that the two parts do not fit together.

At this point we are confronted with a choice. One is to look for lemmas in the target of the completeness proof. We used this strategy, for example, when proving the completeness of chaining, showing that each unfocused rule is admissible in the target.

An alternative is to look for a generalization of the induction hypothesis. Typically, we have to do this when the induction hypothesis is too weak to apply the deduction of the premises of the rules. This does not seem to be the case here.

Our experience has been that a proof is clearer and less subject to mistakes, if we keep the induction hypothesis simpler, when possible, and look to properties in the target of the translation to make the proof go through. Here, both appear to be possible (see Exercise 1).

Here, we pursue a so-called *frame lemma* (stated below) that allows us adjoin unused resources to a given proof. With that, we obtain:

$$\dfrac{\dfrac{\dfrac{\text{i.h.}(i)}{\Delta_1/\cdot \to [G_1]}}{\Delta_1,\Delta_2/\Delta_2 \to [G_1]}\,\mathsf{frame} \qquad \dfrac{\text{i.h.}(i)}{\Delta_2/\cdot \to [G_2]}}{\Delta_1,\Delta_2/\cdot \to [G_1 \otimes G_2]}\,\otimes R$$

This case (and the frame lemma) contain the essence of the argument, so we do not write out any additional cases explicitly.

□

Lemma 3 (Frame)
If $\Delta_I/\Delta_O \to P^-$ *then* $(\Delta_I,\Delta')/(\Delta_O,\Delta') \to P^-$ *for any* Δ'*, and similarly for the other resource management judgments.*

Proof: By straightforward mutual induction on the structure of the given deduction. □

3 May- and Must-Consume Resources

In the judgments above, Δ_I is a context of resources that *may be consumed*. The remaining ones are passed on. At the deactivate rule we check that $x{:}D$ has indeed been used. This check may come somewhat late: perhaps we have missed our last opportunity to use $x{:}D$ somewhere before during proof construction. In that case we should have failed earlier. To account for this we introduce another context Ξ of resources that *must be consumed* during the proof of the judgment. They cannot be passed on. This avoids an a posteriori check that resources have indeed been used.

The generalized judgments are:

Right focus	$\Gamma \; ; \; \Xi \; ; \; \Delta_I / \Delta_O \rightarrow [G]$
Left focus	$\Gamma \; ; \; \Xi \; ; \; \Delta_I / \Delta_O \; ; \; [D] \rightarrow P^-$
Inversion	$\Gamma \; ; \; \Xi \; ; \; \Delta_I / \Delta_O \; ; \; \Omega \rightarrow D$
Stable	$\Gamma \; ; \; \Xi \; ; \; \Delta_I / \Delta_O \rightarrow P^-$

where Ξ is composed of clauses D. Let's reexamine $\otimes R$. Somehow, Ξ must be consumed in the whole proof. But it need not be consumed in the first premise, proving G_1, so we move it to the may-be-consumed context. If it is not consumed there, it *must* be consumed in the proof of the second premise, proving G_2, so we move the remaining resources back.

$$\frac{\cdot \; ; \; (\Xi \cup \Delta_I)/\Delta_M \rightarrow [G_1] \quad (\Delta_M \cap \Xi) \; ; \; (\Delta_M \cap \Delta_I)/\Delta_O \rightarrow [G_2]}{\Xi \; ; \; \Delta_I/\Delta_O \rightarrow [G_1 \otimes G_2]} \otimes R$$

We wrote here $\Xi \cup \Delta_I$ instead of the usual (Ξ, Δ_I) to emphasize the (disjoint) union, from which we recover the components through intersections.

The $\otimes L$ rule does not change, but deactivation now adds a clause to Ξ, and does not need to check that it has been consumed, because this is enforced directly in the proof of the premise.

$$\frac{\Xi, x{:}D \; ; \; \Delta_I/\Delta_O \; ; \; \Omega \rightarrow P^-}{\Xi \; ; \; \Delta_I/\Delta_O \; ; \; D, \Omega \rightarrow P^-} \, \mathsf{deactivate}$$

There is now an additional focus rule, depending on whether we pick the resource to focus on from Δ_I or Ξ

$$\frac{\Xi \; ; \; \Delta_I/\Delta_O \; ; \; [D] \rightarrow P^-}{\Xi \; ; \; \Delta_I, x{:}D/\Delta_O \rightarrow P^-} \, \mathsf{foc}L \qquad \frac{\Xi \; ; \; \Delta_I/\Delta_O \; ; \; [D] \rightarrow P^-}{\Xi, x{:}D \; ; \; \Delta_I/\Delta_O \rightarrow P^-} \, \mathsf{foc}L'$$

In the identity rule (and also the $\mathbf{1}R$ rule), there cannot be any resources left that must be consumed.

$$\frac{}{\cdot \; ; \; \Delta_I/\Delta_I \; ; \; [P^-] \rightarrow P^-} \, \mathsf{id}_{P^-} \qquad \frac{}{\cdot \; ; \; \Delta_I/\Delta_I \rightarrow [\mathbf{1}]} \, \mathbf{1}R$$

The remaining rules are modified in the obvious way.

We need to generalize the soundness and completeness theorem, concentrating on the stable sequents. The others are all similar.

Theorem 4 (Soundness of Resource Management, Version II)
If $\Xi\,;\,\Delta_I/\Delta_O \to P^-$ *then* $\Xi, (\Delta_I - \Delta_O) \to P^-$

Proof: By mutual induction on the given deductions, after generalization to include the other resource management judgments. □

The completeness theorem works as before, with most of the effort isolated into a generalized form of the frame lemma. We will not state it formally (see Exercise 2), but we will need something like the following properties:

(i) If $\Xi\,;\,\Delta_I/\Delta_O \to P^-$ then $\Xi\,;\,(\Delta_I \cup \Delta')/(\Delta_O \cup \Delta') \to P^-$ for any Δ'.

(ii) If $\Xi\,;\,\Delta_I/\Delta_O \to P^-$ then $\cdot\,;\,(\Xi \cup \Delta_I)/\Delta_O \to P^-$.

(iii) If $\Xi\,;\,\Delta_I/\Delta_O \to P^-$ then $\Xi \cup (\Delta_I - \Delta_O)\,;\,\cdot/\cdot \to P^-$.

Theorem 5 (Completeness of Resource Management, Version II)
If $\Delta \to P^-$ *then* $\Delta\,;\,\cdot/\cdot \to P^-$.

4 The Exponential

The exponential introduces little or no complication, since the persistent context is not subject to resource management. But we need to ensure that the empty context is indeed empty where required.

$$\frac{\Gamma\,;\,\Xi\,;\,\Delta_I/\Delta_O\,;\,[D] \to P^- \quad (u{:}D \in \Gamma)}{\Gamma\,;\,\Xi\,;\,\Delta_I/\Delta_O \to P^-}\ \text{copy}$$

$$\frac{\Gamma\,;\,\cdot\,;\,\cdot/\cdot\,;\,\cdot \to D}{\Gamma\,;\,\cdot\,;\,\Delta_I/\Delta_I \to [!D]}\ !R \qquad \frac{\Gamma, u{:}D\,;\,\Xi\,;\,\Delta_I/\Delta_O\,;\,\Omega \to D'}{\Gamma\,;\,\Xi\,;\,\Delta_I/\Delta_O\,;\,!D, \Omega \to D'}\ !L$$

5 Additive Connectives

Additive conjunction (also called alternative conjunction) requires some thought. Both premises of the right rule must be proven with the same resources. The straightforward rendering of that would be

$$\frac{\Xi\,;\,\Delta_I/\Delta_O\,;\,\cdot \to D_1 \quad \Xi\,;\,\Delta_I/\Delta_O\,;\,\cdot \to D_2}{\Xi\,;\,\Delta_I/\Delta_O\,;\,\cdot \to D_1 \,\&\, D_2}\ \&R?$$

While clearly sound and complete, this rule has some unwanted nondeterminism. We know we will search for proofs of the two subgoals in order: first D_1, second D_2. Only after the second one completes can we check is the outputs match up. A potentially less redundant method would be to say that the second subgoal must consume *exactly* what the first one does, namely Ξ and $\Delta_I - \Delta_O$. This reasoning yields the following

$$\frac{\Xi \,;\, \Delta_I/\Delta_O \,;\, \cdot \to D_1 \quad \Xi \cup (\Delta_I - \Delta_O) \,;\, \cdot/\cdot \,;\, \cdot \to D_2}{\Xi \,;\, \Delta_I/\Delta_O \,;\, \cdot \to D_1 \,\&\, D_2}\,\&R$$

The nullary conjunction, $\top$, introduces some new complications. On the left, the usual rule from the focused sequent calculus; on the right a proposed rendering.

$$\frac{}{\Delta \to \top}\,\top R \qquad \frac{\Delta_I \supseteq \Delta_O}{\Xi \,;\, \Delta_I/\Delta_O \,;\, \cdot \to \top}\,\top R$$

The new rule acknowledges that $\top$ could consume any set of resources presented to it: from all inputs (Δ_I) to no inputs ($\cdot$), in addition to all of Ξ (which it must consume). This is clearly an exponential rule, since we do not know which resources from Δ_O might be consumed by subsequent proof steps.

We can avoid the nondeterminism by introducing a new context, Σ_O, which represents those assumptions from Δ_I that *may* be consumed later, while Δ_O contains those that *must* be consumed later. We write

$$\Xi \,;\, \Delta_I/\Delta_O/\Sigma_O \to P^-$$

and similarly for the other resource management judgments.

Then the $\top R$ rule would become:

$$\frac{}{\Xi \,;\, \Delta_I/\cdot/\Delta_I \,;\, \cdot \to \top}\,\top R \qquad \frac{}{\Xi \,;\, \Delta_I/\Delta_I/\cdot \to [\mathbf{1}]}\,\mathbf{1}R$$

We also show the $\mathbf{1}R$ rule for contrast. Now all the other rules need to be reexamined as well. In linear logic without affine resources, any judgment will either have Δ_O or Σ_O while the other is empty. The status of the output resources can therefore be indicated by a boolean flag associated with a sequent. This is spelled out in [CHP00].

When we also have affine resource, it is plausible that we can use both contexts, simultaneously. Working out such a system is the subject of Exercise 3.

Exercises

Exercise 1 (Alternative Completeness Proof) Generalize the induction hypothesis in the proof of completeness of resource management (Theorem 2) to prove the theorem without an explicit frame lemma (Lemma 3).

Exercise 2 (Frame Lemma, Version II) Carefully state the frame lemma for Section 3 in sufficient generality to allow the proof of completeness of resource management to go through, as well as a simple inductive proof of the frame property itself.

Exercise 3 (Affine Resources) Add affine resource to linear logic and give appropriate resource management rules for the judgment

$$\Gamma \,;\, \Psi \,;\, \Xi \,;\, \Delta_I/\Delta_O/\Sigma_O \to P^-$$

and its generalization to the other focusing judgments. We have

Γ	Persistent resources
Ψ	Affine resources
Ξ	Linear resources that must be used
Δ_I	Linear resources that may be used or passed on
Δ_O	Unused resources that must be used later
Σ_O	Unused resources that may be used later

with the invariants that all input contexts (Γ, Ψ, Ξ, Δ_I) are disjoint, as are all output contexts (Δ_O, Σ_O). Furthermore, $\Delta_I \supseteq \Delta_O \cup \Sigma_O$ and $\Psi \supseteq \Sigma_O$.

State completeness and the frame properties in sufficient generality so they follow by simple structural inductions.

References

[CHP00] Iliano Cervesato, Joshua S. Hodas, and Frank Pfenning. Efficient resource management for linear logic proof search. *Theoretical Computer Science*, 232(1–2):133–163, February 2000. Special issue on Proof Search in Type-Theoretic Languages, D. Galmiche and D. Pym, editors.

[HM94] Joshua Hodas and Dale Miller. Logic programming in a fragment of intuitionistic linear logic. *Information and Computation*, 110(2):327–365, 1994. A preliminary version appeared in the Proceedings of the Sixth Annual IEEE Symposium on Logic in Computer Science, pages 32–42, Amsterdam, The Netherlands, July 1991.