Lecture Notes on Ordered Logic

15-816: Linear Logic
Robert J. Simmons

Lecture 15
March 12, 2012

In intuitionistic linear logic, a proposition A can be thought of as a resource which can (and must) be used exactly one time. Linear logic also has the persistent modality $!A$, which represents persistent propositions that can be used any number of times. The midterm asked you to think about an affine modality $@A = A \mathbin{\&} \mathbf{1}$, representing resources that could be used at most once, and a strict modality $\#A = A \otimes !A$, representing resources that you must use, but may use one more more times.

In this note, we'll look at another way to think about the relationship between persistent, affine, strict, and linear logics, based on Gentzen's original presentation of a sequent calculus for persistent intuitionistic logic. Then, we'll use that discussion to motivate ordered logic.

1 Gentzen's presentation of logic

In 1935, Gentzen presented the first sequent calculus for (persistent) intuitionistic and classical logic [Gen35]. That paper is in German, but the various English translations (the translated title is "Investigations into Logical Deduction") are extraordinarily readable for a modern audience. We'll write Gentzen-style sequents as $\Psi \longrightarrow A$. One important distinction is that Gentzen treated Ψ not as a bag or multiset, the way we do, but as a *sequence* – so when we write the usual implication and initial rules as they appear in Gentzen's paper...

$$\frac{}{A \longrightarrow A}\, init \qquad \frac{A, \Psi \longrightarrow A \supset B}{\Psi \longrightarrow A \supset B}\, {\supset}R \qquad \frac{\Psi_1 \longrightarrow A \quad B, \Psi_2 \longrightarrow C}{A \supset B, \Psi_1, \Psi_2 \longrightarrow C}\, {\supset}L$$

…it is *not* the case that we can immediately prove the sequent $C, A, A \supset B \longrightarrow B$, which is true in persistent intuitionistic logic, using rule $\supset L$, because $C, A, A \supset B$ does not match the pattern $A \supset B, \Psi_1, \Psi_2$.

The way Gentzen deals with this is by introducing *structural* rules of weakening (sW), contraction (sC), and exchange (sE).

$$\frac{\Psi \longrightarrow C}{A, \Psi \longrightarrow C}\ s\mathsf{W} \qquad \frac{A, A, \Psi \longrightarrow C}{A, \Psi \longrightarrow C}\ s\mathsf{C} \qquad \frac{\Psi_1, B, A, \Psi_2 \longrightarrow C}{\Psi_1, A, B, \Psi_2 \longrightarrow C}\ s\mathsf{E}$$

By using two instances of the exchange rule, we can apply the $\supset L$ rule, letting $\Psi_1 = C, A$ and letting $\Psi_2 = \cdot$, and thereby prove our example sequent:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \longrightarrow A}\ init}{C, A \longrightarrow A}\ s\mathsf{W} \quad \dfrac{}{B \longrightarrow B}\ init}{A \supset B, C, A \longrightarrow B}\ \supset L}{C, A \supset B, A \longrightarrow B}\ s\mathsf{E}}{C, A, A \supset B \longrightarrow B}\ s\mathsf{E}$$

1.1 Structural properties as explicit or admissible rules

The approach that we've taken in this class is to let weakening, contraction, and exchange be *admissible* rules that don't change the structure of a derivation. Remember that, when we used the induction hypothesis in the cut admissibility proof for linear logic, either the principal formula A had to get smaller, or else the principal formula had to stay the same while one of the derivations got smaller and the other derivation *stayed the same size*. At one point in this cut admissibility proof, we're given a proof $\mathcal{D}$ of the sequent $\Gamma; \Delta \vdash A$ and we need a proof of $\Gamma, B; \Delta \vdash A$. Because of the way our system works, we can weaken $\mathcal{D}$ to obtain $\mathcal{D}'$, a proof of $\Gamma, B; \Delta \vdash A$ with the same structure as $\mathcal{D}$. This allows us to call the induction hypothesis on $\mathcal{D}'$, because it is the same size as $\mathcal{D}$.

Gentzen's system doesn't work this way. Weakening is *not* admissible, because we cannot prove $A, B \longrightarrow A$ directly with the *init* rule; we can only prove $A \longrightarrow A$ with *init*, then $B, A \longrightarrow A$ with the weakening rule W, then $A, B \longrightarrow A$ with the exchange rule E. And that resulting derivation is not structurally the same as one that just uses an *init* rule. These extra rules complicate the cut admissibility theorem for Gentzen-style sequent calculi.

1.2 Multicut

Gentzen's proof ultimately still proceeds by lexicographic induction, first over the size of the principal cut formula A and second over the *rank* of the two derivations, which is a metric that is related to their structure.

The cut admissibility principle as we usually formulate it – "If $\Psi \longrightarrow A$ and $A, \Psi' \longrightarrow C$ then $\Psi, \Psi' \longrightarrow C$" – is not general enough in Gentzen-style logics due to the contraction rule $s\mathsf{C}$. Given that formulation of cut, we are unable to handle this case:

$$\dfrac{\begin{matrix}\mathcal{D} \\ \Psi \longrightarrow A\end{matrix} \quad \dfrac{\begin{matrix}\mathcal{E}_1 \\ A, A, \Psi' \longrightarrow C\end{matrix}}{A, \Psi' \longrightarrow C}\, s\mathsf{C}}{\Psi, \Psi' \longrightarrow C}\, \mathsf{cut}_A$$

If we cut $\mathcal{D}$ against $\mathcal{E}_1$, we would get a derivation $\mathcal{F}_1$ of $\Psi, A, \Psi' \longrightarrow C$. We can't apply the induction hypothesis a second time, because $\mathcal{F}_1$ may be much larger than $\mathcal{D}$ or $\mathcal{E}_1$, and the principal cut formula A is still the same size.

Therefore, Gentzen used the following generalization of the induction hypothesis:

Theorem 1 *If $\Psi \longrightarrow A$, $\Psi' \longrightarrow C$, and if $\Psi'_{/A}$ is Ψ' with all occurrences of A removed, then $\Psi, \Psi'_{/A} \longrightarrow C$.*

2 The family of substructural logics

Let's return to our ordinary linear logic in which contexts are multisets and *add* a weakening rule and a contraction rule. We'll also go back to treating the contexts as multisets, though, which means the exchange rule is trivially admissible, just like it's been all along. We'll distinguish this strange linear logic with explicit weakening and exchange from the linear logic we know and love by writing down the sequents as $\Delta \Vdash^{wc} A$.

$$\dfrac{\Delta \Vdash^{wc} C}{\Delta, A \Vdash^{wc} C}\, s\mathsf{W} \qquad \dfrac{\Delta, A, A \Vdash^{wc} C}{\Delta, A \Vdash^{wc} C}\, s\mathsf{C}$$

By the same token, $\Delta \Vdash^{w} A$ just has the weakening rule $s\mathsf{W}$, and $\Delta \Vdash^{c} A$ just has the contraction rule $s\mathsf{C}$.

$\Delta \Vdash^{w} A$ is affine logic, which we can easily verify by showing that the affine modality see because $@A = A \mathbin{\&} \mathbf{1}$ is interprovable with A when we have the weakening rule.

$$\dfrac{\dfrac{}{A \Vdash^{w} A}\,\mathsf{id}_A}{A \mathbin{\&} \mathbf{1} \Vdash^{w} A}\,\&L_1 \qquad \dfrac{\dfrac{}{A \Vdash^{w} A}\,\mathsf{id}_A \quad \dfrac{\dfrac{}{\cdot \Vdash^{w} \mathbf{1}}\,\mathbf{1}R}{A \Vdash^{w} \mathbf{1}}\,s\mathsf{W}}{A \Vdash^{w} A \mathbin{\&} \mathbf{1}}\,\&R$$

$\Delta \Vdash^{c} A$ is strict logic, as each assumption in Δ can be copied but must be used at least once. However, it is not the case that $\#A = A \otimes !A$ is interprovable with A.

Similarly, $\Delta \Vdash^{wc} A$ is essentially just a different version of Gentzen's presentation of persistent intuitionistic logic where exchange is built in. In persistent intuitionistic logic, the two different versions of conjunction are interprovable: by using weakening, we can prove $A \otimes B \Vdash^{w} A \mathbin{\&} B$, and by using contraction, we can prove $A \mathbin{\&} B \Vdash^{c} A \otimes B$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \Vdash^{w} A}\,\mathsf{id}_A}{A, B \Vdash^{w} A}\,s\mathsf{W} \quad \dfrac{\dfrac{}{B \Vdash^{w} B}\,\mathsf{id}_B}{A, B \Vdash^{w} B}\,s\mathsf{W}}{A, B \Vdash^{w} A \mathbin{\&} B}\,\&R}{A \otimes B \Vdash^{w} A \mathbin{\&} B}\,\otimes L \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A \Vdash^{c} A}\,\mathsf{id}_A}{A \mathbin{\&} B \Vdash^{c} A}\,\&L_1 \quad \dfrac{\dfrac{}{B \Vdash^{c} B}\,\mathsf{id}_B}{A \mathbin{\&} B \Vdash^{c} B}\,\&L_2}{A \mathbin{\&} B, A \mathbin{\&} B \Vdash^{c} A \otimes B}\,\otimes R}{A \mathbin{\&} B \Vdash^{c} A \otimes B}\,s\mathsf{C}$$

This is interesting because, as we have seen in the course's previous discussions of focusing, $A \otimes B$ is a positive proposition whereas $A \mathbin{\&} B$ is a negative proposition. In plain-vanilla persistent intuitionistic logic, we can treat individual instances of conjunction $A \wedge B$ as having a positive character (like $A \otimes B$) or as having a negative character (like $A \mathbin{\&} B$).

It is straightforward to prove that $\Delta \vdash A$ implies $\Delta \Vdash^{c} A$, $\Delta \Vdash^{w} A$, and $\Delta \Vdash^{wc} A$, that $\Delta \Vdash^{w} A$ implies $\Delta \Vdash^{wc} A$ (but not $\Delta \Vdash^{c} A$ or $\Delta \vdash A$), and so on. But what about exchange? Linear logic as we have been using it includes exchange as an admissible concept, but it's also possible to come up with an even more primitive logic that denies not just weakening and contraction, but also exchange. In the next section, we will look at a well-formed logic that admits none of Gentzen's structural rules, not even exchange. The relationship between all of these logics in terms of which structural rules they admit is shown in Figure 1.

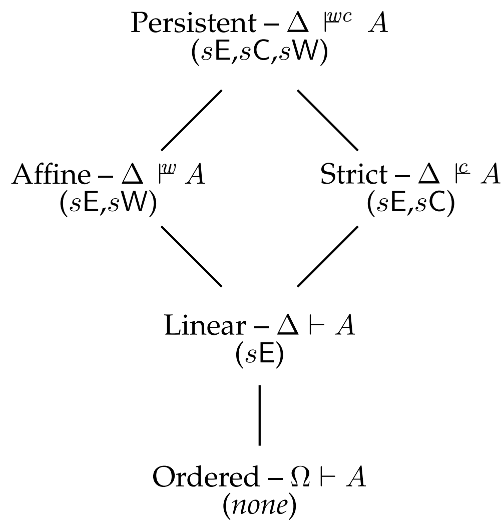

Figure 1: The relationship between persistent, affine, strict, linear, and ordered logics in terms of structural rules.

3 Ordered logic

Ordered logic was first developed by Lambek [Lam58], and was connected to linear logic by Polakow and Pfenning [PP99]. In linear logic we think of propositions in a context as resources to be consumed or processes to be utilized. In ordered logic, we look at propositions in context as tokens in a sequence or sentence; the order in which these tokens appear limits the ways in which they can interact. Sequents in ordered logic are written $\Omega \vdash A$, where the context Ω is either the empty context $\cdot$ or a sequence of comma-separated propositions $A_1, \ldots, A_n$ that are not subject to permutation.

As usual, we can restrict initial sequents to atomic propositions P and demand that cut and identity will both be admissible rules.

$$\frac{}{P \vdash P}\ \mathsf{id}_P \qquad \frac{}{A \vdash A}\ \mathsf{id}_A \qquad \frac{\Omega \vdash A \quad \Omega_L, A, \Omega_R \vdash C}{\Omega_L, \Omega, \Omega_R \vdash C}\ \mathsf{cut}_A$$

The identity theorem in ordered logic is the usual one; the cut admissibility theorem indicates that we can use cut out any proposition A in the context, which in ordered logic means that the context can have the form Ω_L, A, Ω_R.

This pattern – the interesting part of the context being surrounded by an outer Ω_L and Ω_R context – will be repeated for all the left rules in ordered logic. This makes ordered logic different from Gentzen's system, which only applied left rules to propositions on one side of the context. That didn't become a problem for Gentzen, of course, because exchange could be used to permute any proposition to the correct part of the context.

As we introduce the connectives of ordered logic in turn, we will give the associated cases of identity and the principal cuts; left and right commutative cuts also must to be considered in order to actually prove that cut is admissible.

3.1 Fuse

The multiplicative conjunction $A \otimes B$ in linear logic is replaced by the proposition $A \bullet B$ in ordered logic, which embodies concatenation and is pronounced "A fuse B."

$$\frac{\Omega_1 \vdash A \quad \Omega_2 \vdash B}{\Omega_1, \Omega_2 \vdash A \bullet B} \bullet R \qquad \frac{\Omega_L, A, B, \Omega_R \vdash C}{\Omega_L, A \bullet B, \Omega_R \vdash C} \bullet L$$

Like tensor in linear logic, fuse is asynchronous on the left, so we prove the relevant case of the identity theorem by decomposing the proposition on the left first:

$$\frac{\dfrac{A \vdash A \quad B \vdash B}{A, B \vdash A \bullet B} \bullet R}{A \bullet B \vdash A \bullet B} \bullet L$$

As in linear logic, a principal cut on $A \bullet B$ reduces to a cut on A and a cut on B.

$$\frac{\dfrac{\begin{matrix}\mathcal{D}_1 \\ \Omega_1 \vdash A\end{matrix} \quad \begin{matrix}\mathcal{D}_2 \\ \Omega_2 \vdash B\end{matrix}}{\Omega_1, \Omega_2 \vdash A \bullet B} \bullet R \quad \dfrac{\begin{matrix}\mathcal{E}_1 \\ \Omega_L, A, B, \Omega_R \vdash C\end{matrix}}{\Omega_L, A \bullet B, \Omega_R \vdash C} \bullet L}{\Omega_L, \Omega_1, \Omega_2, \Omega_R \vdash C} \mathsf{cut}_{A \bullet B} \quad \Longrightarrow$$

$$\frac{\begin{matrix}\mathcal{D}_1 \\ \Omega_1 \vdash A\end{matrix} \quad \dfrac{\begin{matrix}\mathcal{D}_2 \\ \Omega_2 \vdash B\end{matrix} \quad \begin{matrix}\mathcal{E}_1 \\ \Omega_L, A, B, \Omega_R \vdash C\end{matrix}}{\Omega_L, A, \Omega_2, \Omega_R \vdash C} \mathsf{cut}_B}{\Omega_L, \Omega_1, \Omega_2, \Omega_R \vdash C} \mathsf{cut}_A$$

3.2 Implication

One surprise is that ordered logic has two implications! The right ordered implication $A \twoheadrightarrow B$ throws propositions onto the right side of the context and the left ordered implication $A \rightarrowtail B$ throws propositions on the left side of the context.

$$\frac{\Omega, A \vdash B}{\Omega \vdash A \twoheadrightarrow B}\ {\twoheadrightarrow}R \qquad \frac{A, \Omega \vdash B}{\Omega \vdash A \rightarrowtail B}\ {\rightarrowtail}R$$

The easiest way to figure out what the correct left rules will be is to look at how the identity theorem will have to work:

$$\frac{\dfrac{\dfrac{}{A \vdash A}\ \mathsf{id}_A \quad \dfrac{}{B \vdash B}\ \mathsf{id}_B}{A \twoheadrightarrow B, A \vdash B}\ {\twoheadrightarrow}L}{A \twoheadrightarrow B \vdash A \twoheadrightarrow B}\ {\twoheadrightarrow}R$$

This indicates that the right ordered implication needs to be able to prove the premise A using a piece of the context Ω_A immediately to the *right* of the implication; the left ordered implication, we may safely suspect, works the other way around.

$$\frac{\Omega_A \vdash A \quad \Omega_L, B, \Omega_R \vdash C}{\Omega_L, A \twoheadrightarrow B, \Omega_A, \Omega_R \vdash C}\ {\twoheadrightarrow}L \qquad \frac{\Omega_A \vdash A \quad \Omega_L, B, \Omega_R \vdash C}{\Omega_L, \Omega_A, A \rightarrowtail B, \Omega_R \vdash C}\ {\rightarrowtail}L$$

Cut admissibility for both right and left ordered implication are similar, and assuming that we got the left rules correct, it follows the pattern from linear logic.

$$\frac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \Omega, A \vdash B\end{matrix}}{\Omega \vdash A \twoheadrightarrow B}\ {\twoheadrightarrow}R \quad \dfrac{\begin{matrix}\mathcal{E}_1\\ \Omega_A \vdash A\end{matrix} \quad \begin{matrix}\mathcal{E}_2\\ \Omega_L, B, \Omega_R \vdash C\end{matrix}}{\Omega_L, A \twoheadrightarrow B, \Omega_A, \Omega_R \vdash C}\ {\twoheadrightarrow}L}{\Omega_L, \Omega, \Omega_A, \Omega_R \vdash C}\ \mathsf{cut}_{A \twoheadrightarrow B} \quad \Longrightarrow$$

$$\frac{\dfrac{\begin{matrix}\mathcal{E}_1\\ \Omega_A \vdash A\end{matrix} \quad \begin{matrix}\mathcal{D}_1\\ \Omega, A \vdash B\end{matrix}}{\Omega, \Omega_A \vdash B}\ \mathsf{cut}_A \quad \begin{matrix}\mathcal{E}_2\\ \Omega_L, B, \Omega_R \vdash C\end{matrix}}{\Omega_L, \Omega, \Omega_A, \Omega_R \vdash C}\ \mathsf{cut}_B$$

3.3 Of course!

By uniformly extending our sequents with the same persistent context Γ from linear logic, we can add the "of course!" or "bang" modality from linear logic. We also need a copy rule, analogous to the copy rule from linear logic, which takes a proposition in the persistent context Γ and places a copy of it in the ordered context.

$$\frac{A \in \Gamma \quad \Gamma; \Omega_L, A, \Omega_R \vdash C}{\Gamma; \Omega_L, \Omega_R \vdash C}\,\mathsf{copy} \qquad \frac{\Gamma; \cdot \vdash A}{\Gamma; \cdot \vdash !A}\,!R \qquad \frac{\Gamma, A; \Omega_L, \Omega_R \vdash C}{\Gamma; \Omega_L, !A, \Omega_R \vdash C}\,!L$$

3.4 Somewhere¡

We can also extend our contexts with the *mobile* modality $¡A$, usually pronounced "A mobile" or, unfortunately, "gnab A." Mobile propositions are still resources, but they can move from one part of the ordered context to another. This can be done by extending sequents with a linear context Δ. Importantly, Δ is *not* a sequence like the ordered context Ω, it is a multiset, as in all our previous discussions of linear logic.

The analogue of the copy rule is a place rule that allows propositions in Δ to be placed at any point in the ordered context.

$$\frac{\Gamma; \Delta; \Omega_L, A, \Omega_R \vdash C}{\Gamma; \Delta, A; \Omega_L, \Omega_R \vdash C}\,\mathsf{place} \qquad \frac{\Gamma; \Delta; \cdot \vdash A}{\Gamma; \Delta; \cdot \vdash ¡A}\,¡R \qquad \frac{\Gamma; \Delta, A; \Omega_L, \Omega_R \vdash C}{\Gamma; \Delta; \Omega_L, ¡A, \Omega_R \vdash C}\,¡L$$

3.5 Metatheory with the modalities

The identity theorem changes in an obvious way when we add persistent and linear contexts to ordered logic. Cut admissibility for the modalities $!A$ and $¡A$ requires a generalization of the induction hypothesis. The three parts of the cut admissibility theorem can be seen as three derivable rules cut, cut¡, and cut! (respectively).

Theorem 2 (Identity) *For all A and Γ, we can prove $\Gamma; \cdot; A \vdash A$.*

Proof: By induction on the structure of A. □

Theorem 3 (Cut admissibility)

- *If $\Gamma; \Delta; \Omega \vdash A$ and $\Gamma; \Delta'; \Omega_L, A, \Omega_R \vdash C$, then $\Gamma; \Delta, \Delta'; \Omega_L, \Omega, \Omega_R \vdash C$.*
- *If $\Gamma; \Delta; \cdot \vdash A$ and $\Gamma; \Delta', A; \Omega_L, \Omega_R \vdash C$, then $\Gamma; \Delta, \Delta'; \Omega_L, \Omega_R \vdash C$.*

- *If* $\Gamma; \cdot; \cdot \vdash A$ *and* $\Gamma, A; \Delta'; \Omega_L, \Omega_R \vdash C$*, then* $\Gamma; \Delta'; \Omega_L, \Omega_R \vdash C$.

Proof: By lexicographic induction: either the principal cut formula A gets smaller, or else A stays the same and we call from the cut¡ and cut! cases (which are "bigger") to the cut case (which is "smaller"), or else A stays the same, we stay within the same case, and one of the two given derivations gets smaller while the other stays the same. □

3.6 Other connectives

Ordered logic has other connectives like those in linear logic, the additive (negative) conjunction $A \,\&\, B$ and disjunction $A \oplus B$ and the units $\mathbf{1}$, $\top$, and $\mathbf{0}$. Some presentations of ordered logic also include a second multiplicative conjunction $A \circ B$, or "A esuf B." This second conjunction is definable in terms of fuse: $A \circ B = B \bullet A$.

4 Focusing

As before, we get a (weakly) focused sequent calculus first by assigning propositions as *positive* or *negative* based on whether the left or right rules (respectively) can be applied eagerly during proof search. Atomic propositions P can be assigned an arbitrary polarity as long as the same proposition always gets the same polarity.

$$\begin{aligned} A^+ &::= P^+ \mid\, !A \mid\, ¡A \mid \mathbf{1} \mid A \bullet B \mid A \circ B \mid \mathbf{0} \mid A \oplus B \\ A^- &::= P^- \mid A \twoheadrightarrow B \mid A \rightarrowtail B \mid \top \mid A \,\&\, B \end{aligned}$$

We then extend our sequents $\Gamma; \Delta; \Omega \vdash C$ to allow at most one proposition in Ω or the conclusion C to be *in focus*, written as $[A]$. Now that there are two varieties of atomic proposition, we need two corresponding initial rules to replace the one initial rule from before:

$$\frac{}{\Gamma; \cdot; P^+ \vdash [P^+]}\ \mathsf{id}_{P^+} \qquad \frac{}{\Gamma; \cdot; [P^-] \vdash P^-}\ \mathsf{id}_{P^-}$$

We also need rules for entering into and leaving focus, the *focus* and *blur* rules:

$$\frac{\Gamma; \Delta; \Omega \vdash [A^+]}{\Gamma; \Delta; \Omega \vdash A^+}\ \mathsf{focus}R \qquad \frac{\Gamma; \Delta; \Omega \vdash A^-}{\Gamma; \Delta; \Omega \vdash [A^-]}\ \mathsf{blur}R$$

$$\frac{\Gamma; \Delta; \Omega_L, [A^-], \Omega_R \vdash C}{\Gamma; \Delta; \Omega_L, A^-, \Omega_R \vdash C}\ \mathsf{focus}L \qquad \frac{\Gamma; \Delta; \Omega_L, A^+, \Omega_R \vdash C}{\Gamma; \Delta; \Omega_L, [A^+], \Omega_R \vdash C}\ \mathsf{blur}R$$

Initial

$$\frac{}{\Gamma; \cdot; P^+ \vdash [P^+]}\ \mathsf{id}_{P^+} \qquad \frac{}{\Gamma; \cdot; [P^-] \vdash P^-}\ \mathsf{id}_{P^-}$$

Focusing

$$\frac{\Gamma; \Delta; \Omega \vdash [A^+]}{\Gamma; \Delta; \Omega \vdash A^+}\ \mathsf{focus}R \qquad \frac{\Gamma; \Delta; \Omega \vdash A^-}{\Gamma; \Delta; \Omega \vdash [A^-]}\ \mathsf{blur}R$$

$$\frac{\Gamma; \Delta; \Omega_L, [A^-], \Omega_R \vdash C}{\Gamma; \Delta; \Omega_L, A^-, \Omega_R \vdash C}\ \mathsf{focus}L \qquad \frac{\Gamma; \Delta; \Omega_L, A^+, \Omega_R \vdash C}{\Gamma; \Delta; \Omega_L, [A^+], \Omega_R \vdash C}\ \mathsf{blur}R$$

Conjunction

$$\frac{\Gamma; \Delta_1; \Omega_1 \vdash [A] \quad \Gamma; \Delta_2; \Omega_2 \vdash [B]}{\Gamma; \Delta_1, \Delta_2; \Omega_1, \Omega_2 \vdash [A \bullet B]}\ \bullet R \qquad \frac{\Gamma; \Delta; \Omega_L, A, B, \Omega_R \vdash C}{\Gamma; \Delta; \Omega_L, A \bullet B, \Omega_R \vdash C}\ \bullet L$$

Implication

$$\frac{\Gamma; \Delta; \Omega, A \vdash B}{\Gamma; \Delta; \Omega \vdash A \twoheadrightarrow B}\ {\twoheadrightarrow}R \qquad \frac{\Gamma; \Delta; \Omega_A \vdash [A] \quad \Gamma; \Delta'; \Omega_L, [B], \Omega_R \vdash C}{\Gamma; \Delta, \Delta'; \Omega_L, [A \twoheadrightarrow B], \Omega_A, \Omega_R \vdash C}\ {\twoheadrightarrow}L$$

$$\frac{\Gamma; \Delta; A, \Omega \vdash B}{\Gamma; \Delta; \Omega \vdash A \rightarrowtail B}\ {\rightarrowtail}R \qquad \frac{\Gamma; \Delta; \Omega_A \vdash [A] \quad \Gamma; \Delta'; \Omega_L, [B], \Omega_R \vdash C}{\Gamma; \Delta, \Delta'; \Omega_L, \Omega_A, [A \rightarrowtail B], \Omega_R \vdash C}\ {\rightarrowtail}L$$

Modalities

$$\frac{A \in \Gamma \quad \Gamma; \Delta; \Omega_L, [A], \Omega_R \vdash C}{\Gamma; \Delta; \Omega_L, \Omega_R \vdash C}\ \mathsf{copy} \qquad \frac{\Gamma; \cdot; \cdot \vdash A}{\Gamma; \cdot; \cdot \vdash [!A]}\ !R \qquad \frac{\Gamma, A; \Delta; \Omega_L, \Omega_R \vdash C}{\Gamma; \Delta; \Omega_L, !A, \Omega_R \vdash C}\ !L$$

$$\frac{\Gamma; \Delta; \Omega_L, [A], \Omega_R \vdash C}{\Gamma; \Delta, A; \Omega_L, \Omega_R \vdash C}\ \mathsf{place} \qquad \frac{\Gamma; \Delta; \cdot \vdash A}{\Gamma; \Delta; \cdot \vdash [¡A]}\ ¡R \qquad \frac{\Gamma; \Delta, A; \Omega_L, \Omega_R \vdash C}{\Gamma; \Delta; \Omega_L, ¡A, \Omega_R \vdash C}\ ¡L$$

Figure 2: Focused ordered logic. Γ and Δ are multisets, Ω is a sequence.

The complete presentation of focused ordered logic is given in Figure 2. With just the restriction that at most one proposition be in focus, we get the chaining system. The fully focused system arises when we additionally require that whenever a proposition is in focus Ω must contain only negative propositions and positive atomic propositions and the conclusion C must be either a positive proposition or a negative atomic proposition. This restriction can be phrased as a restriction on the focus$R$, focus$L$, copy, and place rules.

4.1 The atom optimization

The system above is not quite the same as the chaining system for linear logic presented in Lecture 9. In order to be consistent with that presentation, we need to make an extra restriction on the focused copy rule (which is called focus! in Lecture 9) that we cannot copy and focus on a positive atomic proposition:

$$\frac{A \in \Gamma \quad A \text{ not } P^+ \quad \Gamma; \Delta; \Omega_L, [A], \Omega_R \vdash C}{\Gamma; \Delta; \Omega_L, \Omega_R \vdash C}\ \mathsf{copy}'$$

Just making this one change would break the identity property, however: it would be impossible to prove $!P^+ \rightarrowtail P^+$. Therefore, this change to the copy′/focus! rule must be made in tandem with adding the $\mathsf{id!}_{P^+}$ rule.

$$\frac{P^+ \in \Gamma}{\Gamma; \cdot; \cdot \vdash [P^+]}\ \mathsf{id!}_{P^+}$$

A similar change can be made to the place rule in tandem with the addition of a $\mathsf{id¡}_{P^+}$ rule:

$$\frac{A \text{ not } P^+ \quad \Gamma; \Delta; \Omega_L, [A], \Omega_R \vdash C}{\Gamma; \Delta, A; \Omega_L, \Omega_R \vdash C}\ \mathsf{place}' \qquad \frac{}{\Gamma; P^+; \cdot \vdash [P^+]}\ \mathsf{id¡}_{P^+}$$

These changes essentially only have an effect on the way the focusing system treats positive atomic propositions; the treatment described in Figure 2 is more faithful to Chaudhuri's presentation of linear logic [Cha06], and the modified treatment described in this section is more faithful to Andreoli's presentation of linear logic [And92].

Let's call this change the "atom optimization," since it prevents us from ever taking the extra step of focusing on a positive atomic proposition just to get that proposition from the persistent or linear context into the ordered context. The atom optimization is helpful when we want write forward-chaining logic programs, but it reveals that the way we treat positive atomic propositions in linear and ordered logic is a bit ad-hoc. I currently believe that focusing systems with and without this and other, similar optimizations can be given a uniform treatment through what Melliès and Tabareau call *tensor logic* [MT10], but a discussion of this point would take us a bit too far afield.

Exercises

Exercise 1 Is there a Δ and an A such that $\Delta \Vdash^{w} A$ and $\Delta \Vdash^{c} A$ is provable but $\Delta \vdash A$ is not? That is, are there any formulas that are true in both affine and strict logic but not in linear logic?

Exercise 2 What goes wrong if we try to add the rules W and C to our system of session types for the π-calculus?

Exercise 3 We added the W and C rules to linear logic, not ordered logic, for a reason – Gentzen's W and C rules as written ultimately depend on their interaction with the exchange rule E for their sensibility.

(i) Propose an appropriate weakening principle for ordered logic, analogous to the one we presented for linear logic in Section 2, that makes sense in the absence of contraction and weakening.

(ii) Give two sequents that are provable in this "affordered" logic that are not provable in ordered logic.

(iii) Show that affordered logic is consistent by extending the cut admissibility argument (you'll need to deal with one left commutative case and at least one right commutative case).

(iv) Propose at least *two* appropriate (sets of) contraction rules for turning ordered logic into "strict ordered" logic. Show how they differ in what they allow you to prove.

Exercise 4 Are there any sequents that are provable in both linear logic with contraction $\Delta \Vdash^{c} A$ and linear logic with weakening $\Delta \Vdash^{w} A$ but not in linear logic $\Delta \vdash A$?

Exercise 5 *Rigid logic*, or nonassociative ordered logic, does not allow the context combination operator Ω_1, Ω_2 to be associative; in rigid logic, $A \bullet (B \bullet C) \rightarrowtail (A \bullet B) \bullet C$ is not provable in the empty context as it is in ordered logic. Propose a cut admissibility property and rules for fuse and right implication.

Exercise 6 Jason Reed once worked out a *queue logic* in which the only proposition that can be decomposed is the *leftmost* proposition in the context; this is an "ordered" logic that more superficially resembles Gentzen's original formulation without the exchange rule.

(i) Formulate the rules for fuse and right ordered implication in queue logic, as well as the cut admissibility and identity properties; prove the identity property and the principal cases of cut admissibility. (For spoilers, Reed's note on queue logic can be found at `http://www.cs.cmu.edu/~jcreed/papers/queuelogic.pdf`.)

(ii) Why doesn't left ordered implication work in this formulation of queue logic?

(iii) Does an affine queue logic, along the lines of the "affordered" logic in Exercise 3, make sense? Why or why not? What would the explicit weakening rule be?

Exercise 7 Give the rules, identity case, and principal cut elimination cases for disjunction $A \oplus B$ in full ordered logic (sequents $\Gamma; \Delta; \Omega \vdash C$).

Exercise 8 In linear logic, the uncurried version of the proposition $A \multimap B \multimap C \multimap D$ is $(A \otimes B \otimes C) \multimap D$. What is the uncurried version of $A \twoheadrightarrow B \twoheadrightarrow C \twoheadrightarrow D$ in ordered logic? What about $A \rightarrowtail B \rightarrowtail C \rightarrowtail D$? How does this relate to the definable conjunction $A \circ B = B \bullet A$?

References

[And92] Jean-Marc Andreoli. Logic programming with focusing proofs in linear logic. *Journal of Logic and Computation*, 2(3):197–347, 1992.

[Cha06] Kaustuv Chaudhuri. *The Focused Inverse Method for Linear Logic*. PhD thesis, Carnegie Mellon University, December 2006. Available as technical report CMU-CS-06-162.

[Gen35] Gerhard Gentzen. Untersuchungen über das logische Schließen. *Mathematische Zeitschrift*, 39:176–210, 405–431, 1935. English translation in M. E. Szabo, editor, *The Collected Papers of Gerhard Gentzen*, pages 68–131, North-Holland, 1969.

[Lam58] Joachim Lambek. The mathematics of sentence structure. *The American Mathematical Monthly*, 65(3):154–170, 1958.

[MT10] Paul-André Melliès and Nicholas Tabareau. Resource modalities in tensor logic. *Annals of Pure and Applied Logic*, 161(6):632–653, 2010.

[PP99] Jeff Polakow and Frank Pfenning. Relating natural deduction and sequent calculus for intuitionistic non-commutative linear logic. In Andre Scedrov and Achim Jung, editors, *Proceedings of the 15th Conference on Mathematical Foundations of Programming Semantics*, pages 449–466, New Orleans, Louisiana, April 1999. Electronic Notes in Theoretical Computer Science, Volume 20.