Lecture Notes on Natural Deduction

15-816: Linear Logic
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We now turn our attention from the sequent calculus to natural deduction. As we saw, sequent calculus proofs lend themselves to an interpretation as session-typed concurrent processes. Generally speaking, natural deduction is related to the λ-calculus and therefore to functional computation. In the next lecture we will bridge the gap between the two.

1 Linear Hypothetical Judgments

A *linear hypothetical judgment* has the form

$$x_1{:}A_1, \ldots, x_n{:}A_n \Vdash A$$

where we continue to use Δ to denote the hypotheses. All the hypotheses must be labeled uniquely. Their order does not matter, but in accordance with linear logic each must be used in a proof exactly once.

Unlike the sequent calculus, there are no left rules. Instead, the only access to the linear hypotheses is via a *hypothesis* rule

$$\frac{}{x{:}A \Vdash A}\ \mathsf{hyp}$$

The counterpart to the hypothesis rule is *substitution*.

Substitution. *If* $\Delta \Vdash A$ *and* $\Delta', x{:}A \Vdash C$ *then* $\Delta, \Delta' \Vdash C$.

This is called substitution because it describes an operation on proofs where the proof of $\Delta \Vdash A$ is substituted for uses of the hypothesis $x{:}A$ in the proof of $\Delta', x{:}A \Vdash C$. We will see this more clearly when we introduce terms representing proofs. Unlike cut in the sequent calculus, substitution is rarely seen as a rule.

We can also write substitution as an admissible rule:

$$\frac{\Delta \Vdash A \quad \Delta', x{:}A \Vdash C}{\Delta, \Delta' \Vdash C}\ (\mathsf{subst})$$

There is a certain analogy between identity/cut and hypothesis/substitution, but they do not correspond precisely as we may see in a later lecture.

2 Alternative Conjunction

As mentioned above, we do not obtain direct access to the hypotheses, except through the rule hyp. This means that the left rules of the sequent calculus have to be captured differently, somehow. No such difficulties exist for the right rules, since (linear) hypothetical judgments also have a distinguished conclusion.

So the right rules have exact analogues in natural deduction, called *introduction rules*, generally denoted with the letter I.

$$\frac{\Delta \Vdash A \quad \Delta \Vdash B}{\Delta \Vdash A \,\&\, B}\ \& I$$

How can we translate the left rules from the sequent calculus into rules that operate on the conclusion of a hypothetical judgment? Recall that the left rules tell us how to *use* ephemeral resources. So if we have a resource $A \,\&\, B$ we can choose between obtaining A or obtaining B. In the form of two rules:

$$\frac{\Delta \Vdash A \,\&\, B}{\Delta \Vdash A}\ \& E_1 \qquad \frac{\Delta \Vdash A \,\&\, B}{\Delta \Vdash B}\ \& E_2$$

These *elimination rules* are just the inverted left rules, transported to the conclusion of a hypothetical judgment.

3 Proof Terms

Before proceeding with the other connectives of linear logic, we add proof terms to our hypothetical judgments. The intuition behind the Curry-Howard isomorphism for intuitionistic natural deduction is that proof terms correspond to λ-terms. In effect, they form the terms of a functional programming language. Here, it will be a *linear functional language*. We will postpone applications of these until later and concentrate for now on describing the structure of proofs.

The new judgment has the form

$$\Delta \Vdash M : A$$

where M is a *proof term*.

To begin, the use of a hypothesis is just denoted by the variable that labels the hypothesis.

$$\frac{}{x{:}A \Vdash x : A}\ \mathsf{hyp}$$

The substitution principle just performs a substitution on the proof term.

> ***Substitution.*** *If* $\Delta \Vdash M : A$ *and* $\Delta', x{:}A \Vdash N : C$ *then* $\Delta, \Delta' \Vdash [M/x]N$.[1]

4 Pairs and Projections

As we saw earlier, a proof of $A \mathbin{\&} B$ consists of a pair of proofs, one for A and one for B. The proof term assignment for the $\& I$ rule is therefore just a pair of proofs.

$$\frac{\Delta \Vdash M : A \quad \Delta \Vdash N : B}{\Delta \Vdash \langle M, N\rangle : A \mathbin{\&} B}\ \& I$$

Then, the two elimination rules just extract components of these pairs.

$$\frac{\Delta \Vdash M : A \mathbin{\&} B}{\Delta \Vdash \pi_1\, M : A}\ \& E_1 \qquad \frac{\Delta \Vdash M : A \mathbin{\&} B}{\Delta \Vdash \pi_2\, M : B}\ \& E_2$$

5 Local Reductions

In the sequent calculus, a *cut reduction* arises from a cut between a right rule and a left rule on the same proposition. In natural deduction, a corresponding *local reduction* arises when a connective is introduced and then

[1] When we added term-passing to the π-calculus we wrote this as $N\{M/x\}$ which is less common in the λ-calculus.

immediately eliminated. There counterpart to *identity expansion* is *local expansion*, whose discussion will be postponed to a future lecture.

In the case of pairs and projections, the local reductions are easy to see.

$$\dfrac{\dfrac{\Delta \Vdash M : A \quad \Delta \Vdash N : B}{\Delta \Vdash \langle M, N\rangle : A \,\&\, B}\,\&I}{\Delta \Vdash \pi_1\langle M, N\rangle : A}\,\&E_1 \qquad \longrightarrow_R \quad \Delta \Vdash M : A$$

$$\dfrac{\dfrac{\Delta \Vdash M : A \quad \Delta \Vdash N : B}{\Delta \Vdash \langle M, N\rangle : A \,\&\, B}\,\&I}{\Delta \Vdash \pi_2\langle M, N\rangle : B}\,\&E_2 \qquad \longrightarrow_R \quad \Delta \Vdash N : B$$

These local reductions witness the fact that the elimination rules are not too strong with respect to the introduction rules. We do not gain any information because we already have a justification of the conclusion of the elimination rule. Summarizing them on proof terms:

$$\begin{array}{lcl} \pi_1\langle M, N\rangle & \longrightarrow_R & M \\ \pi_2\langle M, N\rangle & \longrightarrow_R & N \end{array}$$

6 Simultaneous Conjunction

For the multiplicative, or simultaneous conjunction, the introduction rule is again the same as the right rule in the sequent calculus.

$$\dfrac{\Delta \Vdash A \quad \Delta' \Vdash B}{\Delta, \Delta' \Vdash A \otimes B}\,\otimes I$$

The left rule is more difficult to derive. Clearly, it would be incorrect to write

$$\dfrac{\Delta \Vdash A \otimes B}{\Delta \Vdash A}\,\otimes E??$$

because both A and B must be used, not just one of them. Instead, we have to relate the fact that we can derive $A \otimes B$ to the hypotheses A and B, which we can only do with a second premise.

$$\dfrac{\Delta \Vdash A \otimes B \quad \Delta', x{:}A, y{:}B \Vdash C}{\Delta, \Delta' \Vdash C}\,\otimes E$$

Actually, this looks somewhat like a sequent calculus rule. If we have a hypothesis $A \otimes B$, we can turn it into the hypotheses A and B as follows:

$$\frac{\overline{z{:}A \otimes B \Vdash A \otimes B}\ \textsf{hyp} \qquad \Delta', x{:}A, y{:}B \Vdash C}{\Delta', z{:}A \otimes B \Vdash C}\ \otimes E$$

Looking now at proof terms, we see that the introduction rule again requires us to record a pair of proofs. This time the two sides do not share any variables. We use the convention to replicate the propositional connective as a term former.

$$\frac{\Delta \Vdash M : A \qquad \Delta' \Vdash N : B}{\Delta, \Delta' \Vdash M \otimes N : A \otimes B}\ \otimes I$$

$$\frac{\Delta \Vdash M : A \otimes B \qquad \Delta', x{:}A, y{:}B \Vdash N : C}{\Delta, \Delta' \Vdash \textsf{let}\ x \otimes y = M\ \textsf{in}\ N : C}\ \otimes E$$

This time the local reduction induced by the elimination following the introduction requires two substitutions.

$$\frac{\dfrac{\Delta_1 \Vdash M_1 : A \qquad \Delta_2 \Vdash M_2 : B}{\Delta_1, \Delta_2 \Vdash M_1 \otimes M_2 : A \otimes B}\ \otimes I \qquad \Delta', x_1{:}A, x_2{:}B \Vdash N : C}{\Delta_1, \Delta_2, \Delta' \Vdash \textsf{let}\ x_1 \otimes x_2 = M_1 \otimes M_2\ \textsf{in}\ N : C}\ \otimes E$$

$$\longrightarrow_R$$

$$\frac{\Delta_1 \Vdash M_1 : A \qquad \dfrac{\Delta_2 \Vdash M_2 : B \qquad \Delta', x_1{:}A, x_2{:}B \Vdash N : C}{\Delta', \Delta_2, x_1{:}A \Vdash [M_2/x_2]N : C}\ \textsf{subst}}{\Delta_1, \Delta_2, \Delta' \Vdash [M_1/x_1][M_2/x_2]N : C}\ \textsf{subst}$$

The reduction on the terms only:

$$\textsf{let}\ x_1 \otimes x_2 = M_1 \otimes M_2\ \textsf{in}\ N \quad \longrightarrow_R \quad [M_1/x_1][M_2/x_2]N$$

7 Operational Semantics

At this point we might speculate a bit on the eventual operational semantics we want to assign to the different pairing constructs. Intuitively, only one component of the pairs $\langle M, N\rangle$ will ever be used, so we should not be computing both of them. In fact, computing both of them could violate linearity, since the same linear variables will occur in M and N. So we might expect pairs to be *lazy* in that $\langle M, N\rangle$ is already a value.

Conversely, pairs $M \otimes N$ split all the variables between them and the values of both components must eventually be used. Therefore we would expect pairs $M \otimes N$ to be *eager*, that is, a pair $M \otimes N$ is a value only if M and N are.

In the next lecture we will come back to this point to see if our expectations are met. Certainly, at this point, the pure reductions should not be considered an operational semantics, although we might expect that they form the core of them, just like the cut reduction formed the core of π-calculus computation.

8 Linear Implication

We recapitulate the $\multimap R$ rule as an introduction rule.

$$\frac{\Delta, x{:}A \Vdash B}{\Delta \Vdash A \multimap B}\ \multimap I$$

The elimination rule derives from the substitution principle and allows us to infer B if we know A.

$$\frac{\Delta \Vdash A \multimap B \quad \Delta' \Vdash A}{\Delta, \Delta' \Vdash B}\ \multimap E$$

With proof terms, $A \multimap B$ is the type of a *linear function* from A to B. *Linear* here means that it should use its argument exactly once. This does not mean that the parameter occurs exactly once. For example, it must occur in each part of pairs $\langle M, N\rangle$. Under the functional interpretation of $A \multimap B$, the elimination rule is just function application.

$$\frac{\Delta, x{:}A \Vdash M : B}{\Delta \Vdash \lambda x.\, M : A \multimap B}\ \multimap I \qquad \frac{\Delta \Vdash M : A \multimap B \quad \Delta' \Vdash N : A}{\Delta, \Delta' \Vdash M\, N : B}\ \multimap E$$

The reduction just uses substitution, as we might expect by now.

$$\dfrac{\dfrac{\Delta, x{:}A \Vdash M : B}{\Delta \Vdash \lambda x.\, M : A \multimap B} \multimap I \quad \Delta' \Vdash N : A}{\Delta, \Delta' \Vdash (\lambda x.\, M)\, N : B} \multimap E$$

$$\longrightarrow_R$$

$$\dfrac{\Delta' \Vdash N : A \quad \Delta, x{:}A \Vdash M : B}{\Delta, \Delta' \Vdash [N/x]M : B} \text{(subst)}$$

This form of reduction is called β*-reduction*, going back to the original development of the λ-calculus by Church [Chu32, Chu33].

$$(\lambda x.\, M)\, N \quad \longrightarrow_R \quad [N/x]M$$

9 Example: Interaction

As an example, we consider the interaction of linear implication with alternative conjunction on the right. One direction of this is

$$A \multimap (B \mathbin{\&} C) \Vdash (A \multimap B) \mathbin{\&} (A \multimap C)$$

We build the proof incrementally. We start using the $\& I$ rule, since we know $\& R$ to be invertible in the sequent calculus.

$$\dfrac{\begin{array}{c}\vdots \\ x{:}A \multimap (B \mathbin{\&} C) \Vdash A \multimap B\end{array} \quad \begin{array}{c}\vdots \\ x{:}A \multimap (B \mathbin{\&} C) \Vdash A \multimap C\end{array}}{x{:}A \multimap (B \mathbin{\&} C) \Vdash (A \multimap B) \mathbin{\&} (A \multimap C)} \& I$$

We continue with the left subproof—the right one will be symmetric.

$$\dfrac{\begin{array}{c}\vdots \\ x{:}A \multimap (B \mathbin{\&} C), y{:}A \Vdash B\end{array}}{x{:}A \multimap (B \mathbin{\&} C) \Vdash A \multimap B} \multimap I$$

At this point we can no longer use introduction rules in the proof of B since it is atomic. So instead of working our way upwards in the proof construction using introduction rules, we have to work our way downwards using elimination rules. This change of direction is characteristic of natural deduction.

We see that we would like to use the hypothesis x.

$$\cfrac{\cfrac{\cfrac{}{x{:}A \multimap (B \,\&\, C) \Vdash A \multimap (B \,\&\, C)}\ \mathsf{hyp}}{\vdots}}{\cfrac{x{:}A \multimap (B \,\&\, C), y{:}A \Vdash B}{x{:}A \multimap (B \,\&\, C) \Vdash A \multimap B}\ {\multimap}I}$$

At this point, the elimination rule for linear implication, ${\multimap}E$ suggests itself. This leads to two new subgoals: we have to prove A on one side and find some use for $B \,\&\, C$ on the other.

$$\cfrac{\cfrac{\cfrac{}{x{:}A \multimap (B \,\&\, C) \Vdash A \multimap (B \,\&\, C)}\ \mathsf{hyp} \qquad \cfrac{\vdots}{\Vdash A}}{x{:}A \multimap (B \,\&\, C) \Vdash B \,\&\, C}\ {\multimap}E \quad \vdots}{\cfrac{x{:}A \multimap (B \,\&\, C), y{:}A \Vdash B}{x{:}A \multimap (B \,\&\, C) \Vdash A \multimap B}\ {\multimap}I}$$

We can prove A using the hypothesis $y{:}A$, discharging one of the proof obligations. This requires us to add $y{:}A$ to the hypotheses in the subproof.

$$\cfrac{\cfrac{\cfrac{}{x{:}A \multimap (B \,\&\, C) \Vdash A \multimap (B \,\&\, C)}\ \mathsf{hyp} \qquad \cfrac{}{y{:}A \Vdash A}\ \mathsf{hyp}}{x{:}A \multimap (B \,\&\, C), y{:}A \Vdash B \,\&\, C}\ {\multimap}E \quad \vdots}{\cfrac{x{:}A \multimap (B \,\&\, C), y{:}A \Vdash B}{x{:}A \multimap (B \,\&\, C) \Vdash A \multimap B}\ {\multimap}I}$$

We can close the gap using the first elimination rule for $\&$.

$$\cfrac{\cfrac{\cfrac{\cfrac{}{x{:}A \multimap (B \,\&\, C) \Vdash A \multimap (B \,\&\, C)}\ \mathsf{hyp} \qquad \cfrac{}{y{:}A \Vdash A}\ \mathsf{hyp}}{x{:}A \multimap (B \,\&\, C), y{:}A \Vdash B \,\&\, C}\ {\multimap}E}{x{:}A \multimap (B \,\&\, C), y{:}A \Vdash B}\ \&E_1}{x{:}A \multimap (B \,\&\, C) \Vdash A \multimap B}\ {\multimap}I$$

It is instructive to annotate the above with proof terms. We obtain the following:

$$x{:}A \multimap (B \mathbin{\&} C) \Vdash \langle \lambda y.\, \pi_1(x\, y), \lambda y.\, \pi_2(x\, y)\rangle : (A \multimap B) \mathbin{\&} (A \multimap C)$$

We can see that a proof term is a much more compact representation of the proof than the two-dimensional format. Some care must be taken to guarantee that there is a bijection between the two, something we will like not explore in this course.

10 Relation to Sequent Calculus

Of course, we would like to verify that the system of natural deduction and the sequent calculus describe the same logic, just using different formats for the inference rules, and therefore different languages for proofs. Fortunately, these are easy to check.

Theorem 1 (From Sequent Calculus to Natural Deduction)
If $\Delta \vdash A$ *then* $\Delta \Vdash A$.

Proof: By induction on the structure of the proof $\mathcal{D}$ of $\Delta \vdash A$. We exemplify the proof with identity, cut, and right and left rules for linear implication.

Case:

$$\mathcal{D} = \frac{}{A \vdash A}\ \mathsf{id}_A$$

We construct:

$$\frac{}{x{:}A \Vdash A}\ \mathsf{hyp}$$

Case:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}_1\\ \Delta_1 \vdash B\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \Delta_2, B \vdash A\end{array}}{\Delta_1, \Delta_2 \vdash A}\ \mathsf{cut}_B$$

We construct:

$$\frac{\begin{array}{c}\text{i.h.}(\mathcal{D}_1)\\ \Delta_1 \Vdash B\end{array} \quad \begin{array}{c}\text{i.h.}(\mathcal{D}_2)\\ \Delta_2, x{:}B \Vdash A\end{array}}{\Delta_1, \Delta_2 \vdash A}\ (\mathsf{subst})$$

Case:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_2 \\ \Delta, A_1 \vdash A_2\end{array}}{\Delta \vdash A_1 \multimap A_2} \multimap R$$

Then we construct

$$\dfrac{\begin{array}{c}\text{i.h.}(\mathcal{D}_2) \\ \Delta, x{:}A_1 \Vdash A_2\end{array}}{\Delta \Vdash A_1 \multimap A_2} \multimap I$$

Case:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Delta_1 \vdash B_1\end{array} \quad \begin{array}{c}\mathcal{D}_2 \\ \Delta_2, B_2 \vdash A\end{array}}{\Delta_1, \Delta_2, B_1 \multimap B_2 \vdash A} \multimap L$$

This is the interesting case, because we somehow have to turn the application of the left rule upside-down. We construct:

$$\dfrac{\dfrac{\dfrac{}{x{:}B_1 \multimap B_2 \Vdash B_1 \multimap B_2}\,\mathsf{hyp} \quad \begin{array}{c}\text{i.h.}(\mathcal{D}_1) \\ \Delta_1 \Vdash B_1\end{array}}{\Delta_1, x{:}B_1 \multimap B_2 \Vdash B_2} \multimap E \qquad \begin{array}{c}\text{i.h.}(\mathcal{D}_2) \\ \Delta_2, y{:}B_2 \Vdash A\end{array}}{\Delta_1, \Delta_2, x{:}B_1 \multimap B_2 \Vdash A}\,(\mathsf{subst})$$

□

The proof in the other direction is quite straightforward as well. The reason is the two sledgehammers of proof theory: cut and substitution.

Theorem 2 (From Natural Deduction to Sequent Calculus)
If $\Delta \Vdash A$ *then* $\Delta \vdash A$

Proof: By induction on the structure of the proof $\mathcal{D}$ of $\Delta \Vdash A$. We show three cases: hypothesis, as well as the introduction and elimination rules for linear implication.

Case:

$$\mathcal{D} = \dfrac{}{x{:}A \Vdash A}\,\mathsf{hyp}$$

We construct:

$$\dfrac{}{A \vdash A}\,\mathsf{id}_A$$

Case:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}' \\ \Delta, x{:}A \Vdash B\end{array}}{\Delta \Vdash A \multimap B} \multimap I$$

We construct:

$$\dfrac{\begin{array}{c}\text{i.h.}(\mathcal{D}') \\ \Delta, A \vdash B\end{array}}{\Delta \vdash A \multimap B} \multimap R$$

Case:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Delta_1 \Vdash B \multimap A\end{array} \quad \begin{array}{c}\mathcal{D}_2 \\ \Delta_2 \Vdash B\end{array}}{\Delta_1, \Delta_2 \Vdash A} \multimap R$$

The elimination rules are the only interesting cases. In order to apply the left rule we need to arrange that the formula in question (here: $B \multimap A$) appears as an antecedent in a sequent. Fortunately, cut together with identity will do the job:

$$\dfrac{\begin{array}{c}\text{i.h.}(\mathcal{D}_1) \\ \Delta_1 \vdash B \multimap A\end{array} \quad \dfrac{\begin{array}{c}\text{i.h.}(\mathcal{D}_2) \\ \Delta_2 \vdash B\end{array} \quad \dfrac{}{A \vdash A}\,\mathsf{id}_A}{\Delta_2, B \multimap A \vdash A} \multimap L}{\Delta_1, \Delta_2 \vdash A} \mathsf{cut}_{B \multimap A}$$

□

11 Disjunction and Units

We only summarize here the rules for disjunction and the units for completeness, together with their proof terms and reductions.

$$\frac{}{\cdot \Vdash \mathbf{1} : \mathbf{1}}\ \mathbf{1}I \qquad \frac{\Delta \Vdash M : \mathbf{1} \quad \Delta' \Vdash N : C}{\Delta, \Delta' \Vdash \mathsf{let}\ \mathbf{1} = M\ \mathsf{in}\ N : C}\ \mathbf{1}E$$

$$\frac{}{\Delta \Vdash \langle\,\rangle : \top}\ \top I \qquad \text{no } \top E \text{ rule}$$

$$\frac{\Delta \Vdash M : A}{\Delta \Vdash \mathsf{inl}\, M : A \oplus B}\ \oplus I_1 \qquad \frac{\Delta \Vdash M : B}{\Delta \Vdash \mathsf{inr}\, M : A \oplus B}\ \oplus I_2$$

$$\frac{\Delta \Vdash M : A \oplus B \quad \Delta', x{:}A \Vdash N : C \quad \Delta', y{:}B \Vdash P : C}{\Delta, \Delta' \Vdash (\mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N \mid \mathsf{inr}\, y \Rightarrow P) : C}\ \oplus E$$

$$\text{no } \mathbf{0}I \text{ rule} \qquad \frac{\Delta \Vdash M : \mathbf{0}}{\Delta, \Delta' \Vdash \mathsf{abort}\, M : C}\ \mathbf{0}E$$

The reductions, on proof terms:

$$\begin{array}{lll}
\mathsf{let}\ \mathbf{1} = M\ \mathsf{in}\ N & \longrightarrow_R & N \\
\text{no reductions for } \top & & \\
(\mathsf{case}\ \mathsf{inl}\, M\ \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N \mid \mathsf{inr}\, y \Rightarrow P) & \longrightarrow_R & [M/x]N \\
(\mathsf{case}\ \mathsf{inr}\, M\ \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N \mid \mathsf{inr}\, y \Rightarrow P) & \longrightarrow_R & [M/x]P \\
\text{no reductions for } \mathbf{0} & &
\end{array}$$

There are no reductions for $\top$ since there is no elimination form, and none for **0** since there is no introduction form.

12 Persistence

In order to encompass persistent hypotheses into natural deduction, we proceed exactly as for the sequent calculus. We create a new form of hypothesis, $u{:}A$, which means that u of type A is a variable that can be used in an unrestricted fashion (that is, not necessarily linearly). As in the sequent calculus, we collect such unrestricted hypotheses into another zone of the linear hypothetical judgment and write

$$\Gamma\, ; \Delta \Vdash A$$

and

$$\Gamma\, ; \Delta \Vdash M : A$$

with proof terms. However, there will be no copy rule. Instead, we can directly use an unrestricted hypothesis.

$$\frac{u{:}A \in \Gamma}{\Gamma \,;\, \cdot \Vdash u : A}\ \mathsf{uhyp}$$

We also have a new form of substitution, which is analogous to cut!.

> ***Unrestricted Substitution.*** *If* $\Gamma \,;\, \cdot \Vdash M : A$ *and* $\Gamma, u{:}A \,;\, \Delta' \Vdash N : C$ *then* $\Gamma \,;\, \Delta' \Vdash [M/u]N : C$.

We use the same notation, $[M/u]N$, even though this form of substitution has quite different properties. In particular, it must descent into all subterms, since u could occur anywhere, and could occur multiple times. In the form of an admissible rule:

$$\frac{\Gamma \,;\, \cdot \Vdash M : A \quad \Gamma, u{:}A \,;\, \Delta' \Vdash N : C}{\Gamma \,;\, \Delta' \Vdash [M/u]N : C}\ \mathsf{usubst}$$

13 Of Course!

The introduction rule follows the right rule, as always.

$$\frac{\Gamma \,;\, \cdot \Vdash A}{\Gamma \,;\, \cdot \Vdash\, !A}\ !I$$

The elimination rule is somewhat like simultaneous conjunction, since it requires a second premise

$$\frac{\Gamma \,;\, \Delta \Vdash\, !A \quad \Gamma, u{:}A \,;\, \Delta' \Vdash C}{\Gamma \,;\, \Delta, \Delta' \Vdash C}\ !E$$

With proof terms, we again use the convention that the term constructor takes the form of the type constructor (viewing here propositions as types of functional programs).

$$\frac{\Gamma \,;\, \cdot \Vdash M : A}{\Gamma \,;\, \cdot \Vdash\, !M : !A}\ !I$$

The elimination rule is somewhat like simultaneous conjunction, since it requires a second premise:

$$\frac{\Gamma \,;\, \Delta \Vdash M : !A \quad \Gamma, u{:}A \,;\, \Delta' \Vdash N : C}{\Gamma \,;\, \Delta, \Delta' \Vdash \mathsf{let}\ !u = M\ \mathsf{in}\ N : C}\ !E$$

It is crucial that the variable in the pattern $!u$ is an unrestricted variable.

These distinctions are completely analogous to the the distinction between linear and shared channels, although here we just think of them as linear and unrestricted variables. Also as before, we have to add Γ to all rules in a systematic way.

The local reduction now appeals to unrestricted substitution.

$$\dfrac{\dfrac{\Gamma\ ;\ \cdot \Vdash M : A}{\Gamma\ ;\ \cdot \Vdash\ !M : !A}\ !I \qquad \Gamma, u{:}A\ ;\ \Delta' \Vdash N : C}{\Gamma\ ;\ \Delta' \Vdash \mathsf{let}\ !u = !M\ \mathsf{in}\ N : C}\ !E$$

$$\longrightarrow_R$$

$$\dfrac{\Gamma\ ;\ \cdot \Vdash M : A \qquad \Gamma, u{:}A\ ;\ \Delta' \Vdash N : C}{\Gamma\ ;\ \Delta' \Vdash [M/u]N : C}\ \mathsf{usubst}$$

Summarizing, just on terms:

$$\mathsf{let}\ !u = !M\ \mathsf{in}\ N \quad \longrightarrow_R \quad [M/u]N$$

The relation to the sequent calculus remains intact (see Exercises 5 and 6).

Exercises

Exercise 1 Consider a version of the copy rule of the sequent calculus in natural deduction.

$$\frac{\Gamma, u{:}A \,;\, \Delta, x{:}A \Vdash C}{\Gamma, u{:}A \,;\, \Delta \Vdash C}\ \mathsf{copy}$$

Is it derivable or admissible? Give an appropriate proof term assignment. How is it related to other rules such as uhyp?

Exercise 2 Provide proof terms for the following linear entailments, one for each direction. You do not need to show any derivations, just the terms.

(i) $A \multimap (B \multimap C) \dashv\Vdash (A \otimes B) \multimap C$

(ii) $A \multimap (B \mathbin{\&} C) \dashv\Vdash (A \multimap B) \mathbin{\&} (A \multimap C)$

(iii) $(A \oplus B) \multimap C \dashv\Vdash (A \multimap C) \mathbin{\&} (B \multimap C)$

(iv) $!(A \mathbin{\&} B) \dashv\Vdash (!A) \otimes (!B)$

(v) $A \otimes \mathbf{1} \dashv\Vdash A$

(vi) $A \mathbin{\&} \top \dashv\Vdash A$

(vii) $A \oplus \mathbf{0} \dashv\Vdash A$

(viii) $A \mathbin{\&} A \dashv\Vdash A$

(ix) $!\top \dashv\Vdash \mathbf{1}$

Exercise 3 Analyze the following alternative elimination rule for $!A$:

$$\frac{\Gamma \,;\, \cdot \Vdash !A}{\Gamma \,;\, \cdot \Vdash A}\ !E'$$

How does it relate to $!E$? Does it create any problems in a calculus of natural deduction?

Exercise 4 Explicate the proof that sequent calculus proofs can be translated to natural deductions in the following way. Annotate antecedents with unique variable names, similar to the way we did for our assignment of π-calculus terms. Then annotate the right-hand side with a *proof term for natural deduction*. The sequent then will have the form

$$u_1{:}B_1, \ldots, u_k{:}B_k \,;\, x_1{:}A_1, \ldots, x_n{:}A_n \vdash M : A$$

Of course, the property we want to achieve is that

$$u_1{:}B_1, \ldots, u_k{:}B_k \,;\, x_1{:}A_1, \ldots, x_n{:}A_n \Vdash M : A$$

This means you will have to express the essence of the proof that we can translate from sequent calcululs to natural deduction in the term assignments M. We show two examples:

$$\frac{\Gamma \,;\, \Delta \vdash M : A \quad \Gamma \,;\, \Delta \vdash N : B}{\Gamma \,;\, \Delta \vdash \langle M, N \rangle : A \mathbin{\&} B} \,\&R \qquad \frac{\Gamma \,;\, \Delta, y{:}A \vdash M : C}{\Gamma \,;\, \Delta, x{:}A \mathbin{\&} B \vdash [\pi_1 x / y] M : C} \,\&L_1$$

You may confine yourselves to the connectives $A \otimes B$, $A \multimap B$ and $!A$ as well as the rules of identity and cut.

Exercise 5 Extend the theorem and proof that one can translate from sequent calculus to natural deduction (Theorem 1) to encompass persistence. Show the cases for $!R$, $!L$, copy and cut!.

Exercise 6 Extend the theorem and proof that one can translate from natural deduction to sequent calculus (Theorem 2) to encompass persistence. Show the cases for $!I$, $!E$, and uhyp.

References

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