Lecture Notes on
Chaining and Focusing

15-816: Linear Logic
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Lecture 9
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In the last lecture we started our discussion on proof search strategies with *inversion*, which applies to negative connectives on the right or positive connectives on the left of a sequent.

Another strategy I call *chaining* (also called *weak focusing* [Lau04]) does not care about inversion, but forces a certain discipline on the non-invertible rules. The idea is that non-invertible rules can can only be applied to formulas in *focus*, and at most one formula can be in focus at a time in a sequent. Chaining, which has *forward chaining* and *backward chaining* as special cases, is at least equally powerful and perhaps more surprising than inversion.

If we put the two together we get a system called *focusing*, which inherits its restrictions from both inversion and chaining. But first, chaining.

1 Chaining

Let's consider a goal $\Delta \Rightarrow A \oplus (B \oplus C)$. Besides the choices for propositions in Δ, this really presents us with three choices on the right: proving A, proving B, or proving C. However, with the rules we have right now, we first have to chose between A and $B \oplus C$. Then we pause and, in principle should again consider any possible choice in Δ, or choose between B and C.

Chaining says that we can make a choice at the beginning to focus on the right-hand side (here: $A \oplus (B \oplus C)$) or on some formula in Δ, and then continue to chose the formulas that come from it. In this example, in case we chose $B \oplus C$ we don't have to re-examine the left-hand side, but only chose between B and C. Perhaps surprisingly, this idea applies to all propositions with non-invertible rules. According to the table from the previous lecture

Negative	$A \multimap B$, $A \mathbin{\&} B$, $\top$
Positive	$A \otimes B$, $\mathbf{1}$, $A \oplus B$, $\mathbf{0}$, $!A$

we can therefore chain together choices for positive formulas occurring on the right-hand side of a sequent, or negative formulas occurring on the left-hand side.

Atomic formulas have a special status. They can be designated as either negative or positive, but we have to deal with them separately on the two sides because, for example, there is no right rule for atomic propositions to invert. We end up working with the following categories:

Negative	A^-	$::=$	$A \multimap B \mid A \mathbin{\&} B \mid \top \mid P^-$
Positive	A^+	$::=$	$A \otimes B \mid \mathbf{1} \mid A \oplus B \mid \mathbf{0} \mid !A \mid P^+$
Formulas	A	$::=$	$A^- \mid A^+$

where P stands for atomic formulas. It is crucial that multiple occurrences of the same atomic formula have the same polarity assigned to them.

We start with the purely linear fragment, concentrating on $A \multimap B$ and $A \otimes B$, the multiplicative connectives. The invertible rules can be applied freely, while noninvertible rules can only applied when the formula is in focus. This leads to a proliferation of judgments: in addition to ephemeral resources and goals, we also have resources and goals *in focus*. We write these as $[A]$, on both sides of the sequent. We obtain:

Antecedents	δ	$::=$	$\cdot \mid \delta, A \mid \delta, [A]$
Succedents	γ	$::=$	$A \mid [A]$
Chaining Sequents	$\Gamma \,;\, \delta \rightarrow \gamma$		

Persistent formulas cannot be in focus, so this part of the sequent remains unchanged (although we will ignore it for the moment). Chaining sequents are subject to the crucial global restriction

> ***Focusing Constraint.*** *In a chaining sequent,* $\Gamma \,;\, \delta \rightarrow \gamma$ *there can be at most formula in focus.*

We now rewrite the rules of the cut-free sequent calculus in order to force chaining.

$$\frac{\delta, A \to B}{\delta \to A \multimap B} \multimap R \qquad \frac{\Delta \to [A] \quad \Delta', [B] \to C}{\Delta, \Delta', [A \multimap B] \to C} \multimap L$$

$$\frac{\Delta \to [A] \quad \Delta' \to [B]}{\Delta, \Delta' \to [A \otimes B]} \otimes R \qquad \frac{\delta, A, B \to \gamma}{\delta, A \otimes B \to \gamma} \otimes L$$

The principle should become apparent in these rules. For each connective, one of the sides (the invertible one) is a standard rule, while the other (the non-invertible one) is restricted to take place only in focus.

But how do we obtain focus? At some point in a derivation we must chose an eligible formula and designate focus. On the right-hand side, only positive formulas are eligible for focus (negative ones are invertible), while on the left-hand side only negative formulas are eligible (and positive ones are invertible).

$$\frac{\Delta \to [A^+]}{\Delta \to A^+} \,\mathsf{focus}R \qquad \frac{\Delta, [A^-] \to C}{\Delta, A^- \to C} \,\mathsf{focus}L$$

The restriction that there is not already another formula in focus in crucial in these rules. We make this manifest by writing Δ instead of δ and C instead of γ if we know the antecedent or succedent cannot contain a formula in focus.

How do we drop out of focus? It happens if after successive decomposition we have reached a proposition of ineligible polarity. On the right, this is a negative formulas; on the left a positive one.

$$\frac{\Delta \to A^-}{\Delta \to [A^-]} \,\mathsf{blur}R \qquad \frac{\Delta, A^+ \to C}{\Delta, [A^+] \to C} \,\mathsf{blur}L$$

This leaves open what happens to atomic formulas. For example, if we have $[P^+ \otimes Q^+]$ in focus on the right, the focus will reduce to $[P^+]$ and $[Q^+]$. But at this point we are stuck: we cannot drop out of focus (wrong polarity: P^+ is not negative) and we cannot continue to decompose P^+ (since it is atomic). One of Andreoli's deep insights is that we can *require* the left-hand side to be exactly P^+, and fail in all other cases. Focus on a negative atomic proposition in the antecedent works symmetrically.

$$\frac{}{P^+ \to [P^+]} \,\mathsf{id}_{P^+} \qquad \frac{}{[P^-] \to P^-} \,\mathsf{id}_{P^-}$$

It is far from obvious that these rules are complete, but we look at some examples first.

2 Example: Negative Atoms

As a first simple example, consider

$$a, b, a \multimap (b \multimap c), c \multimap d \Rightarrow d$$

where a, b, c, and d are atomic formulas. There are a number of proofs of this sequent. For example, we could apply $\multimap L$ to $c \multimap d$, or to $a \multimap (b \multimap c)$, and if choose the later, there two choices in the next step. With focusing (at least in this example), there is *exactly* one proof, no matter which polarity we assign to the atomic propositions. Let's try all negative polarity.

$$a^-, b^-, a^- \multimap (b^- \multimap c^-), c^- \multimap d^- \rightarrow d^-$$

We could try to focus on a^-. But this fails immediately, because in a sequent

$$\cdots, [a^-] \rightarrow d^-$$

there is no applicable rule since the right-hand side d does not match a. If we try to focus on the first implication we can get just a little further. Eliding some irrelevant formulas:

$$\dfrac{\cdots \rightarrow [a^-] \qquad \dfrac{\cdots \rightarrow [b^-] \qquad \cdots, [c^-] \rightarrow d^-}{\cdots, [b^- \multimap c^-] \rightarrow d^-}\multimap L}{\cdots, [a^- \multimap (b^- \multimap c^-)] \rightarrow d^-}\multimap L$$

At this point we must fail, since each inference is forced, and there is no rule for the sequent $\cdots, [c^-] \rightarrow d^-$ since $c \neq d$.

The upshot is that we can only focus on the rightmost antecedent, $c^- \multimap d^-$, because its conclusion is the only one matching the succedent d^-. We show the rest of the proof, each step being uniquely force in that other attempts will fail quickly.

$$\dfrac{\dfrac{\dfrac{a^-, b^-, a^- \multimap (b^- \multimap c^-) \rightarrow c^-}{a^-, b^-, a^- \multimap (b^- \multimap c^-) \rightarrow [c^-]}\mathsf{blur}R \qquad \dfrac{}{[d^-] \rightarrow d^-}\mathsf{id}_{d^-}}{a^-, b^-, a^- \multimap (b^- \multimap c^-), [c^- \multimap d^-] \rightarrow d^-}\multimap L}{a^-, b^-, a^- \multimap (b^- \multimap c^-), c^- \multimap d^- \rightarrow d^-}\mathsf{focus}L$$

The open subproof again has a unique proof, which arises from focusing on the linear implication.

In this example we were prescient in our distribution of resources among the premises of $\multimap L$. In an actual theorem prover, we have to perform *context management*, which can be achieved, for example, via boolean constraints on occurences of resources.

This is an example of what is called *backward chaining*, because it is quite goal-directed. In the example above, the goal of proving d^- is replaced by the goal of proving c^- after focusing on $c^- \multimap d^-$.

3 Example: Forward Chaining

Now we reconsider the example, but make all the atoms positive

$$a^+, b^+, a^+ \multimap (b^+ \multimap c^+), c^+ \multimap d^+ \rightarrow d^+$$

In this case, we cannot focus on $c^+ \multimap d^+$. Let's try:

$$\dfrac{a^+, b^+, a^+ \multimap (b^+ \multimap c^+) \rightarrow [c^+] \quad [d^+] \rightarrow d^+}{a^+, b^+, a^+ \multimap (b^+ \multimap c^+), [c^+ \multimap d^+] \rightarrow d^+}\ \multimap L$$

Now there is no applicable rule in the first premise, since c^+ is not already present among the resources! We also cannot focus on a^+ or b^+ on the left, since they have the wrong polarity. Focusing on the succedent d^+ fails because d^+ is not already available as a resource. So the only possibility is to focus on the other implication.

$$\dfrac{\dfrac{}{a^+ \rightarrow [a^+]}\ \mathsf{id}_{a^+} \quad \dfrac{\dfrac{}{b^+ \rightarrow [b^+]}\ \mathsf{id}_{b^+} \quad \dfrac{c^+, c^+ \multimap d^+ \rightarrow d^+}{[c^+], c^+ \multimap d^+ \rightarrow d^+}\ \mathsf{blur}L}{b^+, [b^+ \multimap c^+], c^+ \multimap d^+ \rightarrow d^+}\ \multimap L}{a^+, b^+, [a^+ \multimap (b^+ \multimap c^+)], c^+ \multimap d^+ \rightarrow d^+}\ \multimap L$$

The open premise can now be proved uniquely, first by focusing on $c^+ \multimap d^+$ and then d^+ on the right. This is quite similar to using the $\multimap L'$ rule from prior exercises, because we can apply a left rule for implication only if its positive, atomic left-hand side is already present in the context.

This is an example of *forward chaining*, where we essentially ignore the succedent of the sequent and see which implications have antecedents already present as resources.

4 Soundness of Chaining

Chaining is sound with respect to the cut-free sequent calculus, and therefore with respect to the sequent calculus. This is almost trivial, since it only imposes a restriction on the application of non-invertible rules, but in other respects the rules are identical. Pictorially, in order to obtain a proof $\Delta \Rightarrow A$ from $\Delta \rightarrow A$ we erase all brackets from the proof of $\Delta \rightarrow A$, which means that the premise and conclusion of the focus and blur rules are identical, and those rules can be contracted. We are left with a proof of $\Delta \Rightarrow A$.

More formally, it follows by a simple structural induction on $\delta \rightarrow \gamma$, where we define a focus erasure on sequents such that $\mathsf{erase}(\delta) \Rightarrow \mathsf{erase}(\gamma)$.

Theorem 1 (Soundness of Chaining) *If $\Delta \rightarrow A$ then $\Delta \Rightarrow A$.*

Proof: By induction on the structure of the given deduction. □

5 Completeness of Chaining

As might be expected, when we impose a restriction on the applications of inference rules, the difficult direction is to show that we can still prove all that we could prove before, that is, with arbitrary cut-free proofs $\Delta \Rightarrow A$.

But how do we show completeness, assuming it even holds? If it is not obvious (and I would be shocked if anyone found it obvious at this point), one way to proceed is to just try it with a straightforward induction to observe what happens. We might be lucky and it goes through.

Theorem 2 (Completeness of Chaining) *If $\Delta \Rightarrow A$ then $\Delta \rightarrow A$*

Proof: Attempt: By induction on the structure of $\overset{\mathcal{D}}{\Delta \Rightarrow A}$.

Case:

$$\mathcal{D} = \frac{}{P \Rightarrow P}\ \mathsf{id}_P$$

Then there are two subcases, depending on whether we assigned a positive or negative polarity to P. In one case we construct

$$\dfrac{\dfrac{}{P^+ \rightarrow [P^+]}\ \mathsf{id}_{P^+}}{P^+ \rightarrow P^+}\ \mathsf{focus}R$$

and in the other

$$\dfrac{\dfrac{}{[P^-] \rightarrow P^-}\ \mathsf{id}_{P^-}}{P^- \rightarrow P^-}\ \mathsf{focus}L$$

Well, that case was easy. Let's try another easy one.

Case:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_2\\ \Delta, A_1 \Rightarrow A_2\end{array}}{\Delta \Rightarrow A_1 \multimap A_2}\ {\multimap}R$$

Then we construct

$$\dfrac{\begin{array}{c}\text{i.h.}(\mathcal{D}_2)\\ \Delta, A_1 \rightarrow A_2\end{array}}{\Delta \rightarrow A_1 \multimap A_2}\ {\multimap}R$$

Clearly, the difficult has to arise when we apply a non-invertible rule.

Case:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_1\\ \Delta_1 \Rightarrow A_1\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \Delta_2, A_2 \Rightarrow C\end{array}}{\Delta_1, \Delta_2, A_1 \multimap A_2 \Rightarrow C}\ {\multimap}L$$

Let see what our situation looks like once we have applied the induction hypothesis:

$$\begin{array}{c}\begin{array}{cc}\text{i.h.}(\mathcal{D}_1) & \text{i.h.}(\mathcal{D}_2)\\ \Delta_1 \rightarrow A_1 & \Delta_2, A_2 \rightarrow C\end{array}\\ \vdots\\ \Delta_1, \Delta_2, A_1 \multimap A_2 \rightarrow C\end{array}$$

At this point we have reached a real impasse. One possibility, is to try to focus on $A_1 \multimap A_2$ in the conclusion, in which case we arrive at

$$\dfrac{\dfrac{\begin{array}{c}\text{i.h.}(\mathcal{D}_1)\\ \Delta_1 \rightarrow A_1\\ \vdots\\ \Delta_1 \rightarrow [A_1]\end{array} \quad \begin{array}{c}\text{i.h.}(\mathcal{D}_2)\\ \Delta_2, A_2 \rightarrow C\\ \vdots\\ \Delta_2, [A_2] \rightarrow C\end{array}}{\Delta_1, \Delta_2, [A_1 \multimap A_2] \rightarrow C}\ {\multimap}L}{\Delta_1, \Delta_2, A_1 \multimap A_2 \rightarrow C}\ \mathsf{focus}L$$

However, we cannot close these gaps. In fact, it is false that if $\Delta \to A$ then $\Delta \to [A]$![1] A simple counterexample would be

$$a^+, a^+ \multimap b^+ \to b^+$$

manifestly holds by focusing first on $a^+ \multimap b^+$ and then on b^+ on the right. But

$$a^+, a^+ \multimap b^+ \to [b^+]$$

fails, because no rule applies (b^+ is not in the antecedent).

Fortunately, we can return to an earlier idea, namely to use cut and identity as admissible rules. So *if* cut and identity were admissible for chaining sequents, then we could proceed as follows:

$$\dfrac{\dfrac{\dfrac{\text{i.h.}(\mathcal{D}_1)}{\Delta_1 \to A_1} \quad A_1, A_1 \multimap A_2 \to A_2}{\Delta_1, A_1 \multimap A_2 \to A_2}\,(\mathsf{cut}_{A_1}) \quad \dfrac{\text{i.h.}(\mathcal{D}_2)}{\Delta_2, A_2 \to C}}{\Delta_1, \Delta_2, A_1 \multimap A_2 \to C}\,(\mathsf{cut}_{A_2})$$

The open premise has a simple proof if identity were admissible.

$$\dfrac{\dfrac{\dfrac{}{A_1 \to [A_1]}\,(\mathsf{id}_{A_1}) \quad \dfrac{}{[A_2] \to A_2}\,(\mathsf{id}_{A_2})}{A_1, [A_1 \multimap A_2] \to A_2}\,{\multimap}L}{A_1, A_1 \multimap A_2 \to A_2}\,\mathsf{focus}$$

□

Assuming we were right, the completeness of focusing comes down to the admissibility of identity and cut. Identity is relatively straightforward.

Theorem 3 (Admissibility of Identity) *The following rules are all admissible.*

$$\dfrac{}{A \to [A]}\,(\mathsf{id}_A[\,]) \qquad \dfrac{}{[A] \to A}\,(\mathsf{id}[\,]_A) \qquad \dfrac{}{A \to A}\,(\mathsf{id}_A)$$

Proof: By mutual induction on the structure of A, where $(\mathsf{id}_A[\,])$ and $(\mathsf{id}[\,]_A)$ can appeal to (id_A) on the same A, but for an appeal in the other direction the formula A must become strictly smaller. □

[1] We missed this observation in lecture.

The admissibility of cut is more difficult, where the difficulty lies in maintaining our central restriction that there be only one formula in focus. Only the first two forms might admit cut reductions, because they match a right rule in focus with a left rule not in focus and vice versa.

$$\frac{\Delta \to [A] \quad \delta, A \to \gamma}{\Delta, \delta \to \gamma}\ (\mathsf{cut}[]_A) \qquad \frac{\delta \to A \quad \Delta, [A] \to C}{\delta, \Delta \to C}\ (\mathsf{cut}_A[])$$

$$\frac{\Delta \to A^- \quad \delta, A^- \to \gamma}{\Delta, \delta \to \gamma}\ (\mathsf{cut}_{A^-}) \qquad \frac{\delta \to A^+ \quad \Delta, A^+ \to C}{\delta, \Delta \to C}\ (\mathsf{cut}_{A^+})$$

In case the we have to blur the focus in the first two cases, we need the next two cases.

Theorem 4 (Admissibility of Cut) *The rules* $\mathsf{cut}[]_A$, $\mathsf{cut}_A[]$, cut_{A^-} *and* cut_{A^+} *are all admissible.*

Proof: By nested induction, first on the structure of A, then simultaneously on the structure of the two given proofs.

If cut meets identity, the proof of the premise opposite the identity can serve as the proof of the conclusion.

If a right rule for the cut formula meets the left rule, as can be the case only for $(\mathsf{cut}[]_A)$ and $(\mathsf{cut}_A[])$, we apply the usual cut reduction and obtain only cuts on the subformulas.

Otherwise, we can commute the cut upwards into the two premises of the $\mathsf{cut}[]_A$ and $\mathsf{cut}_A[]$ rules. Our general restriction cannot be violated by these commutations, because one occurrence of the cut formula always remains in focus.

When we blur focus on the cut formula because it is negative (for $\mathsf{cut}[]_A$) or positive (for $\mathsf{cut}_A[]$), we appeal to (cut_{A^-}) or (cut_{A^+}), respectively, on the subderivations.

For (cut_{A^-}) we fix the proof of the first premise and push the cut upwards in the second one, until A^- comes into focus, at which point we can appeal to the induction hypothesis on $(\mathsf{cut}_A[])$.

The cases for (cut_{A^+}) are symmetric: we fix the second premise and push the cut into the proof of the first premise, until A^+ comes into focus, in which case we appeal to $(\mathsf{cut}[]_A)$. □

Corollary 5

$$\frac{\Delta \to A \quad \Delta', A \to C}{\Delta, \Delta' \to C}\ (\mathsf{cut}_A)$$

Proof: The proposition A must be positive or negative, so the result follows immediately from (cut_{A^+}) or (cut_{A^-}). □

6 Persistent Resources

Adding persistent resources is straightforward in the judgmental formulation we have been following all along in this course. Persistent resources can never be in focus. Copying a persistent resource puts the copy in focus, so copying is a form of the focus rule.

$$\frac{\Gamma \,;\, \Delta \rightarrow [A^+]}{\Gamma \,;\, \Delta \rightarrow A^+}\ \mathsf{focus}R \qquad \frac{\Gamma \,;\, \Delta, [A^-] \rightarrow C}{\Gamma \,;\, \Delta, A^- \rightarrow C}\ \mathsf{focus}L$$

$$\frac{\Gamma, A \,;\, \Delta, [A] \rightarrow C \quad A \text{ not } P^+}{\Gamma, A \,;\, \Delta \rightarrow C}\ \mathsf{focus!}$$

For all three focus rules, it is crucial that there is no focus in the conclusion, which is expressed by writing Δ and C instead of δ and γ. The focus! rule is restricted so that we cannot focus on a positive atom. This means we need a new identity rule in order to prove positive atoms.

$$\frac{}{\Gamma \,;\, P^+ \rightarrow [P^+]}\ \mathsf{id}_{P^+} \qquad \frac{}{\Gamma \,;\, [P^-] \rightarrow P^-}\ \mathsf{id}_{P^-} \qquad \frac{}{\Gamma, P^+ \,;\, \cdot \rightarrow [P^+]}\ \mathsf{id!}_{P^+}$$

Finally, we have to consider the left and right rules for $!A$. From the proof of identity, we conjecture that the right rule requires $!A$ to be in focus, while the left rule works asynchronously.

$$\frac{\Gamma \,;\, \cdot \rightarrow A}{\Gamma \,;\, \cdot \rightarrow [!A]}\ !R \qquad \frac{\Gamma, A \,;\, \delta \rightarrow \gamma}{\Gamma \,;\, \delta, !A \rightarrow \gamma}\ !L$$

A surprising aspect is that we lose focus in the premise of $!R$, while in all other rules so far (except blur), focus is propogated to all premises. See Exercise 2 for a counterexample.

The admissible cut and identity rules now all carry the additional context Γ, and we also have the following

$$\frac{\Gamma \,;\, \cdot \rightarrow A \quad \Gamma, A \,;\, \delta \rightarrow \gamma}{\Gamma \,;\, \delta \rightarrow \gamma}\ (\mathsf{cut!}_A)$$

As in the proof of cut admissibility for the cut-free sequent calculus in Lecture 7, the persistent cut is always pushed into the second premise. When A is copied, we generate two new cuts, one $(\mathsf{cut!}_A)$ with the premise and then a $(\mathsf{cut}_A[])$ with the result. The induction measure is extended as before so that $(\mathsf{cut}_A[])$ is a smaller cut than $(\mathsf{cut!}_A)$.

7 Focusing

Andreoli's system of focusing [And92] incorporates both *inversion* and *chaining*. We can obtain this from the chaining system by an apparently small change. A sequent $\Gamma\ ;\Delta \rightarrow C$ is *stable* if Δ consists only of negative propositions and positive atoms and, symmetrically, C is a positive proposition or negative atom. No inversion applies to a stable sequent, so the first step in a proof attempt must always be a focusing step. We obtain the full system of focusing by requiring the conclusion of the focus rules to be stable.

$$\frac{\Gamma\ ;\Delta \rightarrow [A^+]}{\Gamma\ ;\Delta \rightarrow A^+}\ \mathsf{focus}R^* \qquad \frac{\Gamma\ ;\Delta, [A^-] \rightarrow C}{\Gamma\ ;\Delta, A^- \rightarrow C}\ \mathsf{focus}L^*$$

$$\frac{\Gamma, A\ ;\Delta, [A] \rightarrow C \quad A \text{ not } P^+}{\Gamma, A\ ;\Delta \rightarrow C}\ \mathsf{focus!}^*$$

(*) conclusion must be stable

As a consequence of this restriction, when a formula is in focus, this is the only place in a sequent where a rule can be applied: all the invertible ones are ruled out because the rest of the sequent remains stable.

Despite its simplicity, the proof of completeness of focusing becomes significantly more complicated. Specifically, the identity property is now hard to prove and does not just follow by induction on the structure of A (see Exercise 3).

8 From Propositions to Rules

An important application of focusing is to build the connection between linear inference and linear propositions. Let's reconsider a rule from an early example, changing two dimes and a nickel into a quarter.

$$\frac{\mathsf{d} \quad \mathsf{d} \quad \mathsf{n}}{\mathsf{q}}$$

We have represented this as the persistent proposition

$$\mathsf{d} \otimes \mathsf{d} \otimes \mathsf{n} \multimap \mathsf{q}$$

How does inference proceed if we focus on this persistent proposition, creating a linear copy in focus? Let's assume all atomic propositions are given negative polarity. Eliding other resources, we get something like this:

$$\dfrac{\dfrac{\dfrac{\to \mathsf{d}^-}{\to [\mathsf{d}^-]}\,\mathsf{blur}R \quad \dfrac{\dfrac{\to \mathsf{d}^-}{\to [\mathsf{d}^-]}\,\mathsf{blur}R \quad \dfrac{\to \mathsf{n}^-}{\to [\mathsf{n}^-]}\,\mathsf{blur}R}{\to [\mathsf{d}^- \otimes \mathsf{n}^-]}\,\otimes R}{\to [\mathsf{d}^- \otimes \mathsf{d}^- \otimes \mathsf{n}^-]}\,\otimes R \qquad \dfrac{C = \mathsf{q}^-}{[\mathsf{q}^-] \to C}\,\mathsf{id}_{\mathsf{q}^-}}{[\mathsf{d}^- \otimes \mathsf{d}^- \otimes \mathsf{n}^- \multimap \mathsf{q}^-] \to C}\,{\multimap}L$$

Systematically adding antecedents to each sequent and renaming C to q^- (which is forced if focusing on this formula is to apply), we obtain

$$\dfrac{\dfrac{\dfrac{\Delta_1 \to \mathsf{d}^-}{\Delta_1 \to [\mathsf{d}^-]}\,\mathsf{blur}R \quad \dfrac{\dfrac{\Delta_2 \to \mathsf{d}^-}{\Delta_2 \to [\mathsf{d}^-]}\,\mathsf{blur}R \quad \dfrac{\Delta_3 \to \mathsf{n}^-}{\Delta_3 \to [\mathsf{n}^-]}\,\mathsf{blur}R}{\Delta_2, \Delta_3 \to [\mathsf{d}^- \otimes \mathsf{n}^-]}\,\otimes R}{\Delta_1, \Delta_2, \Delta_3 \to [\mathsf{d}^- \otimes \mathsf{d}^- \otimes \mathsf{n}^-]}\,\otimes R \qquad \dfrac{}{[\mathsf{q}^-] \to \mathsf{q}^-}\,\mathsf{id}_{\mathsf{q}^-}}{\Delta_1, \Delta_2, \Delta_3, [\mathsf{d}^- \otimes \mathsf{d}^- \otimes \mathsf{n}^- \multimap \mathsf{q}^-] \to \mathsf{q}^-}\,{\multimap}L$$

In summary, if we have the persistent $\mathsf{d} \otimes \mathsf{d} \otimes \mathsf{n} \multimap \mathsf{q}$, we have obtained the following derived rule:

$$\dfrac{\Delta_1 \to \mathsf{d}^- \quad \Delta_2 \to \mathsf{d}^- \quad \Delta_3 \to \mathsf{n}^-}{\Delta_1, \Delta_2, \Delta_3 \to \mathsf{q}^-}$$

which is a rendering of the rule we started with at the beginning of the semester.

A powerful consequence of the completeness of focusing is that we can now drop the persistent resource altogether now and just use the above derived rule of inference instead.

Exercises

Exercise 1 In this exercise we work through the material in this lecture using disjunction ($A \oplus B$) and falsehood ($\mathbf{0}$).

(i) Present the rules for the chaining calculus.

(ii) Show the cases for $\oplus R_1$ and $\oplus L$ in the proof of completeness of chaining.

(iii) Show the cases in the admissibility of identity for $A_1 \oplus A_2$ and $\mathbf{0}$.

(iv) Show the case in the admissibility of cut for the chaining calculus where $\oplus R_1$ meets $\oplus L$.

(v) Give the derived rules of inference for $(a \oplus b) \multimap c$ and $a \multimap (b \oplus c)$, once assuming all atoms are positive and once assuming all atoms are negative.

Exercise 2 If we reformulate the $!R$ rule as

$$\frac{\Gamma \,;\, \cdot \rightarrow [A]}{\Gamma \,;\, \cdot \rightarrow [!A]}\ !R??$$

the focusing system becomes incomplete. Demonstrate this with a counterexample. Also, identify the point in the proof of completeness or one of its lemmas which fails under the incorrect version of the rule.

Exercise 3 We explore the admissibility of identity in the fully focused system of Section 7.

(i) Illustrate that the identity property for the system of focusing as presented in Section 7 does not seem to follow by a simple induction on the structure of the formula A.

(ii) Find an alternative proof of identity for the focusing calculus.

Exercise 4 Prove the admissibility of cut for the focused system in Section 7. You do not need to show any actual cases, but you should carefully present the generalized induction hypothesis (that is, all the necessary forms of cut) and the correct form of the induction.

Exercise 5 Determine the derived rules of inference corresponding to the persistent resource $\mathsf{q} \multimap \mathsf{d} \otimes \mathsf{d} \otimes \mathsf{n}$, using the technique of focusing. Consider the cases where all atoms are negative, and where all atoms are positive.

References

[And92] Jean-Marc Andreoli. Logic programming with focusing proofs in linear logic. *Journal of Logic and Computation*, 2(3):197–347, 1992.

[Lau04] Olivier Laurent. A proof of the focalization property of linear logic. Unpublished note, May 2004.