Lecture Notes on Cut Elimination

15-816: Linear Logic
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After presenting an interpretation of linear propositions in the sequent calculus as session types, we now return to studying properties of the sequent calculus itself in order to better understand how to search for proofs. The central theorem here is *cut elimination*: any provable sequent has a proof without using the cut rule. This will have many consequences. To appreciate the import of the theorem, consider the rule of cut.

$$\frac{\Gamma \,;\, \Delta \vdash A \quad \Gamma \,;\, \Delta', A \vdash C}{\Gamma \,;\, \Delta, \Delta' \vdash C}\ \mathsf{cut}$$

If we are trying to use this rule in an attempt to construct a proof of C from resources Δ and Δ' as well as persistent assumptions Γ, we have to conjecture an (arbitrary!) lemma A, prove it, which then licenses its use in the proof of C. If we know cut and cut! are redundant, then all other rules except identity and copy only break down the structure of the given proposition. Identity, of course, completes a subproof, and copy only provides another copy of a proposition already in Γ. This means that proof search never has to invent new propositions, just analyze the structure of ones that already exist in the sequent.

1 Cut Elimination from Cut Admissibility

To state and prove cut elimination we introduce some notation. We write $\Gamma \,;\, \Delta \Rightarrow A$ if we can prove $\Gamma \,;\, \Delta \vdash A$ without the use of the cut or cut! rules. All other rules regarding sequents have an identical counterpart for the '$\Rightarrow$' judgment. The cut elimination is the following theorem:

Theorem 1 (Cut Elimination) *If* $\Gamma \,;\, \Delta \vdash A$ *then* $\Gamma \,;\, \Delta \Rightarrow A$*.*

Because of its fundamental importance, there have been many different kinds of proofs of this theorem for different logics. The first one, which also introduced the sequent calculus, was by Gentzen [Gen35]. We will develop a proof by *structural induction*, by far the most important method of proof in the study of proofs. This technique was developed in [Pfe94] for classical linear logic, adapted to our case by Chang et al. [CCP03]. A good way to organize the proof of cut elimination is first to prove the *admissibility of cut* on cut-free derivations:

Theorem 2 (Admissibility of Cut) *If* $\Gamma \,;\, \Delta \Rightarrow A$ *and* $\Gamma \,;\, \Delta', A \Rightarrow C$ *then* $\Gamma \,;\, \Delta, \Delta' \Rightarrow C$*.*

Recall that a rule is *admissible* if every instance of the rule can be derived. Or, in other words, in every case where there are proofs of all premises there is also a proof of the conclusion. That's different from a *derivable rule* where we must have a closed-form proof of the conclusion from the premises, using the inference rules of the system. A derivable rule has the useful properties that it will remain derivable under any extension of the system under consideration. Admissible rules have to be reconsidered every time we extend a system, say, by adding rules for a new connective. Using the notation introduced in the last lecture, we can write cut as an admissible rule in the cut-free system as

$$\frac{\Gamma \,;\, \Delta \Rightarrow A \quad \Gamma \,;\, \Delta', A \Rightarrow C}{\Gamma \,;\, \Delta, \Delta' \Rightarrow C}\ (\mathsf{cut}_A)$$

We use dashed lines and parenthesized justifications for admissible rules. Of course, we have not yet proved yet that this is really admissible.

Before going forward with the proof of admissibility, it is worth checking that the admissibility of cut really implies cut elimination. Otherwise, we might waste a lot of time and effort proving something that may not be helpful towards our ultimate goal. For the moment, we restrict ourselves to ephemeral truth, postponing persistent truth and propositions $!A$ until Section 3.

Theorem 3 *If cut is admissible for purely linear cut-free sequent calculus, then cut elimination holds for the purely linear sequent calculus.*

Proof: Assume admissibility of cut for cut-free sequent calculus. Our theorem now claims: if $\Delta \vdash A$ then $\Delta \Rightarrow A$. We give these proofs names, so we can refer to them in our justifications.

If $\overset{\mathcal{D}}{\Delta \vdash A}$ then $\overset{\mathcal{D}'}{\Delta \Rightarrow A}$.

The proof is by induction on the structure of $\mathcal{D}$. Except for cut, all cases are straightforward. We show one such case.

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Delta_1 \vdash B} \quad \overset{\mathcal{D}_2}{\Delta_2, C \vdash A}}{\Delta_1, \Delta_2, B \multimap C \vdash A} \multimap L$$

Then

$$\mathcal{D}' = \dfrac{\overset{\text{i.h.}(\mathcal{D}_1)}{\Delta_1 \Rightarrow B} \quad \overset{\text{i.h.}(\mathcal{D}_2)}{\Delta_2, C \Rightarrow A}}{\Delta_1, \Delta_2, B \multimap C \Rightarrow A} \multimap L$$

The remaining case is that of cut. Luckily, we can call on our assumption that cut is admissible.

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Delta_1 \vdash B} \quad \overset{\mathcal{D}_2}{\Delta_2, B \vdash A}}{\Delta_1, \Delta_2 \vdash A} \mathsf{cut}_B$$

Then

$$\mathcal{D}' = \dfrac{\overset{\text{i.h.}(\mathcal{D}_1)}{\Delta_1 \Rightarrow B} \quad \overset{\text{i.h.}(\mathcal{D}_2)}{\Delta_2, B \Rightarrow A}}{\Delta_1, \Delta_2 \Rightarrow A} (\mathsf{cut}_B)$$

□

2 Linear Cut Admissibility

We now look at the admissibility of cut, restricting ourselves for the moment to the purely linear fragment. We have to prove:

Admissibility of Cut. If $\overset{\mathcal{D}}{\Delta \Rightarrow A}$ and $\overset{\mathcal{E}}{\Delta', A \Rightarrow C}$ then $\overset{\mathcal{F}}{\Delta, \Delta' \Rightarrow C}$

In inference rule notation:

$$\dfrac{\begin{matrix}\mathcal{D} \\ \Delta \Rightarrow A\end{matrix} \quad \begin{matrix}\mathcal{E} \\ \Delta', A \Rightarrow C\end{matrix}}{\Delta, \Delta' \Rightarrow C}\ (\mathsf{cut}_A)$$

How do we prove this? As usual in this domain, we expect it to be structural induction of some form. Likely candidates are the proofs $\mathcal{D}$ and $\mathcal{E}$, as well as the proposition A. Less likely would be Δ, Δ' and C.

When the form of induction is not obvious, a good way to proceed is to think about the function that constructs $\mathcal{F}$ when given $\mathcal{D}$ and $\mathcal{E}$. How do we build a (cut-free!) $\mathcal{F}$ when given (cut-free!) $\mathcal{D}$ and $\mathcal{E}$? Think back to the earlier lectures. Does something occur to you? Ponder this for a little while before reading on.

We have already spent considerable effort verifying *cut reductions*, checking that the right and left rules for the connectives are in harmony! Let's reconsider the case for $\multimap$ in this context.

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_2 \\ \Delta, A_1 \Rightarrow A_2\end{matrix}}{\Delta \Rightarrow A_1 \multimap A_2} \multimap R \quad \dfrac{\begin{matrix}\mathcal{E}_1 \\ \Delta'_1 \Rightarrow A_1\end{matrix} \quad \begin{matrix}\mathcal{E}_2 \\ \Delta'_2, A_2 \Rightarrow C\end{matrix}}{\Delta'_1, \Delta'_2, A_1 \multimap A_2 \Rightarrow C} \multimap L}{\Delta, \Delta'_1, \Delta'_2 \Rightarrow C}\ (\mathsf{cut}_{A_1 \multimap A_2})$$

$$\longrightarrow_R$$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{E}_1 \\ \Delta'_1 \Rightarrow A_1\end{matrix} \quad \begin{matrix}\mathcal{D}_2 \\ \Delta, A_1 \Rightarrow A_2\end{matrix}}{\Delta, \Delta'_1 \Rightarrow A_2}\ (\mathsf{cut}_{A_1}) \quad \begin{matrix}\mathcal{E}_2 \\ \Delta'_2, A_2 \Rightarrow C\end{matrix}}{\Delta, \Delta'_1, \Delta'_2 \Rightarrow C}\ (\mathsf{cut}_{A_2})$$

As discussed at length before, this kind of transformation reduces the cut formula, here from $A_1 \multimap A_2$ to A_1 and A_2. This strongly suggests that the cut formula plays a crucial role in showing that the transformation eventually terminates (and therefore the form of induction).

Since we have previously checked all the cases, this means that we have covered all possibilities where the cut formula was just introduced in both premises of the cut.

Which cases remain? We could consider all pairs of inference rules, but it turns out it is easier to organize them in a slightly different way. First, we consider a cut where either of the premises is the identity.

Case:

$$\mathcal{D} = \dfrac{}{A \Rightarrow A}\ \mathsf{id}_A \quad \text{and} \quad \begin{array}{c}\mathcal{E}\\ \Delta', A \Rightarrow C\end{array} \quad \text{arbitrary}$$

We have to construct a proof of $\Delta, \Delta' \Rightarrow C$, but $\Delta = A$, so we can let $\mathcal{F} = \mathcal{E}$.

Case:

$$\begin{array}{c}\mathcal{D}\\ \Delta \Rightarrow A\end{array} \quad \text{arbitrary, and} \quad \mathcal{E} = \dfrac{}{A \Rightarrow A}\ \mathsf{id}_A$$

We have to construct a proof of $\Delta, \Delta' \Rightarrow C$, but $\Delta' = (\cdot)$ and $C = A$, so we can let $\mathcal{F} = \mathcal{D}$.

In the remaining cases the last inference rule applied in the first or second premise of the cut must have been different from the cut formula. We call this a *side formula*. We organize the cases around which rule was applied to which premise. Fortunately, they all go the same way: we "push" up the cut past the inference that was applied to the side formula. We call these *commutative cases* for cut elimination, since we commute the cut with the inference rule in one of the premises. For example:

Case:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_1\\ \Delta_1, B_1, B_2 \Rightarrow A\end{array}}{\Delta_1, B_1 \otimes B_2 \Rightarrow A}\ \otimes L \quad \text{and} \quad \begin{array}{c}\mathcal{E}\\ \Delta', A \Rightarrow C\end{array} \quad \text{arbitrary}$$

In this case we transform

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Delta_1, B_1, B_2 \Rightarrow A\end{array}}{\Delta_1, B_1 \otimes B_2 \Rightarrow A}\ \otimes L \qquad \begin{array}{c}\mathcal{E}\\ \Delta', A \Rightarrow C\end{array}}{\Delta_1, B_1 \otimes B_2, \Delta' \Rightarrow C}\ (\mathsf{cut}_A)$$

to

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Delta_1, B_1, B_2 \Rightarrow A\end{array} \quad \begin{array}{c}\mathcal{E}\\ \Delta', A \Rightarrow C\end{array}}{\Delta_1, B_1, B_2, \Delta' \Rightarrow C}\ (\mathsf{cut}_A)}{\Delta_1, B_1 \otimes B_2, \Delta' \Rightarrow C}\ \otimes L$$

What becomes smaller here? The cut formula A stays the same, but the first premise of the cut is now $\mathcal{D}_1$, which is a subproof of $\mathcal{D}$.

Case:

$$\begin{array}{c}\mathcal{D}\\ \Delta \Rightarrow A\end{array} \quad \text{arbitrary, and} \quad \mathcal{E} = \dfrac{\begin{array}{c}\mathcal{E}_1\\ \Delta', A \Rightarrow C_1\end{array} \quad \begin{array}{c}\mathcal{E}_2\\ \Delta', A \Rightarrow C_2\end{array}}{\Delta', A \Rightarrow C_1 \mathbin{\&} C_2} \,\&R$$

In this case we transform

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \Delta \Rightarrow A\end{array} \quad \dfrac{\begin{array}{c}\mathcal{E}_1\\ \Delta', A \Rightarrow C_1\end{array} \quad \begin{array}{c}\mathcal{E}_2\\ \Delta', A \Rightarrow C_2\end{array}}{\Delta', A \Rightarrow C_1 \mathbin{\&} C_2} \,\&R}{\Delta, \Delta' \Rightarrow C_1 \mathbin{\&} C_2} \,(\mathsf{cut}_A)$$

to

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Delta \Rightarrow A\end{array} \quad \begin{array}{c}\mathcal{E}_1\\ \Delta', A \Rightarrow C_1\end{array}}{\Delta, \Delta' \Rightarrow C_1} \,(\mathsf{cut}_A) \quad \dfrac{\begin{array}{c}\mathcal{D}\\ \Delta \Rightarrow A\end{array} \quad \begin{array}{c}\mathcal{E}_2\\ \Delta', A \Rightarrow C_2\end{array}}{\Delta, \Delta' \Rightarrow C_2} \,(\mathsf{cut}_A)}{\Delta, \Delta' \Rightarrow C_1 \mathbin{\&} C_2} \,\&R$$

What becomes smaller here? One cut on A is turned into two cuts, each on A. But the second premise of each new cut ($\mathcal{E}_1$ and $\mathcal{E}_2$) is a subproof of the original one ($\mathcal{E}$).

Based on the cases considered so far we try to see if we can determine an induction principle we can use to prove admissibility of cut. In the *principal cases*, where a right rule for the cut formula in the first premise is matched up with a left rule for the cut formula in the second premise, the new cuts (which correspond to appeals to the induction hypothesis) use a smaller cut formula. In the *commutative cases* the cut formula and the proof one of the premises remain the same while the other gets smaller. In the *identity cases* where one of the premises is an identity rule, the cut disappears entirely.

This suggests what is called a *nested induction* or *lexicographic induction*. We write it in the proof below as a nested induction.

Theorem 4 (Admissibility of Cut) *The rule*

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \Delta \Rightarrow A\end{array} \quad \begin{array}{c}\mathcal{E}\\ \Delta', A \Rightarrow C\end{array}}{\Delta, \Delta' \Rightarrow C} \,(\mathsf{cut}_A)$$

is admissible in the purely linear sequent calculus.

Proof: By a nested induction, first on the structure of A and second simultaneously on the structures of $\mathcal{D}$ and $\mathcal{E}$. This means we can appeal to the induction hypothesis

1. when the cut formula A becomes smaller, or
2. the cut formula A stays the same and
   (a) either $\mathcal{D}$ becomes smaller and $\mathcal{E}$ stays the same, or
   (b) $\mathcal{D}$ stays the same and $\mathcal{E}$ becomes smaller.

We distinguish three kinds of cases.

Identity cases. When one premise or the other is an instance of the identity rule we can eliminate the cut outright.

Principal cases. When the cut formula A is introduced by the last inference in both premises we can reduce the cut to (potentially several) cuts on strict subformulas of A. We have demonstrated this by cut reductions in previous lectures.

Commutative cases. When the cut formula is a side formula of the last inference in either premise, we can appeal to induction hypothesis on this premise and then re-apply the last inference. These constitute valid appeals to the induction hypothesis because the cut formula and one of the deductions in the premises remain the same while the other becomes smaller.

□

3 Persistent Resources

Now we consider the extension of the previous argument in order to account for persistent resources. Recall that there are two new judgmental rules, and then the left and right rules for $!A$.

$$\frac{\Gamma\,;\cdot\vdash A \quad \Gamma, A\,;\Delta\vdash C}{\Gamma\,;\Delta\vdash C}\,\mathsf{cut!}_A \qquad \frac{\Gamma, A\,;\Delta, A\vdash C}{\Gamma, A\,;\Delta\vdash C}\,\mathsf{copy}_A$$

$$\frac{\Gamma\,;\cdot\vdash A}{\Gamma\,;\cdot\vdash\,!A}\,!R \qquad \frac{\Gamma, A\,;\Delta\vdash C}{\Gamma\,;\Delta, !A\vdash C}\,!L$$

We immediately move to proving the admissibility of cut in the cut-free version of the system, which prohibits both cut and cut!. The proof that cut admissibility implies cut elimination proceeds exactly as before.

First we observe that the cases in the proof that we had so far don't change in any significant way. This is because the persistent resources Γ are propagated from the conclusion of all rules to all premises and are not involved in any inferences except the four shown above: $\mathsf{cut!}_A$, copy_A, $!R$, and $!L$.

The new principal reduction is

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma ; \cdot \Rightarrow A\end{array}}{\Gamma ; \cdot \Rightarrow !A}\,!R \qquad \dfrac{\begin{array}{c}\mathcal{E}\\ \Gamma, A ; \Delta \Rightarrow C\end{array}}{\Gamma ; \Delta, !A \Rightarrow C}\,!L}{\Gamma ; \Delta \Rightarrow C}\ (\mathsf{cut}_{!A})$$

$$\longrightarrow_R$$

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma ; \cdot \Rightarrow A\end{array} \qquad \begin{array}{c}\mathcal{E}\\ \Gamma, A ; \Delta \Rightarrow C\end{array}}{\Gamma ; \Delta \Rightarrow C}\ (\mathsf{cut!}_A)$$

It reduces a cut with formula $!A$ to a cut! with formula A.

Next we examine cut!. In the second premise, the cut formula A is persistent and therefore propagated to all premises, so we can simply push up the cut in the second premise, apply the induction hypothesis, and re-apply the rule we moved it past. The only exception is if the second premise is an application of the copy rule on A. In this case we have to generate two nested cuts, which we already saw when we studied the operational behavior of replicating input under session types in Lecture 5.

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma\,;\,\cdot \Rightarrow A\end{array} \quad \dfrac{\begin{array}{c}\mathcal{E}'\\ \Gamma, A\,;\,\Delta, A \Rightarrow C\end{array}}{\Gamma, A\,;\,\Delta \Rightarrow C}\,\mathsf{copy}_A}{\Gamma\,;\,\Delta \Rightarrow C}\,(\mathsf{cut!}_A)$$

$$\longrightarrow_R$$

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma\,;\,\cdot \Rightarrow A\end{array} \quad \dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma\,;\,\cdot \Rightarrow A\end{array} \quad \begin{array}{c}\mathcal{E}'\\ \Gamma, A\,;\,\Delta, A \Rightarrow C\end{array}}{\Gamma\,;\,\Delta, A \Rightarrow C}\,(\mathsf{cut!}_A)}{\Gamma\,;\,\Delta \Rightarrow C}\,(\mathsf{cut}_A)$$

We see that we need two copies of $\mathcal{D}$, which is okay since $\mathcal{D}$ does not use any ephemeral resources. The new $\mathsf{cut!}_A$ is on a subproof $\mathcal{E}'$ of the second premise of the original $\mathsf{cut!}_A$, so this appears to be a legitimate appeal to an induction hypothesis.

The second cut on A is more problematic. The first premise is the same, but the proof of the second premise is the result of an appeal to the induction hypothesis and could be much larger. So we have to decree that $\mathsf{cut!}_A > \mathsf{cut}_A$. In other words, what is getting smaller here is the kind of cut, while the cut formula stays the same, and the proof may be getting larger.

Fortunately, these are all the new cases we have to consider.

Theorem 5 (Admissibility of Cut) *The rules*

$$\dfrac{\Gamma\,;\,\Delta \Rightarrow A \quad \Gamma\,;\,\Delta', A \Rightarrow C}{\Gamma\,;\,\Delta, \Delta' \Rightarrow C}\,(\mathsf{cut}_A) \qquad \dfrac{\Gamma\,;\,\cdot \Rightarrow A \quad \Gamma, A\,;\,\Delta \Rightarrow C}{\Gamma\,;\,\Delta \Rightarrow C}\,(\mathsf{cut!}_A)$$

are admissible in the cut-free sequent calculus

Proof: By a nested induction, first on the structure of A, second on the order $\mathsf{cut!}_A > \mathsf{cut}_A$, and third simultaneously on the structures of $\mathcal{D}$ and $\mathcal{E}$. This means we can appeal to the induction hypothesis

1. when the cut formula A becomes smaller, or
2. the cut formula A stays the same and $\mathsf{cut!}_A$ appeals to cut_A, or

3. the cut formula A stays the same and the kind of cut stays the same and
   (a) either $\mathcal{D}$ becomes smaller and $\mathcal{E}$ stays the same, or
   (b) $\mathcal{D}$ stays the same and $\mathcal{E}$ becomes smaller.

In addition to *identity cases*, *principal cases*, and *commutative cases*, all of which proceed as before, we also have the *copy case*. Here the second premise of a cut$!_A$ copies the cut formula A, which is reduced as indicated in the text before this theorem. □

Theorem 6 (Cut Elimination) *If* $\Gamma\ ;\ \Delta \vdash A$ *then* $\Gamma\ ;\ \Delta \Rightarrow A$.

Proof: By straightforward induction over the structure of the given deduction, appealing to the admissibility of cut in the cases of cut or cut!. □

Of course, the opposite implication also holds, because the exact proofs can be copied from the cut-free to the system with cut.

4 Consequences of Cut Elimination

There are many important consequences of cut elimination. One class of theorems are so-called *refutations*, showing that certain conjectures can not be proven. Here are a few.

Corollary 7 (Consistency) *It is not the case that* $\cdot\ ;\ \cdot \vdash \mathbf{0}$.

Proof: Assume $\cdot\ ;\ \cdot \vdash \mathbf{0}$. By cut elimination, $\cdot\ ;\ \cdot \Rightarrow \mathbf{0}$. But no rule could have this conclusion (there is no right rule for $\mathbf{0}$). □

Without cut elimination the above proof would not work, because the sequent in question might have been inferred by the cut or cut! rules.

Corollary 8 (Disjunction Property) *If* $\cdot\ ;\ \cdot \vdash A \oplus B$ *then either* $\cdot\ ;\ \cdot \vdash A$ *or* $\cdot\ ;\ \cdot \vdash B$.

Proof: Assume $\cdot\ ;\ \cdot \vdash A \oplus B$. By cut elimination $\cdot\ ;\ \cdot \Rightarrow A \oplus B$. Only two rules could have this conclusion, namely $\oplus R_1$ and $\oplus R_2$, with a premise reading either $\cdot\ ;\ \cdot \Rightarrow A$ or $\cdot\ ;\ \cdot \Rightarrow B$. Therefore either $\cdot\ ;\ \cdot \vdash A$ or $\cdot\ ;\ \cdot \vdash B$ □

Corollary 9 (No Interaction $\multimap/\oplus$) *It is not the case that for arbitary* A, B, *and* C *we have* $A \multimap (B \oplus C) \vdash (A \multimap B) \oplus (A \multimap C)$.

Proof: Assume $A \multimap (B \oplus C) \vdash (A \multimap B) \oplus (A \multimap C)$. By cut elimination, we would then also have

$$A \multimap (B \oplus C) \Rightarrow (A \multimap B) \oplus (A \multimap C)$$

This could have been inferred by three possible rules.

$\multimap L$. Then one premise reads $\cdot \Rightarrow A$, which does not hold for arbitrary A.

$\oplus R_1$. Then the premise reads

$$A \multimap (B \oplus C) \Rightarrow A \multimap B$$

Inferring this by $\multimap L$ fails again as before, so the last rule must have been $\multimap R$ with premise

$$A \multimap (B \oplus C), A \Rightarrow B$$

Now, only $\multimap L$ could conclude this, where A must be propagated the left premise. We would have premises

$$A \Rightarrow A \quad \text{and} \quad B \oplus C \Rightarrow B$$

The first holds by id_A, while the second could have been inferred only by $\oplus L$, with premises

$$B \Rightarrow B \quad \text{and} \quad C \Rightarrow B$$

The first holds by id_B, while the second is manifestly not provable in the given generality.

$\oplus R_2$. Symmetric to the previous case.

□

Exercises

Exercise 1 Write out the following cases in the proof of cut admissibility.

(i) Show the principal case for $\oplus R_1$ matched against $\oplus L$.

(ii) The commutative cases for $\oplus L$ in the first premise matched against $\otimes R$ in the second premise.

(iii) The case of $\mathsf{cut!}_A$ where the rule in the second premise is $\otimes R$.

(iv) The case of $\mathsf{cut!}_A$ where the rule in the second premise is id_C.

Exercise 2 In the reduction where $\mathsf{cut!}_A$ meets copy_A, we create two new cuts: $\mathsf{cut!}_A$ above cut_A. Explore what happens if we swap the order of the two, with cut_A being above $\mathsf{cut!}_A$.

Exercise 3 Reconsider the alternative rule

$$\frac{\Delta, B \vdash C}{\Delta, A, A \multimap B \vdash C} \multimap L'$$

from Exercise L2.3. As noted in some student solutions, deriving $\multimap L$ from this seems to require a cut.

(i) Show which cases in the proof of cut admissibility go awry.

(ii) Prove that cut elimination does *not* hold if $\multimap L$ is replaced by $\multimap L'$.

Exercise 4 Among the following prove those are true and refute those that are not.

(i) $A \dashv\vdash A \mathbin{\&} A$

(ii) $A \dashv\vdash A \otimes A$

(iii) $A \dashv\vdash A \oplus A$

(iv) $\mathbf{1} \dashv\vdash A \multimap A$

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