On the principle of excluded middle

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To a large extent, this paper anticipated not only Heyting's formalization of intuitionistic logic, but also results on the translatability of classical mathematics into intuitionistic mathematics. It provides an important link between intuitionism and other works on the foundations of mathematics.

Two formal systems $\mathfrak{B}$ and $\mathfrak{H}$ (Brouwer's and Hilbert's) of the propositional calculus are proposed. Both contain a (somewhat uneconomical but) complete set of axioms of the positive implicational calculus (I, § 5, (1), Axioms 1, 2, 3, 4) and rules of modus ponens and substitution. In addition, $\mathfrak{B}$ contains also (II, § 6):

Axiom 5. $(A \to B) \to ((A \to \overline{B}) \to \overline{A})$.

The system $\mathfrak{H}$ is obtained from $\mathfrak{B}$ by the addition (II, § 7) of

Axiom 6. $\overline{\overline{A}} \to A$.

It is proved that $\mathfrak{H}$ is equivalent to Hilbert's formulation of the classical propositional calculus.

The system $\mathfrak{B}$ is nowadays known as the minimal calculus and differs from Heyting's system in that the latter contains, in addition to Axioms 1–5, also

Axiom *h*. $A \to (\overline{A} \to B)$.

The status of h in intuitionistic logic is not without question. Thus, according to Kolmogorov, it "is used only in a symbolic presentation of the logic of judgments; therefore it is not affected by Brouwer's critique" (I, § 6), and it "does not have and cannot have any intuitive foundation since it asserts something about the consequences of something impossible: we have to accept B if the true judgment A is regarded as false" (II, § 4). Heyting appears rather diffident about defending the inclusion of h. Thus, according to him (*1956*, p. 102), if we have deduced a contradiction from the supposition that the construction A was carried out, "then, in a sense, this can be considered as a construction.... I shall interpret implication in this wider sense".

Hence it is fair to say that, as a codification of Brouwer's ideas, $\mathfrak{B}$ is no less reasonable than Heyting's propositional calculus.

Kolmogorov makes no attempt in this paper to formalize intuitionistic quantification theory completely. Rather, he just lists as intuitively obvious the rule **P** of generalization, which allows us to prefix (a) to any given formula, and four axioms, I, II, III, IV (V, § 3). In the same context, however, he also argues to the effect that

Axiom *g*. $(a)A(a) \to A(t)$

is intuitively true. If, therefore, to the system $\mathfrak{B}$ we adjoin the explicitly stated rule **P** and Axioms I–IV plus Axiom g, we obtain an adequate axiomatization $\mathfrak{BQ}$ of intuitionistic logic that differs from Heyting's system only in the omission of the questionable Axiom h. This is the extent of the anticipation of Heyting's formalization.

The main purpose of the paper is to

prove that classical mathematics is translatable into intuitionistic mathematics. For this purpose, with each formula $\mathfrak{S}$ of mathematics there is associated a translation $\mathfrak{S}^*$ in a perfectly general manner (IV, § 2). If $\overline{}$, $\to$, (a), and (Ea) are the only symbols we use for forming new formulas from given formulas, the definition amounts to: for atomic $\mathfrak{S}$, $\mathfrak{S}^*$ is its double negation $\overline{\overline{\mathfrak{S}}}$, or $n\mathfrak{S}$; $(\overline{B})^*$ is $n(\overline{B^*})$; $(A \to B)^*$ is $n(A^* \to B^*)$; $((a)A(a))^*$ is $n(a)(A(a))^*$; $((Ea)A(a))^*$ is $n(Ea)(A(a))^*$.

The following results are proved exactly (III, § 3):

Lemma 1. $\vdash_{\mathfrak{B}} n\overline{A} \to \overline{A}$.

Lemma 2. If $\vdash_{\mathfrak{B}} nA \to A$ and $\vdash_{\mathfrak{B}} nB \to B$, then $\vdash_{\mathfrak{B}} n(A \to B) \to (A \to B)$.

Theorem I. If $\mathfrak{S}_1, \ldots, \mathfrak{S}_k$ are all the atomic formulas in A and A is constructed from $\mathfrak{S}_1, \ldots, \mathfrak{S}_k$ by negation and implication only, then, provided

$$n\mathfrak{S}_1 \to \mathfrak{S}_1, \ldots, n\mathfrak{S}_k \to \mathfrak{S}_k$$

are true (or taken as additional axioms), (1) $\vdash_{\mathfrak{B}} nA \to A$ and (2) $\vdash_{\mathfrak{B}} A$ if $\vdash_{\mathfrak{H}} A$.

If $\mathfrak{U} = \{\mathfrak{U}_1, \ldots, \mathfrak{U}_k\}$ is a set of axioms and $\mathfrak{U}^*$ is $\{\mathfrak{U}_1^*, \ldots, \mathfrak{U}_k^*\}$, then (IV, § 3):

Theorem II. If $\mathfrak{U} \vdash_{\mathfrak{H}} \mathfrak{S}$, then $\mathfrak{U}^* \vdash_{\mathfrak{B}} \mathfrak{S}^*$.

The proof of this theorem uses Lemma 2 for the rule of modus ponens and (in IV, § 4) the fact that the translations $\mathfrak{S}_1^*, \ldots, \mathfrak{S}_m^*$ of the atomic formulas have the property

$$\vdash_{\mathfrak{B}} n\mathfrak{S}_1^* \to \mathfrak{S}_1^*, \ldots, \vdash_{\mathfrak{B}} n\mathfrak{S}_m^* \to \mathfrak{S}_m^*.$$

This last fact and Theorem I yield the result that the translation of Axioms 1–6 all are theorems of $\mathfrak{B}$.

Strictly speaking, Theorem II is established only for the case in which $\mathfrak{S}$ is built up by implication and negation (in particular, $\mathfrak{S}$ does not contain quantifiers). However, Kolmogorov does envisage a much stronger result and illustrates by an example the treatment of axioms about quantifiers, axioms that he tends to take as being on the same footing as axioms about numbers and sets (IV, § 5).

If we take his system $\mathfrak{BQ}$ (Axioms 1–5, I–IV, g, with rules of inference) of quantification theory and extend his illustration to cover also the rule **P** and the remaining axioms, and if we let $\mathfrak{HQ}$ be $\mathfrak{BQ}$ plus Axiom 6, then we have also:

Theorem III. If $\mathfrak{U} \vdash_{\mathfrak{HQ}} \mathfrak{S}$, then $\mathfrak{U}^* \vdash_{\mathfrak{BQ}} \mathfrak{S}^*$.

In that case, all the derivations in V, § 4, can be dispensed with because they would follow from Theorem III.

A very suggestive remark (beginning of IV, § 5, and last two paragraphs of IV, § 6) is that every axiom A of mathematics is of type $\mathfrak{K}$, that is, A^* is (intuitionistically) true. From this it would seem to follow that all classical mathematics is intuitionistically consistent (V, § 1). As we know, however, this conclusion, even today, has not yet been firmly established so far as classical analysis and set theory are concerned. On the other hand, it seems not unreasonable to assert that Kolmogorov did foresee that the system of classical number theory is translatable into intuitionistic theory and therefore is intuitionistically consistent. In fact, it is not hard to work out his general indications and verify such a conclusion.

This completes the summary of the anticipations of results on the intuitionistic consistency of classical mathematics. It remains to mention a few of the incidental remarks.

Kolmogorov (V, § 1.1) states that, contrary to a remark by Brouwer, a finitary conclusion established by nonintuitionistic methods is (intuitionistically) true.

Two new examples are given of propositions not provable without the help of illegitimate uses of the principle of excluded middle. One of them, suggested by Novikov, is that every point in the complement C of a closed set is contained in an interval in C (V, § 5). The other example is the Cantor-Bendixson theorem (V, § 6).

With respect to system $\mathfrak{B}$, it is stated explicitly (II, § 6) that "the question whether this axiom system is a complete axiom system for the intuitionistic general logic of judgments remains open".

At the end of II, § 2, the question of the classical completeness of Axioms 1–4 for implication alone is raised. As we know, the answer to the question is negative. In fact, it is now a familiar result that it is necessary to add Peirce's law, $((A \to B) \to A) \to A$, to render the positive implicational calculus classically complete.

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The paper was translated by the editor. When informed of the projected publication of the translation of his paper, Professor Kolmogorov signified his acquiescence.

Introduction

Brouwer's writings have revealed that it is illegitimate to use the principle of excluded middle in the domain of transfinite arguments. Our task here will be to explain why this illegitimate use has not yet led to contradictions and also why the very illegitimacy has often gone unnoticed.

Only the finitary conclusions of mathematics can have significance in applications. But transfinite arguments are often used to provide a foundation for finitary conclusions. Brouwer considers, therefore, that even those who are interested only in the finitary results of mathematics cannot ignore the intuitionistic critique of the principle of excluded middle.

We shall prove that all the finitary conclusions obtained by means of a transfinite use of the principle of excluded middle are correct and can be proved even without its help.

A natural question is whether the transfinite premisses that are used to obtain correct finitary conclusions have any meaning.

We shall prove that every conclusion obtained with the help of the principle of excluded middle is correct provided every judgment that enters in its formulation is replaced by a judgment asserting its double negation. We call the double negation of a judgment its "pseudotruth". Thus, in the mathematics of pseudotruth it is legitimate to apply the principle of excluded middle.

The necessity of introducing such notions as "pseudoexistence" and "pseudotruth" has long been felt in mathematics, if only in connection with the question of Zermelo's axiom. It is only now, however, that one of the forms of pseudotruth has received a strict determination and has been given a firm basis through axioms used in the domain of pseudotruth but not used for truth proper.

I. Formalistic and intuitionistic points of view

§ 1. From the formalistic point of view mathematics is a collection of formulas.[1] Formulas are combinations of elementary symbols taken from a definite supply. At the basis of mathematics lie a certain group of formulas, called axioms, and certain rules that enable us to construct new formulas from given formulas; as rules of this kind we have at the present time the inference according to the schema $\mathfrak{S}, \mathfrak{S} \to \mathfrak{T} \mid \mathfrak{T}$ and the rule of substitution of particular values for the symbols of variables of various kinds.

[1] See *Hilbert 1922a*, p. 152.

In contradistinction to the axioms, wittingly taken as "true", a certain group of formulas are wittingly taken as "false". A system of axioms is said to be "consistent" if no formula considered "false" can be obtained from them by a derivation carried out according to the rules.

§ 2. The formalistic point of view in mathematics asserts that the selection of the axioms constituting its basis is arbitrary and subject only to considerations of practical convenience that lie outside of mathematics and are, of course, more or less conventional.[2] The sole absolute demand made upon every mathematical system is, from the point of view now considered, the demand that the axioms constituting its basis be consistent.

The formulas proved on the basis of the axioms are said to be true, those leading to a contradiction false. The question of the truth or falsity of a consistent but unprovable formula has no meaning from the formalistic point of view. The existence of such formulas shows that the system of axioms is incomplete. We can complete an incomplete system of axioms, if for some reason this is desirable, by taking as an axiom an unprovable and consistent formula, or, with the same right, its contradictory. The selection of the formula taken as the new axiom, from each pair of contradictory formulas, is thus subject only to considerations of convenience.

§ 3. The point of departure of the intuitionistic conception is the recognition of the real meaning of mathematical propositions. The axioms that constitute the basis of mathematics are adopted in order to express facts given to us. This conception tolerates the formalistic method in the study of mathematical constructions as one among other possible methods but goes against the formalistic conception of mathematics as a whole.

To the question of the nature of unprovable but consistent propositions the intuitionistic conception gives an answer completely different from that given by the purely formalistic conception. Suppose that a system of axioms is given for a certain branch of mathematics, geometry for example. These axioms express properties of the subject under investigation, in this particular case, of space. Suppose, further, that a certain proposition of the branch selected cannot be proved on the basis of the given axioms but does not lead to a contradiction either. From the intuitionistic point of view two cases can occur. First, it may happen that the truth or falsity of the proposition considered follows from direct examination; in that case the given proposition, if it is true—or, if it is false, its contradictory—can be taken as a new axiom. Second, it may happen that the proposition is indeterminate, that is, that its truth or falsity does not follow from direct examination; in that case the only thing that we can do is to try to derive the proposition in question from others that are immediately obvious; if this does not succeed, the proposition has to be regarded as indeterminate, since it is possible that we shall subsequently have to adopt as obviously true axioms from which its truth or falsity can be derived; but whether this will be the case is precisely what we do not know at the present time.

[2] See *Whitehead and Russell 1910*, Introduction. Hilbert, too, is close to this point of view; for him absolute truths (*absolute Wahrheiten*) are propositions of "metamathematics" only, that is, assertions of consistency, but, on the other hand, the formulas of ordinary mathematics (*eigentliche Mathematik*), too, are in his opinion expressions of certain thoughts (*Gedanken*). (See *Hilbert 1922a*, pp. 152–153.)

§ 4. The formalistic point of view is advanced in mathematical logic too. In the present paper we confront it precisely in the field of logic. To deny any real meaning to mathematical propositions is, however, what constitutes the basis of the formalistic point of view in mathematical logic. In fact, no one would propose applying to reality any logical formula that has no real meaning. Thus, so long as mathematical logic is regarded only as a formal system whose formulas have no real meaning, it diverges from general logic; the formalistic point of view can exist only in mathematics and mathematical logic but not in the ordinary logic that lays claim to significance in applications to reality.

As for us, we do not isolate a special "mathematical logic" from general logic, but we admit only that the originality of mathematics as a science creates for logic special problems that are investigated by a specialized "mathematical logic". Only in this logic does a doubt arise concerning the unconditional applicability of the principle of excluded middle.

§ 5. The difference between the two points of view presented manifests itself even in the domain of the logic of judgments. In what follows we understand by the general logic of judgments the science that investigates the properties of arbitrary judgments independently of their content, so far as their truth, their falsity, and the ways in which they are derived are concerned. (Each judgment is regarded as an unanalyzable element in the investigation.) The general logic of judgments is formally expressed with the help of symbols for arbitrary judgments, A, B, $C, \ldots$, of the symbol for implication, $A \to B$, and of the symbol for negation, $\overline{A}$.

Hilbert (*1922a*, p. 153) offered the following system of axioms for the logic of judgments:

Axioms of implication

$$(1)\quad \begin{cases} 1.\ A \to (B \to A), \\ 2.\ \{A \to (A \to B)\} \to (A \to B), \\ 3.\ \{A \to (B \to C)\} \to \{B \to (A \to C)\}, \\ 4.\ (B \to C) \to \{(A \to B) \to (A \to C)\}. \end{cases}$$

Axioms of negation

$$(2)\quad \begin{cases} 5.\ A \to (\overline{A} \to B), \\ 6.\ (A \to B) \to \{(\overline{A} \to B) \to B\}. \end{cases}$$

The inner consistency of these axioms can be proved in an extremely elementary way.[3] From the formalistic point of view this is sufficient for accepting them as a basis for the general logic of judgments.

Moreover, Hilbert's system is complete: no new independent axiom can be added without contradiction. More precisely, for every formula written with the symbols of the logic of judgments, even such a formula as

$$\overline{\overline{(A \to B)}} \to (\overline{\overline{A}} \to \overline{\overline{B}}),$$

[3] See *Ackermann 1924*.

either the formula can be proved on the basis of Hilbert's axioms or the consequence

$$A,$$

that is, the truth of an arbitrary judgment, can be derived from it by means of the same axioms.

§ 6. From the intuitionistic point of view the mutual consistency of Hilbert's axioms is by no means sufficient for their acceptance. In the next chapter we shall analyze the source of their significance for judgments in general and for particular forms of judgments.

One of Hilbert's two axioms of negation, Axiom 6, expresses the principle of excluded middle in a somewhat unusual form. Brouwer proved that the application of this principle to arbitrary judgments is without any foundation.[4] Axiom 5 is used only in a symbolic presentation of the logic of judgments; therefore it is not affected by Brouwer's critique, especially since it has no intuitive foundation either.

Thus, together with a critique of Hilbert's axioms, we shall have to present new axioms of negation, whose applicability to arbitrary judgments has been ascertained.

II. Axioms of the logic of judgment

§ 1. The axioms of the general logic of judgments lay claim to having significance for all judgments; therefore, they must follow from the general properties of judgments. To be sure, what comes immediately below is not at all a definition of fundamental notions or a proof of the axioms of the logic of judgments, but a search for their intuitive sources that already uses all the notions and devices of logic.

In the logic of judgments the judgment is considered the ultimate element in the investigation. When we consider the judgment independently of the synthesis of subject and predicate that it contains, there remains the sole characteristic property of a judgment, the one that distinguishes it from other forms of expression and was stated by Aristotle:[5] it can be appraised from the point of view of truth or falsity. It is natural to try to derive the axioms of the general logic of judgments without going beyond its own boundaries, that is, purely from the property of judgments that

[4] See *Brouwer 1923d*, p. 252. Hilbert, too, thinks that the principle of excluded middle is not intuitively obvious when applied to infinite collections of objects. In that case he expresses it symbolically by the two formulas

$$\overline{(a)}A(a)\ \ddot{a}q.\ (Ea)\overline{A}(a),$$
$$\overline{(Ea)}A(a)\ \ddot{a}q.\ (a)\overline{A}(a).$$

(See *Hilbert 1922a*, p. 155.) So far as the principle of excluded middle in the general logic of judgments (Axiom 6) is concerned, Hilbert does not say anything about the question of its intuitive obviousness; apparently he considers this obviousness to be indubitable. These views of Hilbert's are not inseparably linked with the fundamental and purely formal task that he set himself, the investigation of consistency; they seem incorrect to us.

First, Axiom 6 is not intuitively obvious. Its relation to finitary logic (*finite Logik*) is only illusory; while the truth of the axioms of implication (1–4) is perceived independently of the content of the judgments, the truth of Axiom 6, as will be explained in the next chapter, demands for its justification that the content of the judgments be considered, and this content may be transfinite.

Second, if Axioms 1–6 are adopted, the two formulas given above can be proved with the help of a few axioms whose intuitive obviousness cannot be questioned. We shall give the proof in Chapter V, but the first argument is sufficient to justify the investigations that now follow.

[5] *De interpretatione*, 4; *De anima*, III, 6.

was just mentioned. In the next sections of the present chapter we shall investigate to what extent this is possible.

§ 2. The meaning of the symbol $A \to B$ is exhausted by the fact that, once convinced of the truth of A, we have to accept the truth of B too. Or, in the formalistic interpretation: if formula A is written down, we can also write down formula B.[6] Thus, the relation of implication between two judgments does not establish any connection between their contents.

Hilbert's first axiom of implication, which means that "the true follows from anything", results from such a formalistic interpretation of implication: once B is true by itself, then, after having accepted A, we also have to regard B as true. The truth of the remaining three axioms of implication is seen just as easily on the basis of the interpretation given for the notion of implication. Moreover, the character of the judgments considered is not in the least affected; consequently, no doubt can arise about the possibility of applying these axioms to arbitrary judgments.

The question of the completeness of the system of the four axioms of implication is interesting. After what has been said concerning the completeness of Hilbert's full system of axioms for the logic of judgments, the question has to be put thus: A formula proved by means of the axioms of implication and the axioms of negation is said to be true; can every true formula written with the help of only the symbols for arbitrary judgments and implication, without the symbol for negation, be proved on the basis of the four axioms of implication alone?

§ 3. So far as a completed judgment, considered as a whole, is concerned, negation is merely the interdiction from regarding the judgment as true. We can obtain a fuller view of what negation is by considering the judgment as a statement attributing a predicate to a subject; negation then is the assertion that the predicate is incompatible with the subject.

The symbol $\overline{A}$ of the logic of judgments, of course, expresses the first interpretation of negation, that is, the interdiction from considering the judgment A true. However, the usual tradition in logic has been to pass from the first interpretation to the second, regarded as more primitive.[7] In the application to mathematical judgments this turns out to be impossible.

In so far as the negation of a judgment is the product of direct examination, the second interpretation, which takes its point of departure in the idea of the impossibility of the synthesis that creates the judgment, is actually closer to the substance of the matter than the first, which rests upon the purely formal idea of interdiction. But, when a negation is obtained as the result of a derivation, the reduction of the first interpretation to the second is no longer necessary and, in the case of mathematical judgments, is sometimes even impossible. In fact, many negative judgments in mathematics are proved by means of a reduction to a contradiction, according to the schema $\mathfrak{S} \to \mathfrak{T}, \overline{\mathfrak{T}} \mid \overline{\mathfrak{S}}$,[8] and cannot be proved in any other way.

[6] This is precisely what is expressed by the schema $\mathfrak{S}, \mathfrak{S} \to \mathfrak{T} \mid \mathfrak{T}$ in Hilbert's metamathematics. Sigwart, too, regards this schema as the most general schema in any inference (see *Sigwart 1908*, p. 372, [[or *1904*, p. 434]]).

[7] See, for instance, *Sigwart 1908*, pp. 135ff., [[or *1904*, pp. 155ff.]]

[8] See § 6 on the principle of contradiction.

Thus, the first interpretation of negation is independent. It was originally introduced by Brouwer (*1923d*), who defines negation as absurdity. It rests upon the second, since to derive a negative judgment by reduction to a contradiction we must already have some negative judgments, but at the same time it is broader than the second.

§ 4. Hilbert's first axiom of negation, "Anything follows from the false", made its appearance only with the rise of symbolic logic, as did also, incidentally, the first axiom of implication. But, while the first axiom of implication follows with intuitive obviousness from a correct interpretation of the idea of logical implication, the axiom now considered does not have and cannot have any intuitive foundation since it asserts something about the consequences of something impossible: we have to accept B if the true judgment A is regarded as false.

Thus, Hilbert's first axiom of negation cannot be an axiom of the intuitionistic logic of judgments, no matter which interpretation of negation we take as a point of departure. This, of course, does not exclude the possibility that the axiom can be a formula proved on the basis of other axioms.

§ 5. Hilbert's second axiom of negation expresses the principle of excluded middle. The principle is expressed here in the form in which it is used for derivations: if B follows from A as well as from $\overline{A}$, then B is true. Its usual form, "Every judgment is either true or false",[9] is equivalent to that given above.[10]

Clearly, from the first interpretation of negation, that is, the interdiction from regarding the judgment as true, it is impossible to obtain the certitude that the principle of excluded middle is true; incidentally, no such attempts have been made. Consequently, to justify the principle we must turn to the structure of the judgment, the relation of predicate to subject. Even in the very simple case of a judgment of the type "All A are B" the relations of all possible A, the supply of which can be infinite, to the predicate B inevitably enter into consideration. Brouwer showed[11] that in the case of such transfinite judgments the principle of excluded middle cannot, precisely for this reason, be considered obvious.

§ 6. Thus, from the intuitionistic point of view neither of Hilbert's two axioms of negation can be taken as an axiom of the general logic of judgments. We offer here the following axiom, which we shall call the principle of contradiction:

(3) $$5.\ (A \to B) \to \{(A \to \overline{B}) \to \overline{A}\}.$$

[9] This is Leibniz's very simple formulation (see *Nouveaux essais*, IV, 2). The formulation "A is either B or not-B" has nothing to do with the logic of judgments.

[10] Symbolically the second form is expressed thus:

$$A \vee \overline{A},$$

where $\vee$ means "or". The equivalence of the two forms is easily proved on the basis of the axioms of implication and the following axioms, which determine the meaning of the symbol $\vee$ and are taken from *Ackermann 1924*:

1. $A \to A \vee B$,
2. $B \to A \vee B$,
3. $(A \to C) \to (B \to C) \to (A \vee B) \to C$.

[[In these formulas $\vee$ binds more strongly than $\to$, and for $\to$ there is association on the right.]]

[11] See *Brouwer 1923d* or the example of a proposition unprovable except by an illegitimate use of the principle of excluded middle, discussed in detail in *Brouwer 1920*.

Its meaning is: If both the truth and the falsity of a certain judgment B follow from A, the judgment A itself is false.

The usual principle of contradiction, "A judgment cannot be true and false', cannot be formulated in terms of an arbitrary judgment, implication, and negation. Our principle contains something else; namely, from it, together with the first axiom of implication, there follows the principle of *reductio ad absurdum*: If B is true and if the falsity of B follows from A, then A is false.

The truth of the proposed axiom follows from the simplest interpretation of negation, the interdiction from regarding a judgment as true, and does not depend upon whether the content of the judgments is considered.

The system of five axioms, the four axioms of implication (1) and the axiom of negation (3) just adopted, I shall call the system $\mathfrak{B}$. We do not know any formula of the general logic of judgments that possesses intuitive obviousness when applied to arbitrary judgments but is not provable on the basis of this system of axioms. Nevertheless, the question whether this axiom system is a complete axiom system for the intuitionistic general logic of judgments remains open.

§ 7. Although, as we have seen, the principle of excluded middle cannot be regarded as an axiom of the general logic of judgments, it has validity in the limited domain of the judgments that Brouwer calls finitary judgments. We shall not investigate here what the boundaries of the domain of finitary judgments are; this task is not as easy as it may seem. We therefore limit ourselves to the recognition of the fact that such a domain exists.

Besides the principle of excluded middle, the principle of double negation, which is expressed symbolically by

(4) $$6.\ \bar{\bar{A}} \to A,$$

has validity in the domain of the finitary.[12]

It is self-evident that all five axioms of the general logic of judgments (the system $\mathfrak{B}$) are valid in the domain of the finitary too. The system of axioms that consists of the axioms of the system $\mathfrak{B}$—that is, (1) and (3)—and the axiom of double negation (4) we shall call the system $\mathfrak{H}$.

The system $\mathfrak{H}$ is equivalent to the system consisting of Hilbert's axioms (1) and (2). The axioms of implication are common to both. For the proof it is therefore sufficient to prove formulas (3) and (4) on the basis of the formulas (2), and conversely, the axioms of implication being used in both cases. We shall not carry out the proof of formulas (3) and (4) on the basis of Hilbert's axioms (1) and (2), but the converse proof, which rests upon axioms (3) and (4), introduced here for the first time, is carried out in the next section.

For us the system $\mathfrak{H}$ has the advantage of being obtained from the system $\mathfrak{B}$ of the general logic of judgments by the addition solely of the axiom of double negation; this considerably facilitates further investigation.

It is clear that the system $\mathfrak{H}$, just like Hilbert's system, is complete. In it we can derive all the formulas of the traditional logic of judgments. They are all true, if only we replace in them the symbols for arbitrary judgments, $A, B, C, \ldots$, by symbols for

[12] The formula $A \to \bar{\bar{A}}$ is provable on the basis of the system $\mathfrak{B}$. See formula (34) below.

arbitrary finitary judgments, $A^f, B^f, C^f, \ldots$. The proof of this fact meets with some difficulties, which are explained and overcome in the next chapter.[13]

§ 8. We shall designate the axioms of the system $\mathfrak{H}$, (1), (3), and (4), by the numbers 1–6. The numbers of formulas that rest upon Axiom 6 are doubly underlined to indicate that these formulas have validity only in the domain of the finitary, while the others are valid for arbitrary judgments.

(5) $$\frac{A \to (B \to A)}{\bar{B} \to (A \to \bar{B})} \qquad \text{Axiom 1}$$

(6) $$\frac{\{A \to (B \to C)\} \to \{B \to (A \to C)\}}{[(B \to C) \to \{(A \to B) \to (A \to C)\}] \to [(A \to B) \to \{(B \to C) \to (A \to C)\}]} \qquad \text{Axiom 3}$$

$$[(B \to C) \to \{(A \to B) \to (A \to C)\}] \to [(A \to B) \to \{(B \to C) \to (A \to C)\}] \qquad (6)$$

(7) $$\frac{(B \to C) \to \{(A \to B) \to (A \to C)\}}{(A \to B) \to \{(B \to C) \to (A \to C)\}} \qquad \text{Axiom 4}$$

(8) $$\frac{(A \to B) \to \{(B \to C) \to (A \to C)\}}{\{\bar{B} \to (A \to \bar{B})\} \to [\{(A \to \bar{B}) \to \bar{A}\} \to (\bar{B} \to \bar{A})]} \qquad (7)$$

$$\{\bar{B} \to (A \to \bar{B})\} \to [\{(A \to \bar{B}) \to \bar{A}\} \to (\bar{B} \to \bar{A})] \qquad (8)$$

(9) $$\frac{B \to (A \to \bar{B})}{\{(A \to \bar{B}) \to \bar{A}\} \to (\bar{B} \to \bar{A})} \qquad (5)$$

$$(A \to B) \to \{(A \to \bar{B}) \to \bar{A}) \qquad \text{Axiom 5}$$

(10) $$\frac{\{(A \to \bar{B}) \to \bar{A}\} \to (\bar{B} \to \bar{A})}{(A \to B) \to (\bar{B} \to \bar{A})} \qquad (9)$$

(11) $$\frac{(A \to B) \to (\bar{B} \to \bar{A})}{(B \to A) \to (\bar{A} \to \bar{B})} \qquad (10)$$

$$A \to (B \to A) \qquad \text{Axiom 1}$$

(12) $$\frac{(B \to A) \to (\bar{A} \to \bar{B})}{A \to (\bar{A} \to \bar{B})} \qquad (11)$$

(13) $$\frac{A \to (\bar{A} \to \bar{B})}{A \to (\bar{A} \to \bar{\bar{B}})} \qquad (12)$$

(14) $$\frac{\bar{\bar{A}} \to A}{\bar{\bar{B}} \to B} \qquad \text{Axiom 6}$$

(15) $$\frac{(B \to C) \to \{(A \to B) \to (A \to C)\}}{(\bar{\bar{B}} \to B) \to \{(A \to \bar{\bar{B}}) \to (A \to B)\}} \qquad \text{Axiom 4}$$

$$(\bar{\bar{B}} \to B) \to \{(A \to \bar{\bar{B}}) \to (A \to B)\} \qquad (15)$$

[13] See Chapter III, § 4.

$$(16)\qquad \frac{\overline{\overline{B}} \to B}{(A \to \overline{\overline{B}}) \to (A \to B)} \qquad (14)$$

$$(17)\qquad \frac{(A \to \overline{\overline{B}}) \to (A \to B)}{(\overline{A} \to \overline{\overline{B}}) \to (\overline{A} \to B)} \qquad (16)$$

$$A \to (\overline{A} \to \overline{\overline{B}}) \qquad (13)$$

$$(18)\qquad \frac{(\overline{A} \to \overline{\overline{B}}) \to (\overline{A} \to B)}{A \to (\overline{A} \to B).} \qquad (16)$$

Thus Hilbert's first axiom of negation is proved.

$$(19)\qquad \frac{(A \to B) \to (\overline{B} \to \overline{A})}{(\overline{A} \to B) \to (\overline{B} \to \overline{\overline{A}})} \qquad (10)$$

$$(20)\qquad \frac{(A \to B) \to \{(A \to \overline{B}) \to \overline{A}\}}{(\overline{B} \to \overline{A}) \to \{(\overline{B} \to \overline{\overline{A}}) \to \overline{\overline{B}}\}} \qquad \text{Axiom 5}$$

$$(A \to B) \to (\overline{B} \to \overline{A}) \qquad (10)$$

$$(21)\qquad \frac{(\overline{B} \to \overline{A}) \to \{(\overline{B} \to \overline{\overline{A}}) \to \overline{\overline{B}}\}}{(A \to B) \to \{(\overline{B} \to \overline{\overline{A}}) \to \overline{\overline{B}}\}} \qquad (20)$$

$$(22)\qquad \frac{(A \to B) \to \{(B \to C) \to (A \to C)\}}{\{(\overline{A} \to B) \to (\overline{B} \to \overline{\overline{A}})\} \to [\{(\overline{B} \to \overline{\overline{A}}) \to \overline{\overline{B}}\} \to \{(\overline{A} \to B) \to \overline{\overline{B}}\}]} \qquad (7)$$

$$\{(\overline{A} \to B) \to (\overline{B} \to \overline{\overline{A}})\} \to [\{(\overline{B} \to \overline{\overline{A}}) \to \overline{\overline{B}}\} \to \{(\overline{A} \to B) \to \overline{\overline{B}}\}] \qquad (22)$$

$$(23)\qquad \frac{(\overline{A} \to B) \to (\overline{B} \to \overline{\overline{A}})}{\{(\overline{B} \to \overline{\overline{A}}) \to \overline{\overline{B}}\} \to \{(\overline{A} \to B) \to \overline{\overline{B}}\}} \qquad (19)$$

$$(A \to B) \to \{(\overline{B} \to \overline{\overline{A}}) \to \overline{\overline{B}}\} \qquad (21)$$

$$(24)\qquad \frac{\{(\overline{B} \to \overline{\overline{A}}) \to \overline{\overline{B}}\} \to \{(\overline{A} \to B) \to \overline{\overline{B}}\}}{(A \to B) \to \{(\overline{A} \to B) \to \overline{\overline{B}}\}} \qquad (23)$$

$$(25)\qquad \frac{(A \to \overline{\overline{B}}) \to (A \to B)}{\{(\overline{A} \to B) \to \overline{\overline{B}}\} \to \{(\overline{A} \to B) \to B\}} \qquad (16)$$

$$(A \to B) \to \{(\overline{A} \to B) \to \overline{\overline{B}}\} \qquad (24)$$

$$(26)\qquad \frac{\{(\overline{A} \to B) \to \overline{\overline{B}}\} \to \{(\overline{A} \to B) \to B\}.}{(A \to B) \to \{(\overline{A} \to B) \to B\}} \qquad (25)$$

Thus Hilbert's second axiom, too, is proved.

Among the formulas proved with the help of axioms $\mathfrak{B}$, without the axiom of double negation, formulas (12) and (24) are those that are closest to Hilbert's axioms of negation. Formula (24), which comes close to the principle of excluded middle, means: if B follows from the truth of A and from its falsity as well, then it cannot be false. Indeed, if we assume that B is false, A cannot be true, since B would follow from A; but from the falsity of A the truth of B would follow.

III. The special logic of judgments and its domain of applicability

§ 1. The formulas provable on the basis of axioms $\mathfrak{B}$ constitute the general logic of judgments. We shall call the totality of formulas provable on the basis of the six axioms $\mathfrak{H}$ the special logic of judgments.[14] The content of the special logic of judgments is richer than that of the general logic, but its domain of applicability is narrower. Everything below is devoted to the explanation of what the domain of applicability of the special logic of judgments is. This domain is perhaps even somewhat narrower than the domain in which the principle of excluded middle in the Hilbert form is applicable.

§ 2. Let us introduce symbols, $A^{\cdot}$, $B^{\cdot}$, $C^{\cdot}$, ..., to denote arbitrary judgments for which the judgment itself follows from its double negation. The finitary judgments are of that kind. All true judgments are also of that kind; this, however, will not have any application in what follows. Brouwer proved (*1923d*) that all negative judgments are of that kind. The proof, given below, rests only upon the axioms of the system $\mathfrak{B}$.

On the basis of the axioms of implication it is easy to prove the formula

$$(27)\qquad A \to A.$$

$$(28)\qquad \frac{(A \to B) \to \{(A \to \overline{B}) \to \overline{A}\}}{(\overline{A} \to A) \to \{(\overline{A} \to \overline{A}) \to \overline{\overline{A}}\}} \qquad \text{Axiom 5}$$

$$(29)\qquad \frac{A \to (B \to A)}{A \to (\overline{A} \to A)} \qquad \text{Axiom 1}$$

$$A \to (\overline{A} \to A) \qquad (29)$$

$$(30)\qquad \frac{(\overline{A} \to A) \to \{(\overline{A} \to \overline{A}) \to \overline{\overline{A}}\}}{A \to \{(\overline{A} \to \overline{A}) \to \overline{\overline{A}}\}} \qquad (28)$$

$$(31)\qquad \frac{\{A \to (B \to C)\} \to \{B \to (A \to C)\}}{[A \to \{(\overline{A} \to \overline{A}) \to \overline{\overline{A}}\}] \to \{(\overline{A} \to \overline{A}) \to (A \to \overline{\overline{A}})\}} \qquad \text{Axiom 3}$$

$$[A \to \{(\overline{A} \to \overline{A}) \to \overline{\overline{A}}\}] \to \{(\overline{A} \to \overline{A}) \to (A \to \overline{\overline{A}})\} \qquad (31)$$

$$(32)\qquad \frac{A \to \{(\overline{A} \to \overline{A}) \to \overline{\overline{A}}\}}{(\overline{A} \to \overline{A}) \to (A \to \overline{\overline{A}})} \qquad (30)$$

$$(33)\qquad \frac{A \to A}{\overline{A} \to \overline{A}} \qquad (27)$$

$$(\overline{A} \to \overline{A}) \to (A \to \overline{\overline{A}}) \qquad (32)$$

[14] The general logic of judgments also has another, real interpretation (see Chapter I, § 5). The special logic of judgments can for the present be defined merely formally, since the real meaning of its formulas will be established only in what follows.

$$(34)\qquad \frac{\overline{A}\to\overline{A}}{A\to\overline{\overline{A}}} \qquad (33)$$

$$(35)\qquad \frac{(A\to B)\to(\overline{B}\to\overline{A})}{(A\to\overline{\overline{A}})\to(\overline{\overline{\overline{A}}}\to\overline{A})} \qquad (10)$$

$$(A\to\overline{\overline{A}})\to(\overline{\overline{\overline{A}}}\to\overline{A}) \qquad (35)$$

$$(36)\qquad \frac{A\to\overline{\overline{A}}}{\overline{\overline{\overline{A}}}\to\overline{A}}. \qquad (34)$$

The last formula shows that all negative judgments are judgments of type $A^{\cdot}$.

The axiom system $\mathfrak{H}$ differs from the system $\mathfrak{B}$, which is universally applicable, only by the axiom of double negation. For judgments of type $A^{\cdot}$ this axiom is expressed by the following formula:

$$(37)\qquad \overline{\overline{A^{\cdot}}}\to A^{\cdot}$$

We consider only this formula to be true; we consider formula (4) to be unfounded.

But it does not yet follow from what has been said that all formulas of the special logic of judgments are true for judgments of type $A^{\cdot}$; in fact, in the derivation of these formulas the axiom of double negation (4) is applied not only to elementary judgments—which for our case, that of judgments of type $A^{\cdot}$, is justified by formula (37)—but also to complex formulas; whether a formula of type $A^{\cdot}\to B^{\cdot}$, for example, is a formula of type $A^{\cdot}$ is, however, not yet apparent.

§ 3. We shall now prove that every formula expressed by means of the symbols $A^{\cdot}, B^{\cdot}, C^{\cdot}, \ldots$, and the symbols of implication and negation is a formula of type $A^{\cdot}$. For that it suffices to consider two very simple cases.

First, every negative judgment is a judgment of type $A^{\cdot}$ by virtue of Brouwer's formula (36).

Second, we now prove that a judgment of type $A^{\cdot}\to B^{\cdot}$ is also a judgment of type $A^{\cdot}$.

$$(38)\qquad \frac{A\to A}{(A\to B)\to(A\to B)} \qquad (27)$$

$$(39)\qquad \frac{\{A\to(B\to C)\}\to\{B\to(A\to C)\}}{\{(A\to B)\to(A\to B)\}\to[A\to\{(A\to B)\to B\}]} \qquad \text{Axiom 3}$$

$$\{(A\to B)\to(A\to B)\}\to[A\to\{(A\to B)\to B\}] \qquad (39)$$

$$(40)\qquad \frac{(A\to B)\to(A\to B)}{A\to\{(A\to B)\to B\}} \qquad (38)$$

$$(41)\qquad \frac{(A\to B)\to(\overline{B}\to\overline{A})}{(\overline{B}\to\overline{A})\to(\overline{\overline{A}}\to\overline{\overline{B}})} \qquad (10)$$

$$(A\to B)\to(\overline{B}\to\overline{A}) \qquad (10)$$

$$(42)\qquad \frac{(\overline{B}\to\overline{A})\to(\overline{\overline{A}}\to\overline{\overline{B}})}{(A\to B)\to(\overline{\overline{A}}\to\overline{\overline{B}})} \qquad (41)$$

$$(43)\qquad \frac{(A\to B)\to(\overline{\overline{A}}\to\overline{\overline{B}})}{\{(A\to B)\to B\}\to\{\overline{\overline{(A\to B)}}\to\overline{\overline{B}}\}} \qquad (42)$$

$$A\to\{(A\to B)\to B\} \qquad (40)$$

$$(44)\qquad \frac{\{(A\to B)\to B\}\to\{\overline{\overline{(A\to B)}}\to\overline{\overline{B}}\}}{A\to\{\overline{\overline{(A\to B)}}\to\overline{\overline{B}}\}} \qquad (43)$$

$$(45)\qquad \frac{\{A\to(B\to C)\}\to\{B\to(A\to C)\}}{[A\to\{\overline{\overline{(A\to B)}}\to\overline{\overline{B}}\}]\to\{\overline{\overline{(A\to B)}}\to(A\to\overline{\overline{B}})\}} \qquad \text{Axiom 3}$$

$$[A\to\{\overline{\overline{(A\to B)}}\to\overline{\overline{B}}\}]\to\{\overline{\overline{(A\to B)}}\to(A\to\overline{\overline{B}})\} \qquad (45)$$

$$(46)\qquad \frac{A\to\{\overline{\overline{(A\to B)}}\to\overline{\overline{B}}\}}{\overline{\overline{(A\to B)}}\to(A\to\overline{\overline{B}})}. \qquad (44)$$

This formula is true for arbitrary judgments A and B. Replacing A by $A^{\cdot}$ and B by $B^{\cdot}$ and making use of formula (37), we easily obtain the formula

$$(47)\qquad \overline{\overline{(A^{\cdot}\to B^{\cdot})}}\to(A^{\cdot}\to B^{\cdot}),$$

which shows precisely that judgments of type $A^{\cdot}\to B^{\cdot}$ are of type $A^{\cdot}$.

By gradually passing to more complex formulas, we can prove the assertion made at the beginning of the present section.

§ 4. We can now assert that all formulas of the special logic of judgments are true for judgments of type $A^{\cdot}$, including all finitary and all negative judgments. In fact, the symbols $A^{\cdot}, B^{\cdot}, C^{\cdot}, \ldots, A^{\cdot}\to B^{\cdot}$, and $\overline{A^{\cdot}}$ allow all the operations that the symbols of the general logic of judgments do: substitution for the symbols $A^{\cdot}, B^{\cdot}, C^{\cdot}, \ldots$ of an arbitrary formula written by means of the symbols considered and inference according to the schema $\mathfrak{S}\to\mathfrak{T}$, $\mathfrak{T} \mid \mathfrak{T}$; moreover, all six axioms $\mathfrak{H}$ are true for them.

The precise boundary of the domain in which the special logic of judgments is applicable has thus been found; this domain coincides with the domain in which the formula of double negation (4) is applicable.

IV. The mathematics of pseudotruth

§ 1. In the preceding chapter we established that all the formulas of the traditional logic of judgments can actually be proved as formulas of the special logic of judgments. We must merely recognize that they deal only with judgments of type $A^{\cdot}$. Moreover, these formulas themselves turn out to be formulas of type $A^{\cdot}$.

The following question now arises: Can we in a similar way, after we have placed some restrictions on their real interpretation, again give a meaning to all those formulas of mathematics that are proved by an illegitimate use (that is, a use outside the domain in which they are applicable) of formulas of the special logic of judgments, in particular by use of the principle of excluded middle? It turns out that this task can be fulfilled.

§ 2. We shall construct, alongside of ordinary mathematics, a new mathematics, a "pseudomathematics" that will be such that to every formula of the first there corresponds a formula of the second and, moreover, that every formula of pseudomathematics is a formula of type $A^{\cdot}$. For the time being we are not concerned with the question of the truth of the formulas of pseudomathematics; we shall turn to it in § 5 of the present chapter.

A symbol, simple or complex, that expresses a judgment is called a formula. Formulas of which no part is a formula will be called elementary formulas, or formulas of the first order; $a = a$ is such a formula. A formula whose parts are formulas of the $(n\text{-}1)$th order at most will be called a formula of the nth order. For example, the formula

$$a = b \to \{A(a) \to B(a)\}$$

is a formula of the third order since its constituent part $A(a) \to B(a)$ is a formula of the second order.

To an elementary formula $\mathfrak{S}$ there corresponds in pseudomathematics the formula $\mathfrak{S}^*$, which expresses the double negation of $\mathfrak{S}$:

(48) $$\mathfrak{S}^* \equiv \overline{\overline{\mathfrak{S}}}.$$

In what follows we shall, for convenience, denote the double negation of $\mathfrak{S}$ by $n\mathfrak{S}$.

To the formula of the nth order $F(\mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_k)$, where $\mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_k$ are formulas of the $(n\text{-}1)$th order at most, there corresponds in pseudomathematics the formula $F(\mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_k)^*$ such that

(49) $$F(\mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_k)^* \equiv nF(\mathfrak{S}_1^*, \mathfrak{S}_2^*, \ldots, \mathfrak{S}_k^*),$$

$\mathfrak{S}_1^*, \mathfrak{S}_2^*, \ldots, \mathfrak{S}_k^*$ being regarded as already determined. For example, to the formula

$$a = b \to \{A(a) \to B(a)\}$$

there corresponds in pseudomathematics the formula

$$n[n(a = b) \to n\{nA(a) \to nB(a)\}].$$

To every symbol that is not a formula there also corresponds a definite symbol of pseudomathematics. To a symbol, simple or complex, of which no part is a formula there corresponds in pseudomathematics a symbol that is identical with it. To a complex symbol in which formulas enter there corresponds a symbol in which each formula $\mathfrak{S}$ is replaced by the formula $\mathfrak{S}^*$.

§ 3. All the formulas of mathematics are derived from axioms,[15] which we denote by $\mathfrak{U}_1, \mathfrak{U}_2, \ldots, \mathfrak{U}_k$, with the help of the operations of substitution of particular values for variables and of inference according to the schema $\mathfrak{S}, \mathfrak{S} \to \mathfrak{T} \mid \mathfrak{T}$. To the axioms there correspond in pseudomathematics the formulas $\mathfrak{U}_1^*, \mathfrak{U}_2^*, \ldots, \mathfrak{U}_k^*$. We shall prove that every formula of pseudomathematics that corresponds to a formula proved on the basis of the axioms $\mathfrak{U}$ is a consequence of the formulas $\mathfrak{U}^*$. For the proof it suffices to establish the following two facts.

First, if, when particular values are substituted for variables in a formula $\mathfrak{S}$, we obtain a formula $\mathfrak{T}$, then, when the corresponding formulas and symbols are substituted at the corresponding places in the formula $\mathfrak{S}^*$, we obtain the formula $\mathfrak{T}^*$.

[15] Here all the axioms of logic are included among the axioms of mathematics.

Second, in analogy with the schema $\mathfrak{S}, \mathfrak{S} \to \mathfrak{T} \mid \mathfrak{T}$, the schema

(50) $$\mathfrak{S}^*, (\mathfrak{S} \to \mathfrak{T})^* \mid \mathfrak{T}^*$$

is correct.

In fact,

(51) $$(\mathfrak{S} \to \mathfrak{T})^* \equiv \overline{\overline{(\mathfrak{S}^* \to \mathfrak{T}^*)}};$$

since $\mathfrak{S}^*$ and $\mathfrak{T}^*$ are formulas of type $A^{\cdot}$, we have, according to formula (47),

(52) $$\overline{\overline{(\mathfrak{S}^* \to \mathfrak{T}^*)}} \to (\mathfrak{S}^* \to \mathfrak{T}^*),$$

(53) $$\left\{\begin{array}{c} \mathfrak{S}^* \\ (\mathfrak{S} \to \mathfrak{T})^* \to (\mathfrak{S}^* \to \mathfrak{T}^*) \\ \hline \mathfrak{T}^*. \end{array}\right.$$

Thus, we see that to every correct proof in the domain of ordinary mathematics there corresponds a correct proof in the domain of pseudomathematics. From that follows the truth of the proposition advanced at the beginning of the present section.

§ 4. To the five axioms of the general logic of judgments there correspond in pseudomathematics the following formulas:

(54) $$\left\{\begin{array}{l} 1.\ n\{nA \to n(nB \to nA)\}, \\ 2.\ n[n\{nA \to n(nA \to nB)\} \to n(nA \to nB)], \\ 3.\ n[n\{nA \to n(nB \to nC)\} \to n\{nB \to n(nA \to nC)\}], \\ 4.\ n[n(nB \to nC) \to n\{n(nA \to nB) \to n(nA \to nC)\}], \\ 5.\ n[n(nA \to nB) \to n\{(nA \to n(\overline{nB})) \to n(\overline{nA})\}]. \end{array}\right.$$

These formulas can be obtained from

(55) $$\left\{\begin{array}{l} 1.\ n\{A^{\cdot} \to n(B^{\cdot} \to A^{\cdot})\}, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ 5.\ n[n(A^{\cdot} \to B^{\cdot}) \to n\{n(A^{\cdot} \to n\overline{B}^{\cdot}) \to n\overline{A}^{\cdot}\}] \end{array}\right.$$

by substitution of nA, nB, and nC for $A^{\cdot}$, $B^{\cdot}$, and $C^{\cdot}$, respectively. Since formulas (55) are formulas of the special logic of judgments, we have the right to prove them by using all the axioms of $\mathfrak{H}$, or all of Hilbert's axioms. Their proof does not present any difficulty. Thus, all the formulas (54) turn out to be true. It follows from this that all the formulas of pseudomathematics that correspond to the formulas of the general logic of judgments are true.

§ 5. All the axioms of mathematics that we know possess the same property as the axioms of the general logic of judgments, namely, that the formulas corresponding to them in the domain of pseudomathematics are true. For example, to the axiom

$$(a)A(a) \to A(a)$$

there corresponds the true formula

$$n\{n(a)nA(a) \to nA(a)\}.$$

We shall call the axioms that possess the property formulated above axioms of type $\mathfrak{K}$. Further, let us call the formulas provable on the basis of the axioms of type $\mathfrak{K}$ formulas of type $\mathfrak{K}$. All the axioms and formulas of mathematics that we know are of type $\mathfrak{K}$.[16]

By virtue of what was said above, the part of pseudomathematics whose formulas correspond to the formulas of type $\mathfrak{K}$ acquires a real meaning: all its formulas are true, since they are consequences of the true formulas that in pseudomathematics correspond to the axioms of type $\mathfrak{K}$. The name "pseudomathematics" becomes inappropriate for this part, the only one that for the time being exists; as a collection of true formulas, it is part of genuine mathematics.

We shall say that a judgment is pseudotrue if its double negation is true. A judgment of the form $n\mathfrak{S}$ thus asserts the pseudotruth of the judgment $\mathfrak{S}$. The formulas of pseudomathematics always express only judgments about pseudotruth. We have the right, therefore, to call the part of pseudomathematics that has real meaning the mathematics of pseudotruth.

§ 6. In the usual presentation of mathematics a number of conclusions are obtained by an illegitimate use of formulas of the special logic of judgments, for example, by use of the principle of excluded middle. All these cases, as has been shown, can be reduced to the use of the principle of double negation,

(4) $$6.\ \overline{\overline{A}} \to A.$$

Among these conclusions let us consider those that, except for the illegitimate formula (4), rest only upon axioms of type $\mathfrak{K}$. The formulas that express them we shall call formulas of type $\mathfrak{K}'$.

Let us construct the formulas of pseudomathematics that correspond to the formulas $\mathfrak{K}'$. They will all follow from the formulas $\mathfrak{A}^*$ that correspond to axioms of type $\mathfrak{K}$ and from the formula

(56) $$n\{n(\overline{\overline{nA}}) \to nA\},$$

which corresponds to formula (4).

Formula (56) is true. In fact, by virtue of formula (34), it follows from

(57) $$n(\overline{\overline{nA}}) \to nA.$$

Formula (57) can be obtained by means of substitution from a formula of the special logic of judgments,

(58) $$n(\overline{\overline{A^{\cdot}}}) \to A^{\cdot};$$

in the special logic of judgments, as we know, an even number of negations reduces to affirmation.

Thus, while the formulas of ordinary mathematics that we have considered are based on an illegitimate use of formula (4), the corresponding formulas of pseudomathematics rest upon the true formula (56).

So we finally obtain the following result: None of the conclusions of ordinary mathematics that are based on the use, outside the domain of the finitary, of the formula of double negation and of other formulas that depend upon it (like the principle of excluded middle) can be regarded as firmly established. But, for those conclusions whose derivations require, besides this formula [that of double negation], only axioms of type $\mathfrak{K}$ (and no other axioms are known at present), the corresponding formulas of pseudomathematics are true and consequently become part of the mathematics of pseudotruth.

In other words, all the conclusions that rest upon axioms of type $\mathfrak{K}$ and on the formula of double negation are correct if we understand every judgment that enters into them in the sense of the affirmation of its pseudotruth, that is, of its double negation.

[16] In mathematics formulas whose truth is not obvious, for example, what is called Zermelo's axiom, are sometimes taken as axioms. But they, too, possess the property that $\mathfrak{A} \to \mathfrak{A}^*$.

V. Addenda

§ 1. After having shown that it is illegitimate to apply the principle of excluded middle to transfinite judgments, Brouwer set himself the task of providing a foundation for mathematics without the help of that principle, and to a considerable extent he carried out this task.[17] But it then turned out that there exist a number of mathematical propositions that cannot be proved without the help of the principle of excluded middle, rejected by Brouwer. We consider below a few examples of such propositions.

We proved in the preceding chapters that, alongside of the development of mathematics without the help of the principle of excluded middle, we can also retain the usual development. To be sure, a limited interpretation then has to be given to all propositions; namely, every judgment of ordinary mathematics has to be replaced by the affirmation of its pseudotruth. But this development nevertheless retains two remarkable properties.

1. If a finitary conclusion is obtained with the help of arguments based on the use, even in the domain of the transfinite, of the principle of excluded middle, the conclusion is true in the usual sense. In view of what precedes, it can in fact be proved as a conclusion about pseudotruth; but, in the domain of the finitary, pseudotruth coincides with ordinary truth.

2. The use of the principle of excluded middle never leads to a contradiction. In fact, if a false formula were obtained with its help, then the corresponding formula of pseudomathematics would be proved without its help and would also lead to a contradiction.[18]

The first of these assertions contradicts a remark by Brouwer (*1923d*, p. 252, footnote), who thinks that finitary conclusions based on a transfinite use of the principle of excluded middle must also be considered unreliable.

§ 2. Generally, the propositions that we do not know how to prove without an illegitimate use of the principle of excluded middle ordinarily rest directly, not upon the principle of excluded middle of the logic of judgments, but upon another principle that bears the same name. In fact, from the principle of excluded middle in the form

[17] See, for example, *Brouwer 1918* and *1919*.

[18] We assume that all the axioms considered are axioms of type $\mathfrak{K}$ and, moreover, that any formula wittingly taken as false, $\mathfrak{S}$, is such that the corresponding $\mathfrak{S}^*$ is also false.

peculiar to the logic of judgments, namely, "Every judgment is either true or false", we can obtain further conclusions by following the schema that corresponds to Hilbert's formula: If B follows from A as well as from $\bar{A}$, then B is true. But in the case that interests us, that of transfinite judgments, it is difficult to obtain any positive conclusion B from the pure negation $\bar{A}$; for that we must first transform the judgment $\bar{A}$ into some other.

The following type of transfinite judgment is the most customary: for all a, $A(a)$ is true; symbolically, this judgment is written $(a)A(a)$. When we want to derive a positive conclusion from the negation $\overline{(a)}A(a)$ of this judgment, we put it in the form $(Ea)\bar{A}(a)$, that is, there exists an a for which $A(a)$ is not true. The equivalence of the last assertion to the simple negation of the judgment $(a)A(a)$ is expressed by the following two formulas:

(59) $$\overline{(a)}A(a) \to (Ea)\bar{A}(a),$$

(60) $$(Ea)\bar{A}(a) \to \overline{(a)}A(a).$$

In § 4 of the present chapter we shall demonstrate that formula (59) requires for its proof only the acceptance of the principle of double negation (4) in addition to formulas whose intuitive obviousness is unquestionable, among which the axioms of the general logic of judgments are included. But formula (60) can be proved without the help of that principle.

If we adopt formulas (59) and (60), we shall have the right to formulate the principle of excluded middle for judgments of the form $(a)A(a)$ as follows:

$$(a)A(a) \vee (Ea)\bar{A}(a),$$

that is, either $A(a)$ is true for all a or there exists an a for which $A(a)$ is not true.

To formulas (59) and (60) Hilbert adds (*1922a*, p. 157) the following:

(61) $$\overline{(Ea)}A(a) \to (a)\bar{A}(a),$$

(62) $$(a)\bar{A}(a) \to \overline{(Ea)}A(a).$$

In accordance with what was said above, he deems that formulas (59)–(62) justify the application of the principle of excluded middle to infinite collections of objects a. Unlike formulas (59) and (60), formulas (61) and (62) can both be proved on the basis only of axioms that are intuitively obvious. The proof is given in § 4.

Thus, formulas (60)–(62) simply are true formulas, while formula (59) is proved on the basis of the principle of double negation (4); everything presented in the preceding chapter, therefore, applies to the conclusions that rest upon that principle.

§ 3. Let us remark first of all that, having determined the meaning of the symbol $(a)A(a)$ as "for all a, $A(a)$ is true", we understand by "for all a" the same thing as by "for each a", that is, the meaning is that, whatever a may be given, we can assert that $A(a)$ is true.

Every formula of the general logic of judgments, when written by itself, means that it is true for all possible judgments, $A, B, C, \ldots$. Thus, the formula $A \to \bar{\bar{A}}$ means that for any judgment the double negation of the judgment follows from its truth.

Accordingly, it is impossible to assert that the symbolic expression $(a)A(a)$, when introduced, is the first to lead us out of the domain of the finitary; the notion "for all a" is contained in hidden form in all formulas in which there are symbols for variables.

In general, the formula $A(a)$ written by itself means that, whatever the particular meaning of a may be, $A(a)$ is true. From this the following principle, which cannot be expressed symbolically, results: whenever a formula $\mathfrak{S}$ stands by itself [[in a derivation]], we can write the formula $(a)\mathfrak{S}$. When we have to refer to this principle, we denote it by **P**.[19]

Furthermore, we adopt the following axioms,

(63) $$\begin{cases} \text{I. } (a)\{A(a) \to B(a)\} \to \{(a)A(a) \to (a)B(a)\}, \\ \text{II. } (a)\{A \to B(a)\} \to \{A \to (a)B(a)\}, \\ \text{III. } (a)\{A(a) \to C\} \to \{ Ea)A(a) \to C\}, \\ \text{IV. } A(a) \to (Ea)A(a). \end{cases}$$

We believe that all these axioms are intuitively obvious. The choice of these axioms and their number are exclusively determined by our goal, which is to prove formulas (59)–(62).

§ 4.

$$(A \to B) \to \to (\bar{B} \to \bar{A}) \qquad (11)$$

(64) $$\frac{A(a) \to (Ea)A(a)}{\overline{(Ea)}A(a) \to \bar{A}(a)} \qquad \text{Axiom IV}$$

(65) $$(a)\{\overline{(Ea)}A(a) \to \bar{A}(a)\} \qquad (64)\ \mathbf{P}$$

$$(a)\{A \to B(a)\} \to \to \{A \to (a)B(a)\} \qquad \text{Axiom II}$$

(66) $$\frac{(a)\{\overline{(Ea)}A(a) \to \bar{A}(a)\}}{\overline{(Ea)}A(a) \to (a)\bar{A}(a).} \qquad (65)$$

Thus formula (61) is proved.

$$\{A \to (B \to C)\} \to \to \{B \to (A \to C)\} \qquad \text{Axiom 3}$$

(67) $$\frac{A \to (\bar{A} \to \bar{B})}{\bar{A} \to (A \to \bar{B})} \qquad (12)$$

(68) $$\frac{\bar{\bar{A}} \to (A \to \bar{B})}{\bar{A}(a) \to \{A(a) \to \overline{(Ea)}A(a)\}} \qquad (67)$$

(69) $$(a)[\bar{A}(a) \to \{A(a) \to \overline{(Ea)}A(a)\}] \qquad (68)\ \mathbf{P}$$

[19] The symbol (a) can also occur in front of a formula that does not actually contain the variable a. Instead of the principle **P** we can introduce the axiom $(a)V$, where V denotes a true judgment, and the following rule of substitution, not formulated symbolically: any formula that stands by itself can be substituted for V.

$$(a)\{A(a) \to B(a)\} \to \to \{(a)A(a) \to (a)B(a)\} \qquad \text{Axiom I}$$

(70) $$\frac{(a)[\overline{A}(a) \to \{A(a) \to \overline{(Ea)}A(a)\}]}{(a)\overline{A}(a) \to (a)\{A(a) \to \overline{(Ea)}A(a)\}} \qquad (69)$$

(71) $$\frac{(a)\{A(a) \to C\} \to \{(Ea)A(a) \to C\}}{(a)\{A(a) \to \overline{(Ea)}A(a)\} \to \{(Ea)A(a) \to \overline{(Ea)}A(a)\}} \qquad \text{Axiom III}$$

$$(A \to B) \to \to \{(A \to \overline{B}) \to \overline{A}\} \qquad \text{Axiom 5}$$

(72) $$\frac{A \to A}{(A \to \overline{A}) \to \overline{A}} \qquad (27)$$

(73) $$\frac{(A \to \overline{A}) \to \overline{A}}{\{(Ea)A(a) \to \overline{(Ea)}A(a)\} \to \overline{(Ea)}A(a)} \qquad (72)$$

$$(a)\overline{A}(a) \to (a)\{A(a) \to \overline{(Ea)}A(a) \qquad (70)$$

$$(a)\{A(a) \to \overline{(Ea)}A(a)\} \to \{(Ea)A(a) \to \overline{(Ea)}A(a)\} \qquad (71)$$

(74) $$\frac{\{(Ea)A(a) \to \overline{(Ea)}A(a)\} \to \overline{(Ea)}A(a)}{(a)\overline{A}(a) \to \overline{(Ea)}A(a).} \qquad (73)$$

Thus formula (62) is proved.

(75) $$\frac{(a)\overline{A}(a) \to \overline{(Ea)}A(a)}{(a)\overline{\overline{A}}(a) \to \overline{(Ea)}\overline{A}(a)} \qquad (74)$$

(76) $$\frac{A \to \overline{\overline{A}}}{A(a) \to \overline{\overline{A}}(a)} \qquad (34)$$

(77) $$(a)\{A(a) \to \overline{\overline{A}}(a)\} \qquad (76)\ \mathbf{P}$$

$$(a)\{A(a) \to B(a)\} \to \to \{(a)A(a) \to (a)B(a)\} \qquad \text{Axiom I}$$

(78) $$\frac{(a)\{A(a) \to \overline{\overline{A}}(a)\}}{(a)A(a) \to (a)\overline{\overline{A}}(a)} \qquad (77)$$

$$(a)A(a) \to (a)\overline{\overline{A}}(a) \qquad (78)$$

(79) $$\frac{(a)\overline{\overline{A}}(a) \to \overline{(Ea)}\overline{A}(a)}{(a)A(a) \to \overline{(Ea)}\overline{A}(a)} \qquad (75)$$

$$(A \to B) \to \to (\overline{B} \to \overline{A}) \qquad (11)$$

(80) $$\frac{(a)A(a) \to \overline{(Ea)}\overline{A}(a)}{\overline{\overline{(Ea)}}\overline{A}(a) \to \overline{(a)}A(a)} \qquad (79)$$

(81) $$\frac{A \to \overline{\overline{A}}}{(Ea)\overline{A}(a) \to \overline{\overline{(Ea)}}\overline{A}(a)} \qquad (34)$$

$$(Ea)\overline{A}(a) \to \overline{\overline{(Ea)}}\overline{A}(a) \qquad (81)$$

(82) $$\frac{\overline{\overline{(Ea)}}\overline{A}(a) \to \overline{(a)}A(a)}{(Ea)\overline{A}(a) \to \overline{(a)}A(a).} \qquad (80)$$

Thus formula (60) is proved.

The proof of formula (59) cannot be carried out without the help of the axiom of double negation. The numbers of the formulas that rest upon this axiom are doubly underlined.

(83) $$\frac{\overline{(Ea)}A(a) \to (a)\overline{A}(a)}{\overline{(Ea)}\overline{A}(a) \to (a)\overline{\overline{A}}(a)} \qquad (66)$$

$\underline{\underline{(84)}}$ $$\frac{\overline{\overline{A}} \to A}{\overline{\overline{A}}(a) \to A(a)} \qquad \text{Axiom 6}$$

$\underline{\underline{(85)}}$ $$(a)\{\overline{\overline{A}}(a) \to (a)\} \qquad (84)\ \mathbf{P}$$

$$(a)\{A(a) \to B(a)\} \to \to \{(a)A(a) \to (a)B(a)\} \qquad \text{Axiom I}$$

$\underline{\underline{(86)}}$ $$\frac{(a)\{\overline{\overline{A}}(a) \to A(a)\}}{(a)\overline{\overline{A}}(a) \to (a)A(a)} \qquad \underline{\underline{(85)}}$$

$$\overline{(Ea)}\overline{A}(a) \to (a)\overline{\overline{A}}(a) \qquad (83)$$

$\underline{\underline{(87)}}$ $$\frac{(a)\overline{\overline{A}}(a) \to (a)A(a)}{\overline{(Ea)}\overline{A}(a) \to (a)A(a)} \qquad \underline{\underline{(86)}}$$

$$(A \to B) \to \to (\overline{B} \to \overline{A}) \qquad (11)$$

$\underline{\underline{(88)}}$ $$\frac{\overline{(Ea)}\overline{A}(a) \to (a)A(a)}{\overline{(a)}A(a) \to \overline{\overline{(Ea)}}\overline{A}(a)} \qquad \underline{\underline{(87)}}$$

$\underline{\underline{(89)}}$ $$\frac{\overline{\overline{A}} \to A}{\overline{\overline{(Ea)}}\overline{A}(a) \to (Ea)\overline{A}(a)} \qquad \underline{\underline{\text{Axiom 6}}}$$

$$\overline{(a)}A(a) \to \overline{\overline{(Ea)}}\overline{A}(a) \qquad \underline{\underline{(88)}}$$

(90) $$\frac{\overline{\overline{(Ea)}}\overline{A}(a) \to (Ea)\overline{A}(a)}{\overline{(a)}A(a) \to (Ea)\overline{A}(a).} \qquad \underline{\underline{(89)}}$$

Thus formula (59) is proved by means of the axiom of double negation.

§ 5. An excellent example of a proposition unprovable without the help of an illegitimate use of the principle of excluded middle is given by Brouwer (*1920*); he shows that it cannot be considered proved that every real number has an infinite decimal expansion. He even exhibits a definite number for which it is not known whether it has a first digit in its decimal expansion.

Another example is the proposition stating that the complement of a closed set is a region, that is, that every point not belonging to a given closed set is contained in some interval that does not contain any point of the set.[20] The proof, as is well known, is carried out in the following way: by the principle of excluded middle in the form given in § 2 of the present chapter, either all intervals containing the point considered contain points of the set or there exists at least one interval that does not contain any such point; the first assumption leads to a contradiction, since it implies that the point belongs to the set, and therefore the second proposition is true. This example differs from Brouwer's in that here we do not know how to exhibit a definite closed set and a definite point exterior to it for which the existence of the required interval is doubtful.

§ 6. The following example is interesting: without the help of the principle of excluded middle it is impossible to prove any proposition whose proof usually comes down to an application of the principle of transfinite induction. For example, a proposition of that kind is: every closed set is the sum of a perfect set and a denumerable set.

The proof of such propositions is often carried out without the help of the principle of transfinite induction. But all these proofs rest upon the principle of excluded middle, applied to infinite collections, or upon the principle of double negation.

It is important to observe that the principle of transfinite induction itself can be derived without any assumption that, from the point of view of the theory of point sets, is new, but that the principle of excluded middle is necessarily used. It suffices merely to formulate the principle of transfinite induction without the use of the term "transfinite number", whose introduction would demand new axioms. Let us consider, instead, sets of rational numbers that are completely ordered from left to right. For such a set, a part that starts on the left from some point, which may or may not belong to the set, will be called a segment of the set. A segment, too, will always be a completely ordered set. The set of segments will itself be also completely ordered. Let us say that a segment is proper if there exists a point of the set that does not belong to it. The principle of transfinite induction can now be formulated in the following way.

Let a certain property J, which may or may not obtain of a completely ordered set of rational numbers, satisfy the following conditions:

(1) Sets consisting of one point alone have property J;

(2) If all the proper segments of a set have property J, then the set itself has it.

Under these conditions all completely ordered sets of rational numbers possess property J.

This formulation of the principle of transfinite induction can be used in the same cases as the ordinary one. Its proof is carried out thus: either all sets possess property J or there exists a set E that does not possess it; the second assumption leads to a contradiction, for among the segments of E there must be a first one that does not possess property J, and the existence of such a segment contradicts the conditions.

The examples adduced suffice to show that, alongside of the development of mathematics presented by Brouwer without the help of the principle of excluded middle, we must preserve the usual development, which uses this principle, if only as the development of mathematics of pseudotruth.

[20] This example was pointed out by P. S. Novikov.