Introductory note to *1933f*

In this short note Gödel describes an interpretation of intuitionistic propositional logic **IPC** in a system of classical propositional logic enriched with an additional unary operator B. The letter B stands for "beweisbar", i.e., provable, this being the intuitive interpretation of B. Here "provable" should be understood as "provable by any correct means" and not as "provable in a given formal system" (see below). The axioms given for B are now familiar as those for Lewis' system of modal propositional logic **S4** (Gödel's system $\mathfrak{S}$), with B written for the necessity operator N or $\Box$. Gödel's result thus takes the following form:

$$\text{(1)} \qquad \text{if } \mathbf{IPC} \vdash F, \text{then } \mathbf{S4} \vdash F',$$

where F' is formed from F according to his translation table. In addition, Gödel conjectured that the converse of (1) also holds, that is,

$$\text{(2)} \qquad \mathbf{IPC} \vdash F \text{ if and only if } \mathbf{S4} \vdash F'.$$

This conjecture was eventually established by J.C.C. McKinsey and A. Tarski (*1948*), who used algebraic semantics. McKinsey and Tarski also described several alternative interpretations with the same property, for example the interpretation $^-$ given by $p^- := \Box\, p$ (for propositional variables p), $(F \vee G)^- := F^- \vee G^-$, $(F \wedge G)^- := F^- \wedge G^-$, $(F \to G)^- := \Box\, F^- \to \Box\, G^-$, $(\neg F)^- := \Box \neg F^-$. (There is a simple relationship between $'$ and $^-$: $\mathbf{S4} \vdash F^- \leftrightarrow \Box(F')$.)

In the result (2), **S4** can also be replaced by a stronger system, for example, by addition of the following axiom schema introduced by Grzegorczyk (*1967*), where $F' \Rightarrow F''$ abbreviates $\Box(F' \to F'')$:

$$\text{Grz} \qquad ((F \Rightarrow \Box\, G) \Rightarrow \Box\, G) \wedge ((\neg F \Rightarrow \Box\, G) \Rightarrow \Box\, G) \Rightarrow \Box\, G,$$

which is deductively equivalent to the simpler schema

$$\Box(\Box(F \to \Box\, F) \to F) \to F$$

(see *Boolos 1979*, Chapter 13). That is to say, one has

$$\mathbf{IPC} \vdash F \text{ if and only if } \mathbf{S4} + \text{Grz} \vdash F'.$$

According to Hacking (*1963*) one can also weaken **S4** in (2), namely to **S3**. Hacking's proof is based on cut-elimination.

The result (2) was extended to predicate logic by Rasiowa and Sikorski (*1953*), using algebraic semantics, and, independently, by Maehara (*1954*), using cut-elimination. To be precise, let **IQC** be intuitionistic predicate logic, and let **QS4** be **S4** with quantifiers and the usual axioms and rules for quantification. Extend $'$ by stipulating $(\forall x A)' := \forall x A'$, $(\exists x A)' := \exists x \Box\, A'$, or, equivalently, extend $^-$ by $(\forall x A)^- := \Box \forall x A^-$, $(\exists x A)^- := \exists x A^-$. Then

$$\text{(3)} \qquad \mathbf{IQC} \vdash F \text{ if and only if } \mathbf{QS4} \vdash F' \\ \text{if and only if } \mathbf{QS4} \vdash F^-.$$

Prawitz and Malmnäs (*1968*) gave a proof of this result using normalization for suitable natural deduction systems for **IQC** and **QS4**.

Quite recently the result (3) has been extended to systems with mathematical content, namely to intuitionistic arithmetic **HA** on the one hand and to a modal extension of classical first-order arithmetic **PA** on the other (*Mints 1978, Goodman 1984*); the methods used in both cases are proof-theoretical.

As already mentioned above, Gödel's B cannot be interpreted as provability in a given formal system, such as **PA**; for, as Gödel himself observes at the end of his note, this conflicts with the second incompleteness theorem. But it is of interest to see in this connection which laws are preserved by the formal provability interpretation of B. This was accomplished by Solovay (*1976*), who characterized the modal logic (often called **G**) corresponding to formal provability in **PA**. Though the discussion of **G** is really a topic in the history of Gödel's second incompleteness theorem, Solovay's result also leads (as we shall see) to another interpretation of **IPC**.

Let Φ be an interpretation of the formulas in the language of **S4** in **PA** such that: $\Phi(p_i) := \phi_i$ (p_i a proposition letter, ϕ_i a sentence of **PA**); Φ commutes with $\wedge$, $\vee$, $\neg$, $\to$; and $\Phi(\Box\, F) := \text{Prov}(\ulcorner \Phi F \urcorner)$. Here 'Prov' is the canonically defined predicate expressing arithmetized provability in **PA**, and $\ulcorner \Phi F \urcorner$ is the Gödel number of the sentence ΦF. Write $\models^* F$ if we have $\mathbf{PA} \vdash \Phi(F)$ for all possible interpretations Φ.

Let **G** be the modal system containing all classical tautologies, modus ponens, the necessitation rule (from F infer $\Box\, F$), the schema

$$\Box(F \to G) \to (\Box\, F \to \Box\, G)$$

and the following "Löb's axiom":

$$\text{L} \qquad \Box(\Box\, F \to F) \to \Box\, F.$$

The reason for so designating this axiom is that Löb (*1955*) proved in effect that $\models^* F$ holds for all instances F of L.

Solovay's result can now be stated as

$$\text{(4)} \qquad \mathbf{G} \vdash F \text{ if and only if } \models^* F.$$

Let $^\circ$ be the translation of formulas in the language of **S4** such that $p_i^\circ := p_i$; $^\circ$ commutes with $\wedge$, $\vee$, $\rightarrow$, $\neg$, and $(\Box A)^\circ := A^\circ \wedge \Box(A^\circ)$. Goldblatt (*1978*) showed[a]

$$\text{(5)} \qquad \mathbf{IPC} \vdash F \text{ if and only if } \mathbf{G} \vdash (F^-)^\circ,$$

and thus we also obtain a "formal provability" interpretation of **IPC**:

$$\text{(6)} \qquad \mathbf{IPC} \vdash F \text{ if and only if } \models^* (F^-)^\circ.$$

In his note Gödel observes, without proof, that

$$\text{(7)} \qquad \text{If } \mathbf{S4} \vdash \Box F \vee \Box G, \text{ then } \mathbf{S4} \vdash \Box F \text{ or } \mathbf{S4} \vdash \Box G.$$

A proof was given in *McKinsey and Tarski 1948*. By means of (2), this property of **S4** implies the disjunction property for **IPC**:

$$\text{if } \mathbf{IPC} \vdash F \vee G, \text{ then } \mathbf{IPC} \vdash F \text{ or } \mathbf{IPC} \vdash G.$$

[a]In a footnote Goldblatt mentions an earlier proof, which we have not seen, due to A. Kuznetsov and A. Muzavitski.

This fact was proved by Gentzen (*1935*) using cut-elimination.

For Gödel, the interest of his result presumably lay in the fact that it gave for **IPC** an interpretation which was meaningful also from a non-intuitionistic point of view. In this connection it is perhaps significant that Gödel mentions in a footnote Kolmogorov's (*1932*) interpretation of **IPC**, which, although different in character, was also put forward as being meaningful independently of intuitionistic bias.

Heyting's interpretation (*1931*) of intuitionistic logic, which was certainly known to Gödel, suggests the identification of "intuitionistically true" with "(intuitionistically) provable". It may well be that this led Gödel to his interpretation. On the other hand, Heyting's interpretation does not quite prepare us for a result like (2), since **S4** is a system based on *classical* logic.

Historically, Gödel's result was instrumental in the development of Kripke's semantics (*1965*) for intuitionistic logic: once the semantics for modal logic, in particular for **QS4**, had been formulated, Gödel's interpretation with its variants showed how one could obtain a semantics for **IQC** (see the introduction to *Kripke 1965*).

Finally, it should be observed that Gödel's axiomatization of **S4** was new and led to a much simpler and more perspicuous axiomatization of systems of modal logic (see *Lemmon 1977*, pages 6–7).

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Eine Interpretation des intuitionistischen Aussagenkalküls (*1933f*)

Man kann den Heytingschen Aussagenkalkül mittels der Begriffe des gewöhnlichen Aussagenkalküls und des Begriffes "p ist beweisbar" (bezeichnet mit Bp) interpretieren,[1] wenn man für den letzteren das folgende Axiomensystem $\mathfrak{S}$ annimmt:

1. $Bp \rightarrow p$
2. $Bp \rightarrow . B(p \rightarrow q) \rightarrow Bq$
3. $Bp \rightarrow BBp$.

Außerdem sind für die Begriffe $\rightarrow, \sim, ., \vee$ die Axiome und Schlußregeln des gewöhnlichen Aussagenkalküls anzunehmen, ferner die neue Schlußregel: Aus A darf auf BA geschlossen werden.

Die Heytingschen Grundbegriffe sind folgendermaßen zu übersetzen:

$\neg p$	$\sim Bp$
$p \supset q$	$Bp \rightarrow Bq$
$p \vee q$	$Bp \vee Bq$
$p \wedge q$	$p . q.$

Mit dem selben Erfolg könnte man auch $\neg p$ durch $B{\sim}Bp$ und $p \wedge q$ durch $Bp . Bq$ übersetzen. Die Übersetzung einer beliebigen im Heytingschen System gültigen Formel folgt aus $\mathfrak{S}$, dagegen folgt aus $\mathfrak{S}$ nicht die Übersetzung von $p \vee \neg p$ und allgemein keine Formel der Gestalt $BP \vee BQ$, für die nicht schon entweder BP oder BQ aus $\mathfrak{S}$ beweisbar ist. Vermutlich gilt eine Formel im Heytingschen Kalkül dann und nur dann, wenn ihre Übersetzung aus $\mathfrak{S}$ beweisbar ist.

Das System $\mathfrak{S}$ ist mit dem Lewisschen System of Strict Implication äquivalent, wenn Bp durch Np übersetzt wird (vgl. *Parry 1933a*), und wenn man das Lewissche System durch das folgende Beckersche Zusatzaxiom $Np < NNp$ ergänzt.[2]

Es ist zu bemerken, daß für den Begriff "beweisbar in einem bestimmten formalen System S" die aus $\mathfrak{S}$ beweisbaren Formeln nicht alle gelten. Es

[1] Eine etwas andere Interpretation des intuitionistischen Aussagenkalküls gab Kolmogoroff (*1932*) ohne allerdings einen präzisen Formalismus anzugeben.

[2] *Becker 1930*, S. 497.

An interpretation of the intuitionistic propositional calculus (*1933f*)

One can interpret[1] Heyting's propositional calculus by means of the notions of the ordinary propositional calculus and the notion 'p is provable' (written Bp) if one adopts for that notion the following system $\mathfrak{S}$ of axioms:

1. $Bp \rightarrow p$,
2. $Bp \rightarrow . B(p \rightarrow q) \rightarrow Bq$,
3. $Bp \rightarrow BBp$.

In addition, for the notions $\rightarrow, \sim, ., \vee$ the axioms and rules of inference of the ordinary propositional calculus are to be adopted, as well as the new rule of inference: From A, BA may be inferred.

Heyting's primitive notions are to be translated as follows:

$\neg p$	$\sim Bp$
$p \supset q$	$Bp \rightarrow Bq$
$p \vee q$	$Bp \vee Bq$
$p \wedge q$	$p . q.$

One could also translate $\neg p$ by $B{\sim}Bp$ and $p \wedge q$ by $Bp . Bq$ with equal success. The translation of an arbitrary formula that holds in Heyting's system is derivable in $\mathfrak{S}$; on the other hand, the translation of $p \vee \neg p$ is not derivable in $\mathfrak{S}$, nor in general is any formula of the form $BP \vee BQ$ for which neither BP nor BQ is already provable in $\mathfrak{S}$. Presumably a formula holds in Heyting's calculus if and only if its translation is provable in $\mathfrak{S}$.

The system $\mathfrak{S}$ is equivalent to Lewis' system of strict implication if Bp is translated by Np (see *Parry 1933a*) and one supplements Lewis' system by the following additional axiom[2] of Becker: $Np < NNp$.

It is to be noted that for the notion "provable in a certain formal system S" not all of the formulas provable in $\mathfrak{S}$ hold. For example, $B(Bp \rightarrow p)$

[1] Kolmogorov (*1932*) has given a somewhat different interpretation of the intuitionistic propositional calculus, without, to be sure, specifying a precise formalism.

[2] *Becker 1930*, p. 497.

gilt z. B. für ihn $B(Bp \rightarrow p)$ niemals, d. h. für kein System S, das die Arith-
40 metik enthält. Denn andernfalls wäre | beispielsweise $B(0 \neq 0) \rightarrow 0 \neq 0$
und daher auch $\sim B(0 \neq 0)$ in S beweisbar, d. h. die Widerspruchsfreiheit
von S wäre in S beweisbar.

Bemerkung über projektive Abbildungen (*1933g*)

Jede eineindeutige Abbildung ϕ der reellen projektiven Ebene E in sich, welche Gerade in Gerade überführt, ist eine Kollineation. Bezeichnet man für jeden Kegelschnitt der Ebene als die zugehörige *Kegelschnittumgebung* die Menge aller Punkte p von E, für welche jede p enthaltende Gerade genau zwei Punkte mit dem Kegelschnitt gemein hat, so ist das System $\mathfrak{K}$ aller Kegelschnittumgebungen ein unbegrenzt feines Überdeckungssystem[1] von E. Da jeder Kegelschnitt durch zwei Geradenbüschel erzeugt werden kann, welche projektiv (d. h. durch eine endliche Kette von Perspektivitäten) auf einander bezogen sind, und da perspektiv auf einander bezogene Geradenbüschel bei jeder eineindeutigen Abbildung, die Gerade in Gerade überführt, in perspektiv auf einander bezogene Geradenbüschel übergehen, so wird durch die Abbildung ϕ jeder Kegelschnitt auf einen Kegelschnitt abgebildet. Ferner wird wegen der Eineindeutigkeit von ϕ und der obigen Definition einer Kegelschnittumgebung jede Kegelschnittumgebung durch ϕ in eine Kegelschnittumgebung übergeführt. Es wird demnach $\mathfrak{K}$ durch ϕ in sich übergeführt. Da $\mathfrak{K}$ ein unbegrenzt feines Überdeckungssystem von E ist, so ist also ϕ eine *stetige* Abbildung und daher nach einem Fundamentalsatz der projektiven Geometrie eine Kollineation.

[1]So heißt nach Menger ein System von offenen Mengen, in welchem zu jedem Punkt p des Raumes und zu jeder Umgebung U von p eine p enthaltende offene Menge $\subset U$ existiert.

never holds for that notion, that is, it holds for no system S that contains arithmetic. For otherwise, for example, $B(0 \neq 0) \rightarrow 0 \neq 0$ and therefore also $\sim B(0 \neq 0)$ would be provable in S, that is, the consistency of S would be provable in S.

Remark concerning projective mappings (*1933g*)

[The introductory note to *Gödel 1933g*, as well as to related items, can be found on page 272, immediately preceding *1933b*.]

Every one-to-one mapping ϕ of the real projective plane E into itself that carries straight lines into straight lines is a collineation. If for each conic section in the plane one designates as the corresponding *conic-section neighborhood* the set of all points p of E for which every straight line containing p has exactly two points in common with the conic section, then the system $\mathfrak{K}$ of all conic-section neighborhoods is an unboundedly fine covering system[1] of E. Since every conic section can be generated by two pencils of straight lines that are projectively correspondent to each other (that is, correspondent by a finite chain of perspectivities) and since perspectively correspondent pencils of straight lines are transformed, by every one-to-one mapping that carries straight lines into straight lines, into perspectively correspondent pencils of straight lines, every conic section is mapped onto a conic section by the mapping ϕ. Furthermore, on account of the one-to-oneness of ϕ and the above definition of a conic-section neighborhood, every conic-section neighborhood is transformed by ϕ into a conic-section neighborhood. Accordingly, $\mathfrak{K}$ is transformed into itself by ϕ. Since $\mathfrak{K}$ is an unboundedly fine covering of E, ϕ is a *continuous* mapping and therefore, by a fundamental theorem of projective geometry, a collineation.

[1]Menger so calls a system of open sets in case there exists in it, for each point p of the space and for each neighborhood U of p, an open subset of U that contains p.