Lecture Notes on
Decidability and Filtration

15-816: Modal Logic
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1 Introduction to This Lecture

In this lecture we study decidability of propositional modal logics. Also see [Grä99] for an overview.

2 Filtration

Definition 1 *Let $K = (W, \rho, v)$ be a Kripke structure and let Γ be any set of propositional modal formulas. We define an equivalence relation $\sim_\Gamma$ on W by*

$$s \sim_\Gamma t \quad \textit{iff} \quad \textit{for all } F \in S: \ K, s \models F \textit{ iff } K, t \models F$$

and consider equivalence classes $[s]$ of states s with respect to $\sim_\Gamma$. Then the quotient structure $K_\Gamma = (W_\Gamma, \rho_\Gamma, \tau_\Gamma)$ is called filtration *of K with respect to Γ and defined as:*

- *$W_\Gamma := \{[s] \ : \ s \in W_\Gamma\}$ (well-defined because $\sim_\Gamma$ is an equivalence relation)*
- *$[s]\rho_\Gamma[t]$ iff there is a $s_0 \in [s]$ and there is a $t_0 \in [t]$ with $s_0 \rho t_0$*
- *$\tau_\Gamma(q)([s]) := v(q)(s)$ when propositional letter $q \in \Gamma$ (well-defined because $\sim_\Gamma$ is an equivalence relation)*
- *$\tau_\Gamma(q)$ is arbitrary when propositional letter $q \notin \Gamma$*

Lemma 2 *Let K_Γ be the filtration of K with respect to a set of propositional modal formulas Γ that is* closed under subformulas*. Then for all formulas $A \in \Gamma$ and all $s \in W$:*

$$K, s \models A \quad \textit{iff} \quad K_\Gamma, [s] \models A$$

Proof: The proof is by induction on A. For propositional letters, the statement is by construction of K_Γ. Consider the induction step for $\Box B$. Assume $K_\Gamma, [s] \models \Box B$. Then for all $t \in W$ with $[s]\rho_\Gamma[t]$ we know that $K_\Gamma, [t] \models B$. By induction hypothesis, for all $t \in W$ with $[s]\rho_\Gamma[t]$ we have $K, t \models B$ In particular: for all $t \in W$ with $s\rho t$ we have $K, t \models B$. Thus $K, s \models \Box B$.

Conversely assume $K, s \models \Box B$. Consider any $s, t \in W$ with $[s]\rho_\Gamma[t]$. That is, there are $s_0 \in [s]$ and $t_0 \in [t]$ with $s_0 \rho t_0$ and $[s_0] = [s]$ and $[t_0] = [t]$. From $[s_0] = [s]$, the definition of $\sim_\Gamma$ and the fact that $\Box B \in \Gamma$ implies that $K, s_0 \models \Box B$, because $K, s \models \Box B$. Especially, $K, t_0 \models B$. Again, the definition of $\sim_\Gamma$ implies $K, t \models B$. By induction hypothesis, $K_\Gamma, [t] \models B$. Thus, for all $[t] \in W_\Gamma$ with $[s]\rho_\Gamma[t]$ we know $K_\Gamma, [t] \models B$, which immediately implies $K_\Gamma, [s] \models \Box B$. □

Lemma 3 *If Γ in Lemma 2 is finite with $|K| = n$ elements then $|K_\Gamma| \leq 2^n$.*

Proof: There can be at most 2^n equivalence classes for n formulas. □

3 Decidability

Theorem 4 *Validity in the propositional modal logic* **K** *is decidable.*

Proof: Given any input formula A let Γ be the set of all subformulas of $\neg A$. Let Γ have n elements. If $\neg A$ is satisfiable, then, by Lemma 3, it is satisfiable in a Kripke structure with at most 2^n worlds. Simple enumeration of all Kripke structures with at most 2^n worlds can thus decide if $\neg A$ is satisfiable. □

Corollary 5 **T** *and* **S4** *and* **S5** *(and in fact any combinations of their axioms on top of* **K***) is decidable.*

Proof: We have to show that the filtration K_Γ is reflexive / symmetric / transitive whenever the original Kripke structure K is. For reflexive / symmetric, this is a simple check. For transitive, K_Γ does not need to be transitive even though K is. But if we replace ρ_Γ by its transitive closure, then everything can be proven. For that modification, Lemma 2 needs to be proven again though.

Let $K_\Gamma^* = (W_\Gamma, \rho_\Gamma^*, \tau_\Gamma)$ be the filtration where ρ_Γ^* denotes the transitive closure of ρ_Γ. We prove that

$$K, s \models \Box B \quad \text{iff} \quad K_\Gamma^*, [s] \models \Box B$$

Assume $K_\Gamma^*, [s] \models \Box B$. For any $t \in W$ with $s\rho t$ we want to show $K, t \models B$. From $s\rho t$ we conclude both $[s]\rho_\Gamma[t]$ and $[s]\rho_\Gamma^*[t]$. Thus $K_\Gamma^*, [t] \models B$, which, by induction hypothesis, implies $K, t \models B$.

Conversely assume $K, s \models \Box B$. For any $t \in W$ with $[s]\rho_\Gamma^*[t]$ we want to show $K_\Gamma^*, [t] \models B$. Now ρ_Γ^* is the transitive closure of the existential abstraction ρ_Γ. Thus, there are states $t_0, t_1, \ldots, t_n$ such that $[t_0] = [s]$ and $[t_n] = [t]$ and for each $0 \leq i < n$ there are local connecting states $g_i \in [t_i]$ and $h_i \in [t_{i+1}]$ such that $g_i \rho h_i$. Especially $[g_0] = [s]$, $[h_{n-1}] = [t]$ and $[g_{i+1}] = [h_i]$. By induction we show that $K, g_i \models \Box B$ for all i.

0. Because of $[g_0] = [s]$, $K, s \models \Box B$ implies $g_0 \models \Box B$.
1. Assume $K, g_i \models \Box B$. We have assumed that ρ is transitive, thus we even know that $K, g_i \models \Box\Box B$. Now $g_i \rho h_i$ implies that $K, h_i \models \Box B$. But $[g_{i+1}] = [h_i]$, thus also, $g_{i+1} \models \Box B$.

Finally, $K, t \models B$ follows from $g_{n-1} \models \Box B$, because $g_{n-1}\rho h_{n-1}$ and $[h_{n-1}] = [t]$. Thus $K_\Gamma^*, [t] \models B$ by induction hypothesis. Sine t was arbitrary, we have $K_\Gamma^*, [s] \models \Box B$. $\Box$

References

[Grä99] Erich Grädel. Why are modal logics so robustly decidable? *Bulletin of the EATCS*, 68:90–103, 1999.