Lecture Notes on Reconciliation

15-816: Modal Logic

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Lecture 14

March 16, 2010

1 Introduction

We have seen that there are many differences between the intuitionistic and classical approaches to the study of logic. Intuitionistic logic starts with a notion of proof which corresponds to constructions and give rise to computation. To justify the logical laws and define the meaning of the connectives we use verifications. On the other hand, in classical logic the meaning of propositions is defined via an interpretation in mathematical structures of various kind. For modal logic, the most prominent structure is that of Kripke models with multiple worlds. Inference are justified as being sound and complete with respect to classes of models.

Besides the logic itself, the methods for reasoning *about'* the logic can also be classified as either constructive or classical. Typically, intuitionists study logic with constructive means, while classical logicians study logic with classical means. Of course, this is not the only choice. Classical logicians have studied intuitionistic logic with classical means. This line of research develops a class of models such that the inference rule of intuitionistic logic are sound and complete with respect to such models. Conversely, intuitionistic logicians have studied classical logic with constructive means. This line of research develops a proof theory for classical logic and studies its computational meaning, where available.

In this lecture we will try to achieve a kind of reconciliation between intuitionistic and classical logic. Rather than pursuing either of the approaches sketched above, we will develop (constructive!) translations between intuitionistic and classical logic which can explain to an intuitionist what classical logic means in terms she can understand and vice versa. The space of the possible translations is vast, and particular translations are chosen for particular purposes.

For the translation from intuitionistic to classical logic, we choose Gödel's translation [Göd33]. The high-level summary of this translation is that from the classical point of view, intuitionistic logic is a fragment of the classical modal logic S4. Via the Kripke semantics of S4, one can also see this as a Kripke semantics for intuitionistic logic, but we will approach it proof-theoretically instead.

For the translation from classical to intuitionistic logic, we choose Kolmogorov's translation [Kol25]. The high-level summary of this translation is that from the intuitionistic point of view, classical logic is a fragment of intuitionistic logic aimed only at deriving contradictions. Under this translation, any computational content of classical logic would consist only of its internal reductions, because a proof of a contradiction can never deliver a value.

2 A Classical Sequent Calculus

In classical logic, a sequent has the form

$$\Gamma \Longrightarrow \Sigma$$

where the conjunction of the propositions in Γ entails the disjunction of the propositions in Σ. As we will see in Section 5, this can be read intuitionistically as: "*Assuming the propositions in Γ are true and in Σ are false, we can derive a contradiction.*" The intuitionistically-minded reader can keep this interpretation in mind as we write the rules.

We give here a formulation of the classical sequent calculus were the principal formula of a left or right rule does *not* persist in the premises. This can easily be justified (see Exercise 1). It is important to realize, however, that we can no longer view $\Gamma \Longrightarrow \Sigma$ as a hypothetical judgment, even if the judgments on the two sides of the sequent are distinguished. We write the principal formulas of each inference in the middle, next to the sequent arrow $\Longrightarrow$. The rules are listed in Figure 1.

This calculus is characterized by many strong symmetries (see Exercise 2). We have the following properties.

Theorem 1 (Classical Sequent Calculus)

(i) (Left Weakening) If $\Gamma \Longrightarrow \Sigma$ then $\Gamma, A \Longrightarrow \Sigma$.

$$\frac{}{\Gamma, P \Longrightarrow P, \Sigma}\ \mathsf{init}$$

$$\frac{}{\Gamma \Longrightarrow \top, \Sigma}\ \top R \qquad \frac{\Gamma \Longrightarrow \Sigma}{\Gamma, \top \Longrightarrow \Sigma}\ \top L$$

$$\frac{\Gamma \Longrightarrow A, \Sigma \quad \Gamma \Longrightarrow B, \Sigma}{\Gamma \Longrightarrow A \wedge B, \Sigma}\ \wedge R \qquad \frac{\Gamma, A, B \Longrightarrow \Sigma}{\Gamma, A \wedge B \Longrightarrow \Sigma}\ \wedge L$$

$$\frac{\Gamma \Longrightarrow A, B, \Sigma}{\Gamma \Longrightarrow A \vee B, \Sigma}\ \vee R \qquad \frac{\Gamma, A \Longrightarrow \Sigma \quad \Gamma, B \Longrightarrow \Sigma}{\Gamma, A \vee B \Longrightarrow \Sigma}\ \vee L$$

$$\frac{\Gamma \Longrightarrow \Sigma}{\Gamma \Longrightarrow \bot, \Sigma}\ \bot R \qquad \frac{}{\Gamma, \bot \Longrightarrow \Sigma}\ \bot L$$

$$\frac{\Gamma, A \Longrightarrow B, \Sigma}{\Gamma \Longrightarrow A \supset B, \Sigma}\ \supset R \qquad \frac{\Gamma \Longrightarrow A, \Sigma \quad \Gamma, B \Longrightarrow \Sigma}{\Gamma, A \supset B \Longrightarrow \Sigma}\ \supset L$$

$$\frac{\Gamma, A \Longrightarrow \Sigma}{\Gamma \Longrightarrow \neg A, \Sigma}\ \neg R \qquad \frac{\Gamma \Longrightarrow A, \Sigma}{\Gamma, \neg A \Longrightarrow \Sigma}\ \neg L$$

Figure 1: A Classical Sequent Calculus

(ii) *(Right Weakening) If* $\Gamma \Longrightarrow \Sigma$ *then* $\Gamma \Longrightarrow A, \Sigma$*.*

(iii) *(Left Contraction) If* $\Gamma, A, A \Longrightarrow \Sigma$ *then* $\Gamma, A \Longrightarrow \Sigma$*.*

(iv) *(Right Contraction) If* $\Gamma \Longrightarrow A, A, \Sigma$ *then* $\Gamma \Longrightarrow A, \Sigma$*.*

(v) *(Cut) If* $\Gamma \Longrightarrow A, \Sigma$ *and* $\Gamma, A \Longrightarrow \Sigma$ *then* $\Gamma \Longrightarrow \Sigma$*.*

(vi) *(Identity)* $\Gamma, A \Longrightarrow A, \Sigma$ *for any proposition* A*.*

We examine a few classical theorems to see where the multiple conclusions enable a proof, where it cannot be proven in intuitionistic logic. First, the law of excluded middle.

$$\dfrac{\dfrac{\overline{P \Longrightarrow P}\;\mathsf{init}}{\cdot \Longrightarrow P, \neg P}\;\neg R}{\cdot \Longrightarrow P \vee \neg P}\;\vee R$$

Here, it is crucial that we preserve both P and $\neg P$ in the disjunction introduction. Closely related is the following

$$\dfrac{\dfrac{\overline{P \Longrightarrow P, Q}\;\mathsf{init}}{\cdot \Longrightarrow P, P \supset Q}\;\supset R}{\cdot \Longrightarrow P \vee (P \supset Q)}\;\vee R$$

Thinking about the intuitionistic meaning of implication, we can identify the application of $\supset R$ as a *scope violation*: according to the principles of intuitionistic logic, the scope of P should be Q, but at the $\supset R$ rule we expand that scope to include the other succedents (here: P), which allows us to complete the proof.

Let us try to verify this intuition with Pierce's law, which is also only classically true.

$$\dfrac{\dfrac{\dfrac{\overline{P \Longrightarrow Q, P}\;\mathsf{init}}{\cdot \Longrightarrow P \supset Q, P}\;\supset R \qquad \overline{P \Longrightarrow P}\;\mathsf{init}}{(P \supset Q) \supset P \Longrightarrow P}\;\supset L}{\cdot \Longrightarrow ((P \supset Q) \supset P) \supset P}\;\supset R$$

Again, from the intuitionistic point of view, the incorrect inference appears as $\supset R$ in the left branch.

The basic idea of Gödel's translation will be to prevent this scope extrusion through the use of a modality. In order to make to present this formal, we will need to define a classical modal sequent calculus.

3 A Classical Modal Sequent Calculus

A classical modal sequent has the form $\Delta; \Gamma \Longrightarrow \Sigma; \Psi$. Classically, such a sequent stands for $\bigwedge(\Box\Delta) \wedge \bigwedge \Gamma \supset \bigvee \Sigma \vee \bigvee(\Diamond\Psi)$, where the semantics of the modal operators follows S4.

Intuitionstically, it corresponds to judgment that under the assumptions that the propositions in Δ are necessarily true, those in Γ are true, those in Σ are false, and those in Ψ are necessarily false, we can arrive at a contradiction.

We obtain this calculus from the one in Figure 1 by first adding Δ and Ψ systematically to every sequent, and then adding the following rules concerned with the modalities.

$$\frac{\Delta; \bullet \Longrightarrow A, \bullet; \Psi}{\Delta; \Gamma \Longrightarrow \Box A, \Sigma; \Psi}\ \Box R \qquad \frac{\Delta, A; \Gamma \Longrightarrow \Sigma; \Psi}{\Delta; \Gamma, \Box A \Longrightarrow \Sigma; \Psi}\ \Box L$$

$$\frac{\Delta, A; \Gamma, A \Longrightarrow \Sigma; \Psi}{\Delta, A; \Gamma \Longrightarrow \Sigma; \Psi}\ \mathsf{valid}$$

$$\frac{\Delta; \Gamma \Longrightarrow \Sigma; A, \Psi}{\Delta; \Gamma \Longrightarrow \Diamond A, \Sigma; \Psi}\ \Diamond R \qquad \frac{\Delta; \bullet, A \Longrightarrow \bullet; \Psi}{\Delta; \Gamma, \Diamond A \Longrightarrow \Sigma; \Psi}\ \Diamond L$$

$$\frac{\Delta; \Gamma \Longrightarrow A, \Sigma; A, \Psi}{\Delta; \Gamma \Longrightarrow \Sigma; A, \Psi}\ \mathsf{poss}$$

In the judgmental rules, the principal formula A is copied, which is necessary for the rules to be complete with respect to the intended S4 semantics.

The property of weakening, contraction, cut, and identity extend to this calculus (see Exercise 3).

4 Intuitionistic Logic as a Classical Modal Logic

We return to the example from Section 2.

$$\cfrac{\cfrac{\cfrac{}{P \Longrightarrow P, Q}\ \mathsf{init}}{\cdot \Longrightarrow P, P \supset Q}\ {\supset}R}{\cdot \Longrightarrow P \vee (P \supset Q)}\ {\vee}R$$

If we put a $\Box$ modality in front of ever subformula, then the only possible proof attempt will start as follows:

$$\cfrac{\cfrac{\cdot \Longrightarrow \Box P, \Box(\Box P \supset \Box Q)}{\cdot \Longrightarrow \Box P \vee \Box(\Box P \supset \Box Q)}\ {\vee}R}{\cdot \Longrightarrow \Box(\Box P \vee \Box(\Box P \supset \Box Q))}\ \Box R$$

Now we have to apply the $\Box R$ rule again, but no matter which principal formula we choose we cannot complete the proof because the other succedent is erased as part of the application of $\Box R$.

We define two mutually inductive translations

$$
\begin{array}{lcl}
A^{\Box} & = & \Box A^{+} \\
\\
(P)^{+} & = & P \\
(A \wedge B)^{+} & = & A^{\Box} \wedge B^{\Box} \\
(\top)^{+} & = & \top \\
(A \vee B)^{+} & = & A^{\Box} \vee B^{\Box} \\
(\bot)^{+} & = & \bot \\
(A \supset B)^{+} & = & A^{\Box} \supset B^{\Box} \\
(\neg A)^{+} & = & \neg A^{\Box}
\end{array}
$$

We would like to show that $\cdot \Longrightarrow A$ in intuitionistic sequent calculus iff $\cdot; \cdot \Longrightarrow A^{+}; \cdot$ in classical modal sequent calculus. It requires some experience to appropriately generalize this statement in each direction. We do the easy direction first.

Theorem 2 (From Intuitionistic to Classical Modal Logic) *If* $\Gamma \Longrightarrow A$ *then* $\Gamma^{+}; \bullet \Longrightarrow A^{+}; \cdot$.

Proof: By induction on the structure of the given sequent derivation. We show some representative cases. Because the rightmost context of the classical sequent will always be empty, we elide it in the proof below following our general conventions. In each case, we imitate the intutionistic rules with the classical rules, interweaving $\Box R$ and $\Box L$ rules. In addition, we need weakening for the cases $\vee R_1$, $\vee R_2$, $\wedge L_1$, $\wedge L_2$ to mediate the differences between the intuitionistic and classical rules.

Case:

$$\frac{}{\Gamma_0, P \Longrightarrow P}\ \mathsf{init}$$

$\Gamma_0^{+}, P; P \Longrightarrow P$	By rule init
$\Gamma_0^{+}, P; \bullet \Longrightarrow P$	By rule valid

Case:

$$\frac{\Gamma \Longrightarrow A_1}{\Gamma \Longrightarrow A_1 \wedge A_2}\ \vee R_1$$

$$\begin{array}{ll}
\Gamma^+;\bullet \Longrightarrow A_1^+ & \text{By i.h.} \\
\Gamma^+;\bullet \Longrightarrow \Box A_1^+ & \text{By rule } \Box R \\
\Gamma^+;\bullet \Longrightarrow \Box A_1^+, \Box A_2^+ & \text{By right weakening} \\
\Gamma^+;\bullet \Longrightarrow (A_1 \vee A_2)^+ & \text{By rule } \vee R
\end{array}$$

Case:

$$\frac{\Gamma_0, B_1 \vee B_2, B_1 \Longrightarrow A \quad \Gamma_0, B_1 \vee B_2, B_2 \Longrightarrow A}{\Gamma_0, B_1 \vee B_2 \Longrightarrow A} \vee L$$

$$\begin{array}{ll}
\Gamma_0^+, (B_1 \vee B_2)^+, B_1^+;\bullet \Longrightarrow A^+ & \text{By i.h.} \\
\Gamma_0^+, (B_1 \vee B_2)^+;\bullet, \Box B_1^+ \Longrightarrow A^+ & \text{By } \Box L \\
\Gamma_0^+, (B_1 \vee B_2)^+, B_2^+;\bullet \Longrightarrow A^+ & \text{By i.h.} \\
\Gamma_0^+, (B_1 \vee B_2)^+;\bullet, \Box B_2^+ \Longrightarrow A^+ & \text{By } \Box L \\
\Gamma_0^+, (B_1 \vee B_2)^+;\bullet, \Box B_1^+ \vee \Box B_2^+ \Longrightarrow A^+ & \text{By } \vee L \\
\Gamma_0^+, (B_1 \vee B_2)^+;\bullet \Longrightarrow A^+ & \text{By rule valid}
\end{array}$$

Case:

$$\frac{\Gamma, A_1 \supset A_2}{\Gamma \Longrightarrow A_1 \supset A_2} \supset R$$

$$\begin{array}{ll}
\Gamma^+, A_1^+;\bullet \Longrightarrow A_2^+ & \text{By i.h.} \\
\Gamma^+, A_1^+;\bullet \Longrightarrow \Box A_2^+ & \text{By } \Box R \\
\Gamma^+;\bullet, \Box A_1^+ \Longrightarrow \Box A_2^+ & \text{By } \Box L \\
\Gamma^+;\bullet \Longrightarrow (A_1 \supset A_2)^+ & \text{Rule } \supset R
\end{array}$$

Case:

$$\frac{\Gamma_0, B_1 \supset B_2 \Longrightarrow B_1 \quad \Gamma_1, B_1 \supset B_2, B_2 \Longrightarrow A}{\Gamma_0, B_1 \supset B_2 \Longrightarrow A} \supset L$$

$$\begin{array}{ll}
\Gamma_0^+, (B_1 \supset B_2)^+;\bullet \Longrightarrow B_1^+ & \text{By i.h.} \\
\Gamma_0^+, (B_1 \supset B_2)^+;\bullet \Longrightarrow \Box B_1^+ & \text{By } \Box R \\
\Gamma_0^+, (B_1 \supset B_2)^+, B_2^+;\bullet \Longrightarrow A^+ & \text{By i.h.} \\
\Gamma_0^+, (B_1 \supset B_2)^+;\Box B_2^+ \Longrightarrow A^+ & \text{By } \Box L \\
\Gamma_0^+, (B_1 \supset B_2)^+;\Box B_1^+ \supset \Box B_2^+ \Longrightarrow A^+ & \text{By } \supset L \\
\Gamma_0^+, (B_1 \supset B_2)^+;\bullet \Longrightarrow A^+ & \text{By rule valid}
\end{array}$$

□

The other direction is more difficult, because we temporarily have to contend with multiple succedents in the classical calculus. The key is to consider how general the induction hypothesis needs to be to accomodate any kind of rule that might be applied in a classical proof. For example, if we have $\Gamma^+; \bullet \Longrightarrow A^+$ then the valid rule copies a formula B^+ from Γ^+ into the truth context, and this can be done multiple times, so we have to allow $\Gamma^+; \Gamma_2^+ \Longrightarrow A^+$. Now we can break down a proposition in Γ_2^+, for example, $(A \supset B)^+ = \Box A^+ \supset \Box B^+$, by applying a left rule. This puts formulas $\Box A^+$ into the succedent and $\Box B^+$ into the antecedent.

In the succedent, the situation is a bit tricker. If we allow mixing of formulas A^+ and $\Box A^+$, then the theorem fails. Consider, for example, the succedent $\cdot; \cdot \Longrightarrow \Box P \supset \Box Q, \Box P$. Classically, this is perfectly true, but in the preimage of the translation we have $\cdot \Longrightarrow (P \supset Q) \vee P$, which is not intuitionistically true. The idea for avoiding this problem is to prove inversion properties for connectives on the right, excepting $\Box$ (since $\Box R$ is clearly not invertible). There may be a more elegant way to achieving this, but since it is a relatively common pattern in proof theory, we measure the size of proofs and use this measure in a later induction argument. The size of a proof is simply measured by the total number of inference rules applied.

Lemma 3 (Inversion)

(i) *If* $\Delta; \Gamma \vdash A \wedge B, \Sigma; \Psi$ *then* $\Delta; \Gamma \vdash A, \Sigma; \Psi$ *and* $\Delta; \Gamma \vdash B, \Sigma; \Psi$*, both with proofs of smaller or equal size.*

(ii) *If* $\Delta; \Gamma \vdash A \vee B, \Sigma; \Psi$ *then* $\Delta; \Gamma \vdash A, B, \Sigma; \Psi$ *with a proof of smaller or equal size.*

(iii) *If* $\Delta; \Gamma \vdash \bot, \Sigma; \Psi$ *then* $\Delta; \Gamma \vdash \Sigma; \Psi$ *with a proof of smaller or equal size.*

(iv) *If* $\Delta; \Gamma \vdash A \supset B, \Sigma; \Psi$ *then* $\Delta; \Gamma, A \vdash B, \Sigma; \Psi$ *wth a proof of smaller or equal size.*

(v) *If* $\Delta; \Gamma \vdash \neg A, \Sigma; \Psi$ *then* $\Delta; \Gamma, A \vdash \Sigma; \Psi$ *with a proof of smaller or equal size.*

Proof: See Exercise 4. $\Box$

We define $\bigvee(A_1, \ldots, A_n) = A_1 \vee \cdots \vee A_n$ where $\bigvee(\cdot) = \bot$ and $\bigvee(A) = A$. Finally we have the pieces assembled to prove the main theorem.

Theorem 4 (From Classical Modal to Intuitionistic Sequent Calculus)

(i) *If* $\Gamma_1^+; \Gamma_2^+, \Gamma_3^\Box \Longrightarrow \Sigma^\Box; \cdot$ *then* $\Gamma_1, \Gamma_2, \Gamma_3 \Longrightarrow \bigvee \Sigma$

(ii) *If* $\Gamma_1^+; \bullet \Longrightarrow A^+, \bullet; \cdot$ *then* $\Gamma_1 \Longrightarrow A$.

Proof: By mutual induction on the size of the given deduction. When part (i) appeals to part (ii) the size must be strictly smaller; when part (ii) appeals to part (i) the size must be smaller or equal.

We show some representative cases. For part (i) we distinguish cases based on the last inference applied; for part (ii) we distinguish cases based on the structure of A.

Case (i):

$$\frac{\Gamma_0^+, A; \Gamma_2^+, A^+, \Gamma_3^\Box \Longrightarrow \Sigma^\Box}{\Gamma_0^+, A; \Gamma_2^+, \Gamma_3^\Box \Longrightarrow \Sigma^\Box}\ \mathsf{valid}$$

$\Gamma_0, A, \Gamma_2, A, \Gamma_3 \Longrightarrow \bigvee \Sigma$ — By i.h.(i)
$\Gamma_0, A, \Gamma_2, \Gamma_3 \Longrightarrow \bigvee \Sigma$ — By contraction

Case (i):

$$\frac{\Gamma_1^+; \bullet \Longrightarrow A^+}{\Gamma_1^+; \Gamma_2^+, \Gamma_3^\Box \Longrightarrow \Box A^+, \Sigma_0^\Box}\ \Box R$$

$\Gamma_1 \Longrightarrow A$ — By i.h.(ii)
$\Gamma_1, \Gamma_2, \Gamma_3 \Longrightarrow A$ — By weakening
$\Gamma_1, \Gamma_2, \Gamma_3 \Longrightarrow A \vee \bigvee \Sigma_0$ — By repeated $\vee I$

Case (i):

$$\frac{\Gamma_1^+, A^+; \Gamma_2^+, \Gamma_0^\Box \Longrightarrow \Sigma^\Box}{\Gamma_1^+; \Gamma_2^+, \Gamma_0^\Box, \Box A^+ \Longrightarrow \Sigma^\Box}\ \Box L$$

$\Gamma_1, A, \Gamma_2, \Gamma_0 \Longrightarrow \bigvee \Sigma$ — By i.h.(i)

Case (i):

$$\frac{\Gamma_1^+; \Gamma_0^+, \Gamma_3^\Box \Longrightarrow A_1^\Box, \Sigma^\Box \quad \Gamma_1^+; \Gamma_0^+, \Gamma_3^\Box, A_2^\Box \Longrightarrow \Sigma^\Box}{\Gamma_1^+; \Gamma_0^+, A_1^\Box \supset A_2^\Box, \Gamma_3^\Box \Longrightarrow \Sigma^\Box}\ {\supset} L$$

$$
\begin{array}{ll}
\Gamma_1, \Gamma_0, A_1 \supset A_2, \Gamma_3, A_1, A_2 \Longrightarrow \bigvee \Sigma & \text{By i.h.(i) and weakening} \\
\Gamma_1, \Gamma_0, A_1 \supset A_2, \Gamma_2, A_1 \Longrightarrow A_1 & \text{By identity} \\
\Gamma_1, \Gamma_0, A_1 \supset A_2, \Gamma_3, A_1 \Longrightarrow \bigvee \Sigma & \text{By } {\supset}L \\
\Gamma_1, \Gamma_0, A_1 \supset A_2, \Gamma_3, \bigvee \Sigma \Longrightarrow \bigvee \Sigma & \text{By identity} \\
\Gamma_1, \Gamma_0, A_1 \supset A_2, \Gamma_3, A_1 \vee \bigvee \Sigma \Longrightarrow \bigvee \Sigma & \text{By } {\vee}L \\
\Gamma_1, \Gamma_0, A_1 \supset A_2, \Gamma_3 \Longrightarrow A_1 \vee \bigvee \Sigma & \text{By i.h.(i)} \\
\Gamma_1, \Gamma_0, A_1 \supset A_2, \Gamma_3 \Longrightarrow \bigvee \Sigma & \text{By cut}
\end{array}
$$

Case (ii): $A^+ = A_1^\Box \vee A_2^\Box$.

$$
\begin{array}{ll}
\Gamma_1^+; \bullet \Longrightarrow A_1^\Box, A_2^\Box & \text{By inv.(ii), with a proof of smaller or equal size} \\
\Gamma_1 \Longrightarrow A_1 \vee A_2 & \text{By i.h.(i)}
\end{array}
$$

Case (ii): $A^+ = A_1^\Box \supset A_2^\Box$.

$$
\begin{array}{ll}
\Gamma_1^+; A_1^\Box \Longrightarrow A_2^\Box & \text{By inv.(v), with proof of smaller or equal size} \\
\Gamma_1, A_1 \Longrightarrow A_2 & \text{By i.h.(i)} \\
\Gamma_1 \Longrightarrow A_1 \supset A_2 & \text{By } {\supset}R
\end{array}
$$

□

5 Classical Logic as a Fragment of Intuitionistic Logic

In the lecture so far we have demonstrated that a classical logician can view intuitionistic logic as a fragment the classical modal logic S4. In this section we will see that an intuitionist can already consider classical logic as a fragment of intuitionistic logic by a simple reinterpretation of classical formulas. In the next section we extend this to the classical case.

The basic idea is that we view a classical sequent

$$A_1, \ldots, A_n \Longrightarrow B_1, \ldots B_m$$

as establishing a contradiction from the assumption that the A_i are true and the B_j are false:

$$A_1\ \mathit{true}, \ldots A_n\ \mathit{true}, B_1\ \mathit{false}, \ldots B_m\ \mathit{false} \vdash \mathit{contra}$$

This would explain several aspects of classical sequents puzzling to an intuitionist. First, we have multiple succedents because they are just hypotheses about falsehood in disguise. Second, classical proofs don't seem to have computational contents, and this is now explained because classical proofs are really proofs of a contradiction (which doesn't have corresponding values). Of course that latter doesn't mean that there is no computational content at all to be found in classical proofs; it just means that it is not apparent by constructions as it is in the case of intuitionstic logic.

To make the intuition above precise we use an intuitionistic sequent calculus as the target of the translation. Recall the rules for negation, if taken as a primitive.

$$\frac{\Gamma, A \Longrightarrow \cdot}{\Gamma \Longrightarrow \neg A}\ \neg R \qquad \frac{\Gamma \Longrightarrow A}{\Gamma, \neg A \Longrightarrow \gamma}\ \neg L$$

whhere γ stands either for the empty succedent $(\cdot)$ or a proposition C. Also recall from Exercise 8.10 that there is a right weakening principle: If $\Gamma \Longrightarrow \cdot$ then $\Gamma \Longrightarrow C$ for any C. For the translation below we only need the case of $\gamma = (\cdot)$ for $\neg L$ and we do not need right weakening.

As much as possible we would like to interpret conjunction with conjunction, implication with implication, disjunction with disjunction, etc., so we propose a translation $(A)^o$ which distributes over all connectives and applies a second translation to the subformulas. For example $(A \wedge B)^o = A^? \wedge B^?$ and $(A \supset B)^o = A^? \wedge B^?$. What we eventually want to prove is: $\Gamma \Longrightarrow \Sigma$ iff $\Gamma^o, \neg\Sigma^o \Longrightarrow \cdot$ where the first is in classical sequent calculus and the second in intuitionistic sequent calculus.

We consider two cases to see what the translation should be.

Case:

$$\frac{\Gamma, A \Longrightarrow B, \Sigma}{\Gamma \Longrightarrow A \supset B, \Sigma}\ {\supset}R$$

We construct

$$\cfrac{\cfrac{\begin{array}{c}\text{i.h.}\\ \Gamma^o, A^o, \neg B^o, \neg\Sigma^o \Longrightarrow \cdot \\ \vdots \\ \Gamma^o, \neg(A^? \supset B^?), A^? \Longrightarrow B^?\end{array}}{\Gamma^o, \neg(A^? \supset B^?) \Longrightarrow A^? \supset B^?}\ {\supset}R}{\Gamma^o, \neg(A^? \supset B^?) \Longrightarrow \cdot}\ \neg L$$

To close this gap we need weakening (which we have), and we should have $B^? = \neg\neg B^o$, while $A^? = A^o$ would be sufficient.

Case:

$$\frac{\Gamma \Longrightarrow A, \Sigma \quad \Gamma, B \Longrightarrow \Sigma}{\Gamma, A \supset B \Longrightarrow \Sigma} \supset L$$

We construct

$$\frac{\begin{array}{c}\text{i.h.}\\ \Gamma^o, \neg A^o, \neg\Sigma^o \Longrightarrow \cdot \\ \vdots \\ \Gamma^o, A^? \supset B^?, \neg\Sigma^o \Longrightarrow A^?\end{array} \quad \begin{array}{c}\text{i.h}\\ \Gamma^o, B^o, \neg\Sigma^o \Longrightarrow \cdot \\ \vdots \\ \Gamma^o, B^?, \neg\Sigma^o \Longrightarrow \cdot\end{array}}{\Gamma^o, A^? \supset B^?, \neg\Sigma^o \Longrightarrow \cdot} \supset L$$

Here we see that $A^? = \neg\neg A^o$ and $B^? = B^o$ would be sufficient.

The insight now is that if we uniformly set $A^? = \neg\neg A^o$ we can complete both of these case (and in fact all the other cases). This is because we can infer

$$\frac{\dfrac{\Gamma^o, \neg\neg B^o, B^o, \neg\Sigma^o \Longrightarrow \cdot}{\Gamma^o, \neg\neg B^o, \neg\Sigma^o \Longrightarrow \neg B^o} \neg R}{\Gamma^o, \neg\neg B^o, \neg\Sigma^o \Longrightarrow \cdot} \neg L$$

in effect transitioning from B^o on the left to $\neg\neg B^o$ on the left in two steps when we are in the process of deriving a contradiction.

Based on this we now define:

$$\begin{array}{lcl} A^{\perp\perp} & = & \neg\neg A^o \\ \\ (P)^o & = & P \\ (A \wedge B)^o & = & A^{\perp\perp} \wedge B^{\perp\perp} \\ (\top)^o & = & \top \\ (A \supset B)^o & = & A^{\perp\perp} \supset B^{\perp\perp} \\ (A \vee B)^o & = & A^{\perp\perp} \vee B^{\perp\perp} \\ (\perp)^o & = & \perp \\ (\neg A)^o & = & \neg A^{\perp\perp} \end{array}$$

Theorem 5 (From Classical to Intuitionistic Logic) *If $\Gamma \Longrightarrow \Sigma$ in classical sequent calculus, then $\Gamma^o, \neg\Sigma^o \Longrightarrow \cdot$ in intuitionistic sequent calculus.*

Proof: By induction on the structure of the given sequent derivation. The cases for $\supset R$ and $\supset L$ follow as sketched above. We show only cases for disjunction and falsehood on the right.

Case:

$$\frac{\Gamma \Longrightarrow A, B, \Sigma_1}{\Gamma \Longrightarrow A \vee B, \Sigma_1} \vee R$$

We construct

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{i.h.}}{\Gamma^o, \neg A^o, \neg B^o, \neg \Sigma_1^o \Longrightarrow \cdot}}{\Gamma^o, \neg(A^{\perp\perp} \vee B^{\perp\perp}), \neg A^o, \neg B^o, \neg \Sigma_1^o \Longrightarrow \cdot} \text{(weakening)}}{\Gamma^o, \neg(A^{\perp\perp} \vee B^{\perp\perp}), \neg A^o, \neg \Sigma_1^o \Longrightarrow B^{\perp\perp}} \neg R}{\Gamma^o, \neg(A^{\perp\perp} \vee B^{\perp\perp}), \neg A^o, \neg \Sigma_1^o \Longrightarrow A^{\perp\perp} \vee B^{\perp\perp}} \vee R_2}{\Gamma^o, \neg(A^{\perp\perp} \vee B^{\perp\perp}), \neg A^o, \neg \Sigma_1^o \Longrightarrow \cdot} \neg L}{\Gamma^o, \neg(A^{\perp\perp} \vee B^{\perp\perp}), \neg \Sigma_1^o \Longrightarrow A^{\perp\perp}} \neg R}{\Gamma^o, \neg(A^{\perp\perp} \vee B^{\perp\perp}), \neg \Sigma_1^o \Longrightarrow A^{\perp\perp} \vee B^{\perp\perp}} \vee R_1}{\Gamma^o, \neg(A^{\perp\perp} \vee B^{\perp\perp}), \neg \Sigma_1^o \Longrightarrow \cdot} \neg L$$

Case:

$$\frac{\Gamma \Longrightarrow \Sigma_1}{\Gamma \Longrightarrow \perp, \Sigma_1} \perp R$$

We construct

$$\frac{\Gamma^o, \neg \Sigma_1^o \Longrightarrow \cdot}{\Gamma^o, \neg \perp^o, \neg \Sigma_1^o \Longrightarrow \cdot} \text{(weakening)}$$

□

For the translation in the other direction, we observe that, classically, $A^{\perp\perp} \equiv A^o \equiv A$.

Lemma 6 *In classical logical, $A^{\perp\perp} \equiv A$ and $A^o \equiv A$.*

Proof: By induction on the translation, exploiting that, classically, $\neg\neg A \equiv A$. □

Theorem 7 (From Intuitionistic Logic to Classical Logic) *If $\Gamma^o, \neg \Sigma^o \Longrightarrow \cdot$ in intuitionistic sequent calculus, then $\Gamma \Longrightarrow \Sigma$ in classical sequent calculus*

Proof: Any intuitionistic sequent proof can be translated to a classical sequent proof, possibly using some instances of contraction and weakening, but otherwise keeping the structure of the proof intact. From this we have $\Gamma^o, \neg\Sigma^o \Longrightarrow \cdot$ in classical sequent calculus. By repeated application of $\neg R$ we obtain $\Gamma^o \Longrightarrow \neg\neg\Sigma^o$ which is the same as $\Gamma^o \Longrightarrow \Sigma^{\perp\perp}$. Now we use the preceding lemma to conclude $\Gamma \Longrightarrow \Sigma$. □

The translation in this section can be extended to include the modalities. We leave the detailed development as Exercise 5.

Exercises

Exercise 1 *Prove the left and right contraction principles for the classical sequent calculus. Further, demonstrate that this principle would fail if we did not retain the principal formula in the* valid *and* poss *rules of the classical modal sequent calculus.*

Exercise 2 *We note that for classical sequent calculi there is a strong symmetry between the left- and right-hand side of a sequent, except that a conjunction on the left corresponds to disjunction on the right, and similarly for other connectives.*

State a theorem which expresses this duality succinctly for the classical modal sequent calculus and sketch the proof. You only need to show a few representative cases.

Exercise 3 *Write out the properties of weakening, contraction, cut, and identity, for the classical modal sequent calculus in Section 3.*

Present the sequence in which these properties can be proven and state the necessary induction in each case. For example, in intuitionistic sequent calculus we would say that weakening follows by induction on the structure of the given derivation, and that identity follows by induction on the structure of the proposition, etc.

Exercise 4 *Prove parts (ii) and (iv) in Lemma 3 (Inversion).*

Exercise 5 *Show how to extend Kolmogorov's double negation translation to interpret classical modal logic in intuitionistic modal logic.*

State the generalization of Theorem 5 and show the cases concerned with $\Box$, $\Diamond$ *as well as the judgmental rules* valid *and* poss.

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