Lecture Notes on Noncorrespondence

15-816: Modal Logic
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1 Introduction to This Lecture

In lecture 7, we have seen how axiomatics and semantics of modal logic fit together in soundness proofs and correspondence proofs. We have seen several examples of classes of Kripke frames that are characterized by formulas of propositional modal logic. These were several special cases. But we are looking for a general correspondence result.

Can we find a full correspondence result? For any formula of propositional modal logic, associate a class of Kripke frames that it characterizes? And for any class of Kripke frames, associate a formula of propositional modal logic that characterizes it? Certainly not! Even for Kripke frames with countable sets of worlds, classes of Kripke frames are not countable, and cannot possibly be matched with the countable set of formulas.

Let us try a more modest general correspondence between propositional modal logic and first-order definable classes of Kripke frames. For any class of Kripke frames given by a first-order formula on the frame, associate a formula of propositional modal logic that characterizes it? And for any formula of propositional modal logic, associate a class of Kripke frames—defined by a formula in first-order logic—that it characterizes?

It turns out that even that is impossible!

2 Noncorrespondence

We will prove that the class of irreflexive Kripke frames that is defined by the first-order formula $\forall x\, \neg\rho(x, x)$ cannot be characterized in propositional modal logic. For that we will find two Kripke structures that are indistinguishable in propositional modal logic, but one of them is irreflexive, while the other one is not.

Theorem 1 *The class of all irreflexive Kripke frames (defined by the first-order formula $\forall x \neg\rho(x,x)$) cannot be characterized by a formula in propositional modal logic.*

In order to prove this, we construct a "parallel universe" W^* with copy states and choose an accessibility relation ρ^* that conceals whether a state is real or part of the parallel reality, which are indistinguishable by propositional modal means.

Lemma 2 (Irreflexivization) *For Kripke structure $K = (W, \rho, v)$ we define the irreflexively split Kripke structure $K^* = (W^*, \rho^*, v^*)$ as*

$$\begin{aligned} W^* &= \{s^i \,:\, s \in W, i \in \{1,2\}\} \\ s^i\rho^*t^j &\text{ iff } s\rho t \text{ and } i,j \in \{1,2\} \qquad \text{for } s \neq t \\ s^i\rho^*s^j &\text{ iff } s\rho t \text{ and } i \neq j \\ v^*(s^i)(q) &= v^*(s)(q) \end{aligned}$$

Then, for any propositional modal logic formula ϕ we have

$$\begin{aligned} K^*, s^1 \models \phi &\quad \text{iff} \quad K, s \models \phi \\ K^*, s^2 \models \phi &\quad \text{iff} \quad K, s \models \phi \\ K^* \models \phi &\quad \text{iff} \quad K \models \phi \end{aligned}$$

Proof: The proof is by simultaneous induction on ϕ. The last equivalence is a simple consequence of the first two equivalences.

0. For propositional letters q, the equivalence is by definition of v^*.

1. The propositional operators are easy.

2. Let ϕ be of the form $\Box\psi$. For the one direction, assume $K^*, s^1 \models \Box\psi$. We have to show $K, s \models \Box\psi$. Thus consider any t with $s\rho t$. The construction of K^* ensures that $s^1\rho^*t^2$ regardless of whether $s = t$ or $s \neq t$. Thus, because of $K^*, s^1 \models \Box\psi$ we have that $K^*, t^2 \models \psi$. Now the induction hypothesis on ψ for 2 implies that $K, t \models \psi$. Yet t was arbitrary, hence $K, s \models \Box\psi$.
   For the converse direction, assume $K, s \models \Box\psi$. We want to show $K^*, s^i \models \Box\psi$ for $i \in \{1,2\}$. Consider any t^j with $s^i\rho^*t^j$. We have to show $K^*, t^j \models \psi$. We consider the two cases of the construction.

- Case $t \neq s$ and $j \in \{1, 2\}$ and $s\rho t$. Then because of $K, s \models \Box\psi$, we have $K, t \models \psi$. Hence $K^*, t^j \models \psi$ by induction hypothesis.
- Case $t = s$ and $s\rho s$. Then $i \neq j$ by construction. Because of $K, s \models \Box\psi$, we have $K, s \models \psi$. By induction hypothesis, we obtain $K^*, s^j \models \psi$ a.k.a. $K^*, t^j \models \psi$.

□

Proof: [of Theorem 1] Suppose ϕ was a formula characterizing irreflexive Kripke frames. Consider the simple-most reflexive singleton Kripke frame $(W, \rho) = (\{s\}, \rho)$ with $s\rho s$. Then (W, ρ) is reflexive, but, by construction in Lemma 2, (W^*, ρ^*) is irreflexive. We supposed that, for any truth-map v:

$$\text{for } K = (W, \rho, v) \text{ we have } K \models \phi$$

but

$$\text{for } K^* = (W^*, \rho^*, v^*) \text{ we have } K^* \not\models \phi$$

Yet this contradicts Lemma 2. □

Thus, not all first-order definable Kripke frames can be characterized in propositional modal logic. Conversely, not all propositional modal formulas characterize first-order definable Kripke frames. Recall from lecture 7:

Theorem 3 (Lecture 7) *The conjunction of the following two multimodal formulas*

$$\Box_a p \to (p \land \Box_a \Box_b p)$$
$$\Box_a (p \to \Box_b p) \to (p \to \Box_a p)$$

characterizes the set of all multimodal Kripke frames (W, ρ_a, ρ_b) such that ρ_a is the reflexive, transitive closure of ρ_b.

In particular, the above conjunction of propositional multimodal formulas is not definable in first-order logic, by a simple consequence of the compactness theorem.

Theorem 4 (Compactness) *First-order logic is* compact, *i.e.*

$$\Gamma \vDash A \quad \textit{iff} \quad \textit{there is a finite } E \subseteq \Gamma \ \ E \vDash A$$

Consequently, there can only be partial forms of general correspondence results.

3 A Simple Correspondence Result

We next consider a well-known (yet still special) class of correspondence results of general confluence. It captures most of the correspondence results we have seen so far and is essentially an n-step generalization. By $\Box^n$ we denote n nested $\Box$-operators, and by $\Diamond^n$ we abbreviate n nested $\Diamond$-operators:

$$\Box^0\phi \equiv \phi \qquad\qquad \Diamond^0\phi \equiv \phi$$
$$\Box^{n+1}\phi \equiv \Box\Box^n\phi \qquad\qquad \Diamond^{n+1}\phi \equiv \Diamond\Diamond^n\phi$$

Similarly, for the predicate ρ representing the accessibility relation, we abbreviate the n-fold composition of the relation by ρ^n:

$$\rho^n(x,y) \equiv \exists x_1 \ldots \exists x_{n-1} \big(\rho(x,x_1) \land \rho(x_2,x_3) \land \ldots \rho(x_{n-2},x_{n-1}) \land \rho(x_{n-1},y)\big)$$

Proposition 5 *For any natural numbers m, n, j, k the modal formula*

$$\Diamond^m\Box^n q \to \Box^j\Diamond^k q$$

characterizes the class of all Kripke frames satisfying

$$\forall x\,\forall y\,\forall y' \left(\rho^m(x,y) \land \rho^j(x,y') \;\to\; \exists z\,(\rho^n(y,z) \land \rho^k(y',z))\right)$$

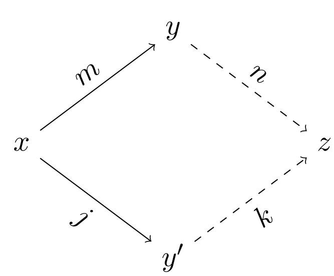

Figure 1: Principle behind general confluence property

Proof: See Fig. 1 for an illustration of the general confluence property that we abbreviate as $C(m,n,j,k)$:

$$C(m,n,j,k) \equiv \forall x\,\forall y\,\forall y' \left(\rho^m(x,y) \land \rho^j(x,y') \;\to\; \exists z\,(\rho^n(y,z) \land \rho^k(y',z))\right)$$

Consider a Kripke frame (W, ρ) with $(W, \rho) \models C(m, n, j, k)$. Let $K = (W, \rho, v)$ be any Kripke structure extending the Kripke frame with any v. Let $s \in W$ with $K, s \models \Diamond^m \Box^n q$. We have to show $K, s \models \Box^j \Diamond^k q$. From $K, s \models \Diamond^m \Box^n q$, we know that there is a $t \in W$ that is reachable from s in n steps such that $K, t \models \Box^n q$. Now consider any $t' \in W$ that is reachable from s in j steps. By translating notations, it is easy to see that $(W, \rho) \models C(m, n, j, k)$ implies that there is a $z \in W$ that is reachable from t in n steps and that is reachable from t' in k steps. Since $K, t \models \Box^n q$, we have $K, z \models q$. Hence $K, t' \models \Diamond^k q$. Thus $K, s \models \Box^j \Diamond^k q$, because t' was arbitrary. $\Box$

Exercises

Exercise 1 *Using your knowledge about first-order logic, give a simple proof of the compactness theorem.*