Lecture Notes on First-Order Reductions of First-Order Modal Logic

15-816: Modal Logic
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1 Introduction to This Lecture

In this lecture, we will consider the relationship of (first-order) modal logic and propositional modal logic to classical first-order logic [Car46, Kri63, Sch03].

2 First-Order Reductions

First-order modal logic is clearly more than classical first-order logic, because modalities can be used to phrase more advanced necessity and possibility properties. At the same time, first-order modal logic and non-modal first-order logic are not entirely different either. Modalities quantify over all ($\Box$) or some ($\Diamond$) worlds. Quantifiers also quantify over all ($\forall$) or some ($\exists$) objects (yet not usually over worlds). So all we need to do to relate first-order modal with first-order non-modal logic is to make worlds and their accessibility relation "accessible" in the object language. Indeed, quantified modal logic can be embedded in first-order modal logic in the following sense. Here we assume constant domain for simplicity.

For every first-order modal formula ϕ, we construct a reduced formula $\phi^\flat(s)$ in non-modal first-order logic with one extra free variable s. To every signature Σ of first-order modal logic, we associate a reduced signature $\Sigma^\flat$. Furthermore, to every first-order Kripke structure K (we assume constant domain), we define a corresponding first-order structure $K^\flat$ of first-order logic. That is we embed first-order modal logic into first-order logic by a reduction:

$$\begin{array}{lrcl} & \text{FOML} & \rightarrow & \text{FOL} \\ \text{signature} & \Sigma & \mapsto & \Sigma^\flat \\ \text{formula} & \phi & \mapsto & \phi^\flat(s) \\ \text{interpretation} & K & \mapsto & K^\flat \end{array}$$

Well so much associating is a good start, but the interesting property is that the reduction preserves truth! For any world $s \in W$ of K:

$$K, s \models \phi \quad \text{iff} \quad K^\flat \models \phi^\flat(s)$$

and $\phi^\flat(s)$ is a first-order formula over the signature $\Sigma^\flat$ when ϕ is a first-order modal formula over the signature Σ. Let us build the proper reductions that satisfy this equivalence. Note that we mostly ignore function symbols here and discuss these in an exercise.

In modal logic, truth depends on the current world, which it does not in non-modal logic. In order to capture the different, world-dependent truth-values, we thus add an extra argument s (as argument number 0) to all predicate symbol that is intended to capture a symbolic dependency on the world. We also add a predicate symbol $r/2$ for the internalization of the accessibility relation and $w/1$ to single out worlds from among the objects. Thus we define the reduced signature as:

$$\Sigma^\flat := \{r/2, w/1\} \cup \{p/(n+1) : p \in \Sigma \text{ predicate}\} \cup \{f \in \Sigma : f \text{ function}\}$$

The *reduced formula* $\phi^\flat(s)$ is obtained from ϕ essentially by adding s as a first argument to all predicate symbols and by translating modalities into quantifiers:

$$\begin{array}{rcll} (p(\theta_1, \ldots, \theta_n))^\flat & = & p(s, \theta_1, \ldots, \theta_n) \wedge w(s) & \\ (\phi \wedge \psi)^\flat & = & \phi^\flat \wedge \psi^\flat & \\ (\phi \vee \psi)^\flat & = & \phi^\flat \vee \psi^\flat & \\ (\neg\phi)^\flat & = & \neg\phi^\flat & \\ (\Box\phi)^\flat & = & \forall z\, (r(s,z) \rightarrow {\phi^\flat}_s^z) & \text{where } z \text{ is new} \\ (\Diamond\phi)^\flat & = & \exists z\, (r(s,z) \wedge {\phi^\flat}_s^z) & \text{where } z \text{ is new} \\ (\forall x\, \phi)^\flat & = & \forall x\, (\neg w(x) \rightarrow \phi^\flat) & \\ (\exists x\, \phi)^\flat & = & \exists x\, (\neg w(x) \wedge \phi^\flat) & \end{array}$$

The translation of $\phi \equiv \Diamond\Box q \rightarrow \Box\Diamond q$ is the formula $\phi^\flat(s)$:

$$\exists z\, \big(r(s,z) \wedge \forall y\, (r(z,y) \rightarrow q(y) \wedge w(y))\big) \rightarrow \forall z\, \big(r(s,z) \rightarrow \exists y\, (r(z,y) \rightarrow q(y) \wedge w(y))\big)$$

This formula has one extra free variable s.

The translation of $\phi \equiv \Diamond\exists x\, q(x) \rightarrow \exists x\, \Diamond q(x)$ is the following formula $\phi^\flat(s)$, which is provable:

$$\exists y\, \big(r(s,y) \wedge \exists x\, (q(y,x) \wedge w(y))\big) \rightarrow \exists x\, \exists y\, \big(r(s,y) \wedge q(y,x) \wedge w(y)\big)$$

So far, we have transformed syntax but destroyed meaning. How can we connect ϕ with its reduction $\phi^\flat$? We have a syntactical connection, but no semantical connection yet. Let us go for a semantical relation next and define $K^\flat$ for K.

Given a Kripke structure $K = (W, \rho, M)$, the corresponding *reduced first-order structure* $K^\flat$ is defined as follows: The domain of $K^\flat$ is the disjoint union $D \dot{\cup} W$ of the domain D of K and the worlds W of K. The interpretation $K^\flat(f)$ of any function symbol f is defined as in K on D and is an arbitrary element of W outside D. The interpretation of predicate symbol q of arity n is:

$$K^\flat(q) := \{(s, a_1, \dots, a_n) : M(s) \models p(a_1, \dots, a_n)\}$$

The interpretation of the special predicate symbol r is $K^\flat(r) = \rho$ and the interpretation of the special predicate symbol w is $K^\flat(r) = W$.

With this reduction of the formulas and the Kripke structures to first-order logic, we can show that the semantics is equivalent in the following sense.

Lemma 1 *Let $\phi(x_1, \dots, x_n)$ be a modal formula with n free variables $x_1, \dots, x_n$. For any Kripke structure K, any $s \in W$, and any $a_1, \dots, a_n$ in the domain of K, we have that*

$$K, s \models \phi(a_1, \dots, a_n) \quad \textit{iff} \quad K^\flat \models \phi^\flat(s, a_1, \dots, a_n)$$

Proof: The proof is by structural induction on ϕ.

0. If $\phi(x_1, \dots, x_n)$ is of the form $p(x_1, \dots, x_n)$ for a predicate symbol p, then the claim is just the definition of $K^\flat$.

1. If $\phi(x_1, \dots, x_n)$ is of the form $\phi_1(x_1, \dots, x_n) \vee \phi_2(x_1, \dots, x_n)$, the proof is obvious.

2. If $\phi(x_1, \dots, x_n)$ is of the form $\Diamond\psi(x_1, \dots, x_n)$, then we consider both directions. Assume $K, s \models \Diamond\psi(a_1, \dots, a_n)$. Then, there is a state t

with $s\rho t$ such that $K, t \models \psi(a_1, \ldots, a_n)$. Thus, by induction hypothesis,

$$K^\flat \models \psi^\flat(t, a_1, \ldots, a_n)$$

By the choice of the interpretation of r as $K^\flat r = \rho$ we have $K^\flat \models r(s, t)$. Thus $K^\flat \models r(s, t) \land \psi^\flat(t, a_1, \ldots, a_n)$ which implies

$$K^\flat \models \exists z \left(r(s, z) \land \psi^\flat(z, a_1, \ldots, a_n)\right)$$

i.e., $K^\flat \models \phi^\flat(a_1, \ldots, a_n)$.
Conversely, assume $K^\flat \models (\lozenge\phi)^\flat(s, a_1, \ldots, a_n)$. Thus

$$K^\flat \models \exists z \left(r(s, z) \land \psi^\flat(z, a_1, \ldots, a_n)\right)$$

Hence, there is an object a_0 in the universe of $K^\flat$ with $K^\flat \models r(s, a_0)$ and $K^\flat \models \psi^\flat(a_0, a_1, \ldots, a_n)$. By the choice of the semantics of r as $K^\flat r = \rho$ in $K^\flat$, this implies that $a_0 \in W$. By induction hypothesis, this implies that $K, a_0 \models \psi(a_1, \ldots, a_n)$. By the choice of r as $K^\flat r = \rho$ we also obtain $s\rho a_0$ from $K^\flat \models r(s, a_0)$. Consequently,

$$K, s \models \lozenge\psi(a_1, \ldots, a_n)$$

3. If $\phi(x_1, \ldots, x_n)$ is of the form $\exists x_0 \psi(x_0, x_1, \ldots, x_n)$, then we consider both directions. Assume $K, s \models \exists x_0 \psi(x_0, a_1, \ldots, a_n)$. Then there is an object a_0 in the universe of K such that $K, s \models \psi(a_0, a_1, \ldots, a_n)$. By induction hypothesis, we have $K^\flat \models (\psi(a_0, a_1, \ldots, a_n))^\flat$. With the domain of $K^\flat$ being a disjoint union, we know that $K^\flat \models \neg w(a_0)$ and we know that a_0 also is an object in the universe of $K^\flat$. Thus, we have $K^\flat \models \neg w(a_0) \land (\psi(a_0, a_1, \ldots, a_n))^\flat$, which implies

   $$K^\flat \models \exists x_0 \left(\neg w(x_0) \land (\psi(x_0, a_1, \ldots, a_n))^\flat\right)$$

   Hence $K^\flat \models \phi^\flat$.
   Conversely, assume $K^\flat \models \phi^\flat(s, a_1, \ldots, a_n)$, which, for the particular quantified formula at hand gives

   $$K^\flat \models \exists x_0 \big(\neg w(x_0) \land (\psi(x_0, a_1, \ldots, a_n))^\flat\big)$$

   Hence, there is an object a_0 in the universe of $K^\flat$ such that

   $$K^\flat \models \neg w(a_0) \land (\psi(a_0, a_1, \ldots, a_n))^\flat$$

   Thus, with $K^\flat \models (\psi(a_0, a_1, \ldots, a_n))^\flat$ the induction hypothesis implies $K, s \models \psi(a_0, a_1, \ldots, a_n)$. Because of $K^\flat \models \neg w(a_0)$, we know that a_0 also is an object in the universe of K, hence $K, s \models \exists x_0 \psi(x_0, a_1, \ldots, a_n)$.

$\square$

Theorem 2 (Reduction) *Let ψ be a quantified modal formula and Φ a set of quantified modal formulas without free variables. Then*

$$\Gamma_{\boldsymbol{K}} \cup \{\forall s\,(w(s) \to \phi^\flat(s)) \;:\; \phi \in \Phi\} \vDash^{FOL} \forall s\,(w(s) \to \psi^\flat(s)) \quad \textit{iff} \quad \Phi \vDash_g^{\boldsymbol{K}} \psi$$

Proof: Let us first prove the direction from left to right. Let K be a Kripke structure and $K \models \Phi$. We have to show that $K \models \psi$. First of all, we know $K \models \Gamma_{\mathbf{K}}$ by Lemma 3. Because of $K \models \Phi$, i.e., $K, s \models \Phi$ for all $s \in W$ and all $\phi \in \Phi$, Lemma 1 implies that $K^\flat \models \phi^\flat(s)$ for all $s \in W$ and all $\phi \in \Phi$. In particular, by the interpretation of w as $K^\flat(w) = W$ we have $K^\flat \models \forall s\,(w(s) \to \phi^\flat(s))$ for all $\phi \in \Phi$. Consequently, the assumptions imply $K^\flat \models \forall s\,(w(s) \to \psi^\flat(s))$. Consider any state $s \in W$. Because W is a subset of the domain of $K^\flat$, we find $K^\flat \models w(s) \to \psi^\flat(s)$. With $K^\flat(w) = W$ this implies $K^\flat \models \psi^\flat(s)$. Using Lemma 1, we thus obtain $K, s \models \psi$ as intended.

Conversely, consider the direction from right to left. Let I be an interpretation with $I \models \Gamma_{\mathbf{K}}$ and $I \models \forall s\,(w(s) \to \phi^\flat(s))$ for all $\phi \in \Phi$. We have to show that $I \models \forall s\,(w(s) \to \psi^\flat(s))$. First of all, we know that there is a Kripke structure K with $K^\flat = I$, because of $I \models \Gamma_{\mathbf{K}}$ and Lemma 3. Now because of $K^\flat \models \forall s\,(w(s) \to \phi^\flat(s))$, we have that $K, s \models \phi$ for all $\phi \in \Phi$ and all $s \in W$ by Lemma 1, similar to the reasoning above. Consequently, because of $\Phi \vDash_g^{\mathbf{K}} \psi$, we have for all $s \in W$, that $K, s \models \psi$. Then Lemma 1 implies that $K^\flat \models \psi^\flat(s)$ for all $s \in W$. Therefore, $K^\flat \models \forall s\,(w(s) \to \psi^\flat(s))$, because $K^\flat(w) = W$. $\square$

Lemma 3 *Let $\Gamma_{\mathbf{K}}$ be the following set of formulas in first-order logic:*

$$\begin{aligned}
&\{\exists s\, w(s)\}\\
&\cup \{\forall s\,\forall t\,\big(r(s,t) \to w(s) \land s(t)\big)\}\\
&\cup \{\forall s\,\forall x_1 \dots \forall x_n\,\big(p(s, x_1, \dots, x_n) \to w(s) \land \neg w(x_1) \land \neg w(x_n)\big) \;:\; p/n \in \Sigma\}\\
&\cup \{\forall x_1 \dots \forall x_n\,\big(\neg w(x_1) \land \dots \land \neg w(x_n) \to \neg w(f(x_1, \dots, x_n))\big) \;:\; f/n \in \Sigma\}\\
&\cup \{\forall x_1 \dots \forall x_n\,\big(w(x_1) \lor \dots \lor w(x_n) \to f(x_1, \dots, x_n) = c \land w(c)\big) \;:\; f/n \in \Sigma\}
\end{aligned}$$

where c is a fresh constant symbol. Then for any interpretation I:

$$I \models \Gamma_{\mathbf{K}} \quad \textit{iff} \quad \textit{there is a Kripke structure } K \textit{ such that } K^\flat = I$$

Exercises

Exercise 1 *There is an imprecision in the definition of $K^\flat$. What is the problem, why is it a problem, and what can be done to fix it?*

Exercise 2 *We did not handle function symbols in Section 2. Did we forget them? What do we need to do to fix this?*

References

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