Lecture Notes on
Modal Tableaux

15-816: Modal Logic
André Platzer

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1 Introduction to This Lecture

The Hilbert calculus for modal logic from the last lectures is incredibly simple, but it is not entirely simple to find a proof in it. In this lecture, we introduce a modal tableau calculus that is more amenable to systematic proof construction and automated theorem proving.

Tableaux calculi for modal logic can be found in the work of Fitting [Fit83, Fit88] and the manuscript by Schmitt [Sch03].

2 The Petite Modal Zoo

In previous lectures, we have mainly seen the propositional modal logic **S4** and its Hilbert-style axiomatization. This is, by far, not the only modal logic of interest. The minimal (normal) modal logic is modal logic **K**. The axiomatisation of **K** is a subset of the axioms of **S4** and the same proof rules of **S4**; see Figure 1. In fact, normal modal logics share the same proof rules (MP and G) and mostly differ in the choice of axioms.

Extensions of logic **K** are shown in Figure 2.

3 Modal Tableaux

For proving formulas in propositional modal logic, we develop a tableau calculus. Tableaux often give very intuitive proof calculi. Here we choose prefix tableaux, where every formula on the tableau has a *prefix* σ, which

(P)	all propositional tautologies
(K)	$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$
(MP)	$\dfrac{\phi \quad \phi \to \psi}{\psi}$
(G)	$\dfrac{\phi}{\Box\phi}$

Figure 1: Modal logic **K**

T	is system **K** plus	(T)	$\Box\phi \to \phi$
S4	is system **T** plus	(4)	$\Box\phi \to \Box\Box\phi$

Figure 2: Some other modal logics

is a finite sequence of natural numbers. In addition, every formula on the tableau has a *sign* $Z \in \{F, T\}$ that indicates the truth-value we currently expect for the formula in our reasoning. That is, a formula in the modal tableaux is of the form

$$\sigma Z A$$

where the prefix σ is a finite sequence of natural numbers, the sign Z is in $\{F, T\}$, and F is a formula of modal logic. At this point, we understand a prefix σ as a symbolic name for a world in a Kripke structure.

Definition 1 (K prefix accessibility) *For modal logic **K**, prefix σ' is* accessible *from prefix σ if σ' is of the form σn for some natural number n.*

For every formula of a class α with a top level operator and sign (T or F for true and false) as indicated, we define two successor formulas α_1 and α_2:

α	α_1	α_2
$TA \wedge B$	TA	TB
$FA \vee B$	FA	FB
$FA \to B$	TA	FB
$F\neg A$	TA	TA

β	β_1	β_2
$TA \vee B$	TA	TB
$FA \wedge B$	FA	FB
$TA \to B$	FA	TB
$T\neg A$	FA	FA

For the following cases of formulas we define one successor formula

ν	ν_0
$T\Box A$	TA
$F\Diamond A$	FA

π	π_0
$T\Diamond A$	TA
$F\Box A$	FA

Every combination of top-level operator and sign occurs in one of the above cases. Tableau proof rules by those classes are shown in Figure 3. A tableau is *closed* if every branch contains some pair of formulas of the form σTA and σFA. A *proof* for modal logic formula consists of a closed tableau starting with the root $1FA$.

$$(\alpha)\ \frac{\sigma\alpha}{\begin{array}{c}\sigma\alpha_1\\ \sigma\alpha_2\end{array}} \qquad (\beta)\ \frac{\sigma\beta}{\sigma\beta_1 \quad \sigma\beta_2} \qquad (\nu^*)\ \frac{\sigma\nu}{\sigma'\nu_0}\,{}^{1} \qquad (\pi)\ \frac{\sigma\pi}{\sigma'\pi_0}\,{}^{2}$$

[1] σ' accessible from σ and σ' occurs on the branch already

[2] σ' is a simple unrestricted extension of σ, i.e., σ' is accessible from σ and no other prefix on the branch starts with σ'

Figure 3: Tableau proof rules for QML

The tableau rules can also be used to analyze $F\Box A \to \Diamond A$ as follows:

1	$F\Box A \to \Diamond A$	(1)
1	$T\Box A$	(2) from 1
1	$F\Diamond A$	(3) from 1
	stop	

No more proof rules can be used because the modal formulas are ν rules, which are only applicable for accessible prefixes that already occur on the branch. If we drop this restriction, we can continue to prove and close the tableau:

1	$F\Box A \to \Diamond A$	(1)
1	$T\Box A$	(2) from 1
1	$F\Diamond A$	(3) from 1
1.1	TA	(4) from 2
1.1	FA	(5) from 3
	$*$	

But this is bad news, because the formula $\Box A \to \Diamond A$ that we set out to prove in the first place is not even valid in **K**. Consequently, the side condition on the ν rule is necessary for soundness!

As an example proof in **K**-tableaux we prove $\Box A \land \Box B) \to \Box(A \land B)$:

1	$F(\Box A \wedge \Box B) \to \Box(A \wedge B)$	(1)
1	$T\Box A \wedge \Box B$	(2) from 1
1	$F\Box(A \wedge B)$	(3) from 1
1	$T\Box A$	(4) from 2
1	$T\Box B$	(5) from 2
1.1	$FA \wedge B$	(6) from 3

1.1	FA	(7) from 6	1.1	FB	(8) from 6
1.1	TA	(9) from 4	1.1	TB	(10) from 5
	$*$	7 and 9		$*$	10 and 8

Let us prove the converse $\Box(A \wedge B) \to (\Box A \wedge \Box B)$ in **K**-tableaux:

1	$F\Box(A \wedge B) \to (\Box A \wedge \Box B)$	(1)
1	$T\Box(A \wedge B)$	(2) from 1
1	$F\Box A \wedge \Box B$	(3) from 1

1	$F\Box A$	(4) from 3	1	$F\Box B$	(5) from 3
1.1	FA	(6) from 4	1.1	FB	(10) from 5
1.1	$TA \wedge B$	(7) from 2	1.1	$TA \wedge B$	(11) from 2
1.1	TA	(8) from 7	1.1	TA	(12) from 11
1.1	TB	(9) from 7	1.1	TB	(13) from 11
	$*$	6 and 8		$*$	10 and 13

Let us try to prove $\Box(A \vee B) \to \Box A \vee \Box B$:

1	$F\Box(A \vee B) \to \Box A \vee \Box B$	(1)
1	$T\Box(A \vee B)$	(2) from 1
1	$F\Box A \vee \Box B$	(3) from 1
1	$F\Box A$	(4) from 3
1	$F\Box B$	(5) from 3
1.1	FA	(6) from 4
1.2	FB	(7) from 5
1.1	$TA \vee B$	(8) from 2
1.2	$TA \vee B$	(9) from 2

1.1 TA (10) from 8	1.1 TB (11) from 8	1.2 TA (12) from 9	1.2 TB (13) from 9
$*$ 10 and 6	open	open	$*$ 13 and 7

This tableau does not close but remains open, which is good news because the formula we set out to prove is not valid in **K**.

Exercises

Exercise 1 *Prove or disprove using modal tableaux:* $\Diamond(A \land B) \to \Diamond A \land \Diamond B$.

Exercise 2 *Are the side conditions on the prefixes for the* ν^**-rule and the* π*-rule necessary or not? Prove or disprove each case.*

Exercise 3 *Use a tableau procedure to prove or disprove the formulas*

$$\Box A \to \Box(\Box A \lor B)$$

and

$$\Box\Box A \leftrightarrow \Box A$$

in the modal logic ***S4****. Explain your solution.*

Exercise 4 *Use a tableau procedure to prove or disprove the formula*

$$\Box\Diamond A \to \Diamond\Box A$$

in the modal logic ***S4****. Explain your solution and which difficulties exist in comparison to classical propositional cases.*

References

[Fit83] Melvin Fitting. *Proof Methods for Modal and Intuitionistic Logic*. Reidel, 1983.

[Fit88] Melvin Fitting. First-order modal tableaux. *J. Autom. Reasoning*, 4(2):191–213, 1988.

[Sch03] Peter H. Schmitt. Nichtklassische Logiken. Vorlesungsskriptum Fakultät für Informatik , Universität Karlsruhe, 2003.