Lecture Notes on Classical Modal Logic

15-816: Modal Logic
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1 Introduction to This Lecture

The goal of this lecture is to develop a starting point for classical modal logic.

Classical logic studies formulas that are true (especially those that are true in all interpretations, i.e., valid) and how truth is preserved in reasoning such that true premisses only have true consequences. These valid formulas are characterized semantically as those that are true in all interpretations I. We write $I \models A$ if formula A holds in interpretation I. We just write $\vDash A$ if formula A holds in all interpretations and say that A is valid. The most crucial criterion in classical logic is that logical reasoning from valid assumptions should only lead us to valid conclusions for otherwise there is something wrong with the reasoning schemes. Formulas of the form $A \vee \neg A$ are always trivially valid in classical logic, because each interpretation I satisfies either $I \models A$ or $I \models \neg A$. Consequently, $\vDash A \vee \neg A$.

Intuitionistic logic takes a more fine-grained view at logic and studies formulas that are justified (by some argument) and how justification is preserved in reasoning. If there is a proof of A, intuitionistic logic would accept both A and $A \vee \neg A$, but not without such a justification of either of the two disjuncts. Thus, intuitionistic logic has a more fine-grained view than just true/false signified in the classical axiom $\phi \vee \neg\phi$ or law of excluded middle.

To some extent, modal logics also take a more fine-grained view. Classical model logics do not dispose off the law of excluded middle, though.

They still accept the axiom $\phi \vee \neg\phi$ and are a perfectly conservative extension of basic classical logic (they do not accept less formulas as valid than classical propositional logic). But they allow distinctions between modes of truth, i.e., between formulas that are true, necessarily true, possibly true, possibly false, false. In fact, this similarity of intuitionistic logic and classical modal logic is not a mere coincidence, but can be made formally precise by a translation of intuitionistic logic into classical modal logic where, obviously, the new concept of necessity plays an important role.

The formal study of modal logic was founded by C. I. Lewis [Lew18]. Modal logic is an area with numerous results. As an excellent background on modal logic, these notes are also partly based on a manuscript by Schmitt [Sch03] and the book by Hughes and Cresswell [HC96]. Further background on modal logic can be found in the book by Fitting and Mendelsohn [FM99]. Further material on the connections of modal an intuitionistic logic can be found in [Fit83].

2 The Power of Knowledge in a Logic of Knowledge

Classical modal logics come in multifarious styles and variations. Here we first introduce the classical propositional modal logic S4 and study variations later. We first follow an axiomatic approach to classical modal logic and save the model-theoretic approach due to Kripke [Kri63] for later in this course.

We start with an informal introduction and consider a well-known puzzle:

> Three wise men are told to stand in a circle. A hat is put in each of their heads. The hats are either red or black and everyone knows that there is at least one black hat. Every wise man can see the color of the other hats except his own. They are asked to deduce the color of their own hat without cheating with a mirror or something of that sort. After some time went by, one of the wise men says: "I don't know which hat I have." With some more thought, another wise man says: "I don't know mine either." "Then I know that my hat is black." says the third one.

The solution to this puzzle is a matter of knowledge, not just a matter of truth. After the first wise man admits he doesn't know the color of his hat, the third wise man can conclude that wise men 2 and 3 cannot possibly both have had red hats on, otherwise the first white man would have seen that and concluded that he must wear a black hat. The third wise man also knows that the second wise man will be able to do the same reasoning and know the same for he is wise. But once the second wise man admits he doesn't know the color of his hat either, the third wise man is now sure not to wear a red hat.

Let us use the following propositional variables for $i \in \{1,2,3\}$:

B_i wise man i wears a black hat
R_i wise man i wears a red hat

We use a formula $\Box_i \phi$ to say that wise man number i knows that formula ϕ holds true. Formula $\Box_i \phi$ clearly represents something else than ϕ being true for ϕ might still be true, but wise man i may just not know that. The operator $\Box_i$ is what we call a *modality*.

What would we want to allow as valid reasoning schemes in a logic of knowledge? Certainly, we want to allow all classical propositional reasoning, because our analysis is allowed to use all logical reasoning that we know about in classical logic already. Modal instances of propositional tautologies are perfectly acceptable. We want to accept $\Box_i A \to B \lor \Box_i A$ and $\Box_i A \lor \neg\Box_i A$, for instance. But what kind of reasoning with the knowledge (or modalities) themselves do we admit?

The wise men know about all basic facts. This includes all tautologies and all basic rules of the hat game, for instance, that there is at least one black hat. We thus allow the proof rule called *generalization rule*:

$$\text{(G)} \quad \frac{\phi}{\Box \phi}$$

We write this proof rule with modality $\Box$ in uni-modal logic. In the knowledge logic case, we allow to use it for any instance where $\Box$ is replaced by one of the $\Box_i$.

The wise men are truly wise and can draw conclusions. If a wise man knows that he must wear a black hat if he doesn't wear a red hat, and if he knows that he does not wear a red hat, then he also knows that he wears a black hat, because he is able to draw this conclusion. This is an instance of the Kripke axiom:

$$\text{(K)} \quad (\Box\phi \land \Box(\phi \to \psi)) \to \Box\psi$$

If wise man i knows ϕ and he knows that ϕ implies ψ, then he also knows consequence ψ, otherwise he wouldn't be called a particularly wise man.

This axiom is often stated in the following elegant form, which is easily obtained by propositional equivalences:

$$\text{(K)} \quad \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$$

From these rules and axioms, we can easily derive the following rules with G,K and modus ponens.

$$\text{(I)} \quad \frac{\phi \to \psi}{\Box\phi \to \Box\psi}$$

$$\text{(E)} \quad \frac{\phi \leftrightarrow \psi}{\Box\phi \leftrightarrow \Box\psi}$$

An extremely useful axiom that we can derive from the previous rules and axioms is

$$(\Box\wedge) \quad \Box\phi \wedge \Box\psi \;\leftrightarrow\; \Box(\phi \wedge \psi)$$

The derivation is slightly more involved.

Now we know how to reason about knowledge, or at least have one way of reasoning about who knows what, consider the wise men with hats. The general facts from the puzzle are facts like $B_1 \vee B_2 \vee B_3$ because there is at least one black hat, and $B_1 \to \Box_2 B_1 \wedge \Box_3 B_1$, because wise men 2 and 3 see and know if the first wise man wears a black hat. Moreover, the third wise man knows that the second wise man can see the color of the first wise guy and will know if it's black: $B_1 \to \Box_3\Box_2 B_1$. Most importantly, the fact that the first two wise men admit they do not know anything contributes to what the third wise man will know. So we have:

$$\neg\Box_1 B_1 \tag{1}$$
$$\neg\Box_2 B_2 \tag{2}$$
$$\neg\Box_1 R_1$$
$$\neg\Box_2 R_2$$

From this the fact $\Box_3 B_3$ can be derived.

Although not necessary for solving the particular puzzle about the hats, there are two more axioms that can make sense in a knowledge context:

$$\text{(T)} \quad \Box\phi \to \phi$$

$$\text{(4)} \quad \Box\phi \to \Box\Box\phi$$

The first axiom T makes sense, because the wise men are wise: they should only know things that are actually true. If the third wise man knows that he has a black hat, then he should actually be wearing a black hat, otherwise he is not very wise. So T could be called the "wise men only know what's true" axiom. When modeling belief rather than knowledge or when modeling faulty knowledge, T will be dropped.

Axiom 4, instead, says that there is no passive knowledge. If a wise man knows something, then he also knows that he knows it, and will not say later on "oh I knew that but I just didn't know I knew it". Thus axiom 4 represents an assumption on perfect and flawless knowledge and introspection. The logic S4, for instance, is a classical modal logic with the axioms K,T,4 and rules G, modus ponens and all propositional tautologies.

The meaning of $\Box$ set forth in this section is that of epistemic modal logic in a logic of knowledge. Formula $\Box\phi$ is taken to mean that some entity "knows ϕ".

3 Classical Propositional Modal Logic

Let Σ be a set of propositional letters or atomic propositions. The syntax of classical propositional modal logic is defined as follows:

Definition 1 (Propositional modal formulas) *The set* $\mathrm{Fml}_{PML}(\Sigma)$ *of* formulas of classical propositional modal logic *is the smallest set with:*

- *If* $A \in \Sigma$ *is a propositional letter, then* $A \in \mathrm{Fml}_{PML}(\Sigma)$.
- *If* $\phi, \psi \in \mathrm{Fml}_{PML}(\Sigma)$, *then* $\neg\phi, (\phi \wedge \psi), (\phi \vee \psi), (\phi \rightarrow \psi) \in \mathrm{Fml}_{PML}(\Sigma)$.
- *If* $\phi \in \mathrm{Fml}_{PML}(\Sigma)$ *and* $x \in V$, *then* $(\Box\phi), (\Diamond\phi) \in \mathrm{Fml}_{PML}(\Sigma)$.

The informal meaning of $\Box\phi$ would be that ϕ is necessary (holds in all possible worlds). Formula $\Diamond\phi$, instead, would mean that ϕ is possible (holds in some possible world). This is the alethic meaning of $\Box$, where $\Box\phi$ is taken to mean that ϕ is necessary.

For reference, Figure 1 summarizes the axioms and rules we have identified for modal logic so far: The Kripke axiom K, the T axiom, the 4 axiom, modus (ponendo) ponens MP, and the Gödel or necessitation rule G. Note, however, that there are many different variations of modal logic.

We say that a formula ψ is provable or derivable from a set of formulas if there is a Hilbert-style proof:

(P)	all propositional tautologies
(K)	$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$
(T)	$\Box\phi \to \phi$
(4)	$\Box\phi \to \Box\Box\phi$
(MP)	$\dfrac{\phi \quad \phi \to \psi}{\psi}$
(G)	$\dfrac{\phi}{\Box\phi}$

Figure 1: Modal logic **S4**

Definition 2 (Provability) *Let **S** be a system of modal logic, i.e., a set of proof rules (including axioms) like, e.g., **S4**. For a formula ψ and a set of formulas Φ, we write $\Phi \vdash_{\mathbf{S}} \psi$ and say that ψ can be derived from Φ (or is provable from Φ), iff there is a proof of ψ that uses only the formulas of Φ and the axioms and proof rules of **S**. That is, we define $\Phi \vdash_{\mathbf{S}} \psi$ inductively as:*

$$\Phi \vdash_{\mathbf{S}} \psi$$

iff $\psi \in \Phi$ or there is an instance

$$\frac{\phi_1 \quad \dots \quad \phi_n}{\psi}$$

*of a proof rule of **S** with conclusion ψ and some number $n \geq 0$ of premisses such that for all $i = 1, \dots, n$, the premiss ϕ_i is derivable, i.e.:*

$$\Phi \vdash_{\mathbf{S}} \phi_i$$

Note that the case $n = 0$ is permitted, which corresponds to axioms.

4 Gödel Translation

Intuitionistic logic takes a more fine-grained view than classical truth or false with its law of excluded middle or tertium-non-datur. In classical (two-valued) logic, where the central constructions are about truth and preservation of truth, every formula is either true or false in a given interpretation. In particular, $A \vee \neg A$ is a classical tautology for A either has to be true or false.

In intuitionistic logic, the central constructions are about justification and preservation of justification. For the formula $A \vee \neg A$, there is (usually) no justification of A, nor a justification of $\neg A$. The law of excluded middle is thus not accepted.

In the realm of modal logics, however, there is a way to understand intuitionistic logic in a modal setting. After all, modal logic also takes a more fine-grained view of modes of truth.

The intuition behind understanding intuitionistic logic in a classical setting is to identify intuitionistic truth (being justified) with classical provability. The Gödel translation $\mathcal{G}$ maps formulas of intuitionistic logic to modal logic by prefixing all formulas with the modality $\Box$, which is understood as "provable". This translation $\mathcal{G}$ is defined inductively:

$$\begin{aligned}
\mathcal{G}(a) &= \Box a && \text{if } a \in \Sigma \text{ is a propositional letter}\\
\mathcal{G}(\phi \supset \psi) &= \Box\big(\mathcal{G}(\phi) \to \mathcal{G}(\psi)\big)\\
\mathcal{G}(\phi \wedge \psi) &= \Box\big(\mathcal{G}(\phi) \wedge \mathcal{G}(\psi)\big)\\
\mathcal{G}(\phi \vee \psi) &= \Box\big(\mathcal{G}(\phi) \vee \mathcal{G}(\psi)\big)
\end{aligned}$$

Translation $\mathcal{G}$ captures the idea that we would accept a in an intuitionistic setting if a is provable. Likewise, we would accept an intuitionistic implication $\phi \supset \psi$ if the (translated) implication $\mathcal{G}(\phi) \to \mathcal{G}(\psi)$ is provable.

The question is, if there is a way to characterize the formulas obtained by Gödel translation $\mathcal{G}$ from provable formulas of intuitionistic logic. In fact, it turns out that an intuitionistic propositional formula is provable (intuitionistically) if and only if its translation is provable in propositional modal logic, provided that we have the right set of axioms. What properties should $\Box$ satisfy for a provability interpretation?

With the provability interpretation for the Gödel translation, we expect that the K axiom makes sense. If $\phi \to \psi$ is provable, and ϕ is provable, then we should be able to glue their proofs together to a proof of ψ:

$$(\text{K}) \quad \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$$

Moreover, we expect to be able to prove only properties that are actually true, otherwise we would not venture to call it a proof. Thus if ϕ is provable, it should be true:

$$(\text{T}) \quad \Box\phi \to \phi$$

If a formula is provable, then it should be provable that it is provable, for the proof itself already is a very good proof of provability. If ϕ is provable, then it should be provably provable:

$$(4) \quad \Box\phi \to \Box\Box\phi$$

Provability is a rational notion, so we expect the notion to be closed both under arbitrary propositional inferences and the modus ponens. After all, these only glue together proofs:

$$(\text{MP}) \quad \frac{\phi \quad \phi \to \psi}{\psi}$$

Finally, if we have proven any formula ϕ, then it should be provable, for otherwise, we would not call it proven:

$$(\text{G}) \quad \frac{\phi}{\Box\phi}$$

In summary, the axioms and rules we need in this provability interpretation of $\Box$ directly coincide with those of the modal logic **S4**, i.e., Figure 1.

In fact, it can be shown that an intuitionistic formula F is provable in intuitionistic logic if and only if their translation $\mathcal{G}F$ is provable in **S4**. The proof of this statement requires more techniques than we have at this stage of the lectures.

5 Kripke Structures

Another introduction to modal logic follows transition systems and finite automata.

Consider the example of a transition structure in Figure 2. The names of the state are not of relevance to us here, only what values two signals or internal state variables have in these states. We consider those state variables as propositional variables p and q. Their actual values in the respective states of the transition system are as indicated in Figure 2. For this transition system, we want to express that p is false in all successor states of a state in which both p and q are true. Likewise, p is still false in all successors of all successors of states in which p and q are true. This property does not generalize to all third successor states though. Similarly, if p and q are both false, then p is true in all successor states.

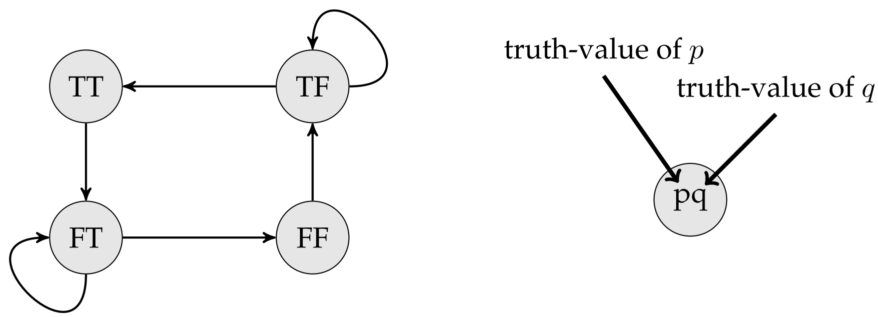

Figure 2: A transition system

In order to formalize these properties, propositional logic is not quite accurate, because it is not only important what is true and false, but also in which states something is true and false. In addition, the notion of successor states or a means to refer to them does not exist in propositional logic. Now consider the modality $\Box$ with the intended semantics being that $\Box\phi$ holds true in a state, if ϕ holds true in all of its successors. The modality $\Diamond$ would be taken to mean that $\Diamond\phi$ holds true in a state, if ϕ holds true in at least one of its successors. Then we can phrase the above properties quite naturally:

$$
\begin{aligned}
p \land q &\rightarrow \Box\neg p \\
p \land q &\rightarrow \Box\Box\neg p \\
p \land q &\rightarrow \Box\Box\Box\neg p \\
\neg p \land \neg q &\rightarrow \Box p \\
\neg p \land \neg q &\rightarrow \Box\Box p \\
\neg p \land \neg q &\rightarrow \Box\Box\Box p
\end{aligned}
$$

Note that the nesting of $\Box$ refers to all successors of all successors (double nesting), or all successors of all successors of all successors (triple nesting), respectively. Some of these formulas are true in some states of Figure 2, others are true in all states of Figure 2. Yet another class of formulas may even be true in all states of all transition systems, and not just in the particular transition system depicted in Figure 2.

6 Kripke Semantics

The meaning of formulas in propositional modal logic is defined in terms of truth in possible worlds, due to Kripke [Kri63], following suggestions of Leibniz for the understanding of necessity as truth in all possible worlds. An interpretation consists of a non-empty set W of possible worlds. For each world $s \in W$ we need an assignment of a truth-value to each propositional letter $A \in \Sigma$. The notions of possibility and necessity depend on which worlds are possible or conceivable from which other world. For that, an interpretation also consists of an accessibility relation $\rho \subseteq W \times W$ among worlds. The relation $(s,t) \in \rho$ would hold if world t is accessible from world s. Interchangeably, we also write just $s\rho t$ iff $(s,t) \in \rho$. A different way to explain ρ is that it defines—from the perspective of world s—which world t is possible or conceivable.

Definition 3 (Kripke frame) *A* Kripke frame *(W, ρ) consists of a non-empty set W and a relation $\rho \subseteq W \times W$ on worlds. The elements of W are called possible worlds and ρ is called accessibility relation.*

Definition 4 (Kripke structure) *A* Kripke structure *$K = (W, \rho, v)$ consists of Kripke frame (W, ρ) and a mapping $v : W \to \Sigma \to \{\mathit{true}, \mathit{false}\}$ that assigns truth-values to all the propositional letters in all worlds.*

By an abuse of notation, you will sometimes find the notation $s(A)$ instead of $v(s)(A)$. See exercise.

Definition 5 (Interpretation of propositional modal formulas) *Given a Kripke structure $K = (W, \rho, v)$, the interpretation $\models$ of modal formulas in a world s is defined as*

1. *$K, s \models A$ iff $v(s)(A) = \mathit{true}$.*
2. *$K, s \models \phi \land \psi$ iff $K, s \models \phi$ and $K, s \models \psi$.*
3. *$K, s \models \phi \lor \psi$ iff $K, s \models \phi$ or $K, s \models \psi$.*
4. *$K, s \models \neg\phi$ iff it is not the case that $K, s \models \phi$.*
5. *$K, s \models \Box\phi$ iff $K, t \models \phi$ for all worlds t with $s\rho t$.*
6. *$K, s \models \Diamond\phi$ iff $K, t \models \phi$ for some world t with $s\rho t$.*

When K is clear from the context, we also often abbreviate $K, s \models \phi$ by $K, s \models \phi$.

Definition 6 (Validity) *Given a Kripke structure $K = (W, \rho, v)$, formula ϕ is* valid in *K, written $K \models \phi$, iff $K, s \models \phi$ for all worlds $s \in W$.*

Let K be the Kripke structure corresponding to Figure 2, then

$$
\begin{aligned}
K &\models p \land q \to \Box\neg p \\
K &\models p \land q \to \Box\Box\neg p \\
K &\not\models p \land q \to \Box\Box\Box\neg p \\
K &\models \neg p \land \neg q \to \Box p \\
K &\models \neg p \land \neg q \to \Box\Box p \\
K &\not\models \neg p \land \neg q \to \Box\Box\Box p \\
K &\models \neg p \land q \to \Diamond p \\
K &\models \neg p \land q \to \Diamond\neg p \\
K &\models \neg p \land q \to \Diamond(\neg p \land q) \\
K &\models \neg(p \leftrightarrow q) \to \Diamond\neg(p \leftrightarrow q) \\
K &\models \neg(p \leftrightarrow q) \to \neg\Box\neg(p \leftrightarrow q) \\
K &\models (p \leftrightarrow q) \to \neg\Diamond(p \leftrightarrow q)
\end{aligned}
$$

7 Consequences

For defining consequences of formulas in modal logic, we need to distinguish if the assumptions are meant to hold locally in the current world, or globally for all worlds.

Definition 7 (Local consequence) *Let ψ be a formula and Φ a set of formulas. Then we write $\Phi \vDash_l \psi$ if and only if, for each Kripke structure $K = (W, \rho, v)$ and each world $s \in W$:*

$$K, s \models \Phi \textit{ implies } K, s \models \psi$$

Likewise, we write $\Phi \vDash_l^{\mathsf{C}} \psi$ if the local consequence holds for all Kripke structures of a class C (instead of all Kripke structures by and large). This will be of relevance if we are not interested in all Kripke structures but only those of a certain shape, say, all reflexive Kripke structures.

Definition 8 (Global consequence) *Let ψ be a formula and Φ a set of formulas. Then we write $\Phi \vDash_g \psi$ if and only if, for each Kripke structure $K = (W, \rho, v)$:*

$$\textit{if for all world } s \in W: \; K, s \models \Phi$$

then

$$\textit{for all world } s \in W : \; K, s \models \psi$$

Again, we write $\Phi \vDash_g^{\mathsf{C}} \psi$ if the global consequence holds for all Kripke structures of a class C.

Definition 9 (Tautology) *A formula ϕ is* valid *or a tautology, iff $\emptyset \vDash_l \phi$, which we write $\vDash \phi$. A set of formulas Φ is called* satisfiable, *iff there is a Kripke structure K and a world s with $K, s \models \Phi$.*

Again, we write $\vDash^{\mathsf{C}} \phi$ if formula ϕ is valid for all Kripke structures of a class C.

Lemma 10 (Local deduction theorem) *For formulas ϕ, ψ we have*

$$\phi \vDash_l \psi \;\; \textit{iff} \;\; \vDash_l \phi \to \psi$$

8 Modal Logic and Finite Automata

Consider the finite automaton in Figure 3 over the alphabet $\{0, 1\}$ with initial state p and accepting state F. Consider its corresponding transition

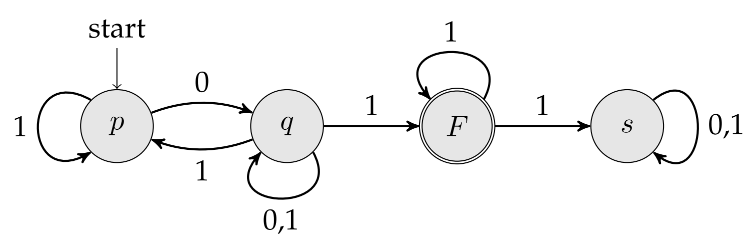

Figure 3: A finite automaton / acceptor

structure as a Kripke structure, where the assignment of propositional letter at states is as indicated. That is, at the left-most state only propositional letter p holds, at the right-most, only s holds and so on. With this, the states of the finite automaton are captured in the Kripke structure.

The finite automaton has labels on the edges also, which cannot (really) be captured in the states. Instead, we consider a labelled transition structure where the input 0,1 is represented as labels on the accessibility relation. Now we have two accessibility relations $\rho(0)$ and $\rho(1)$ for the accessibility under input 0 and under input 1, respectively. To access these two separate accessibility relations in logical formulas, we use two separate pairs of modalities, which are also labelled with input 0 or input 1, respectively: the modality pair $\Box_0$ and $\Diamond_0$ referring to the accessibility relation $\rho(0)$, and the modality pair $\Box_1$ and $\Diamond_1$ for the accessibility relation $\rho(1)$.

Let K be the Kripke structure corresponding to Figure 3, then

$K \models \neg\Diamond_0 F$	does not end with 0
$K \models p \to \Diamond_0 p$	p has a 1-loop
$K \models \Diamond_0 true$	never stuck with input 0
$K \models \Diamond_1 true$	never stuck with input 1
$K \models F \to \Box_0(\neg\Diamond_0 F \wedge \neg\Diamond_1 F)$	no end one step after seeing 0 from F

The last formula is a bit cumbersome to write. So we introduce a third pair of modal operators $\Box_{01}$ and $\Diamond_{01}$ that we bind to refer to transition under any input (0 or 1) by assuming the following axiom (for all instantiations of formula ϕ):

$$\Diamond_{01}\phi \;\leftrightarrow\; \Diamond_0\phi \vee \Diamond_1\phi$$

With this we find that:

$K \models F \to \Box_0\neg\Diamond_{01}F$	no end one step after seeing 0 from F
$K \models F \to \Box_0\neg\Diamond_{01}\Diamond_{01}F$	no end two steps after seeing 0 from F
$K \models p \to \Diamond_{01}q$	p has a q successor
$K \models F \to \Box_1 F$	stay final on 1s

Supposing we do not know the transition system, but only the above modal formulas. What other formulas can we infer about the system? Let us assume the following set of formulas Γ:

$$\begin{array}{l} \neg\Diamond_0 F \\ p \to \Diamond_0 p \\ \Diamond_0 true \\ \Diamond_1 true \\ F \to \Box_0(\neg\Diamond_0 F \wedge \neg\Diamond_1 F) \end{array}$$

Can we conclude any of the following consequences?

$$\Gamma \overset{?}{\vDash_l} F \to \Diamond_1 F\ ?$$

$$\Gamma \overset{?}{\vDash_g} F \to \Diamond_1 F\ ?$$

$$\Gamma \overset{?}{\vDash_l} F \to \Diamond_1 \Diamond_1 F\ ?$$

$$\Gamma \overset{?}{\vDash_g} F \to \Diamond_1 \Diamond_1 F\ ?$$

It turns out that the first two consequences hold using $F \to \Box_1 F$ and $\Diamond_1 true$ from Γ. The third one is *not* a consequence, because the local facts are not sufficient. The fourth consequence, instead, is justified using again $F \to \Box_1 F$ and $\Diamond_1 true$ from Γ, but needs these facts globally.

Another question is if we can characterize the finite automaton in Figure 3 using a finite set of modal formulas?

Exercises

Exercise 1 *Give a Hilbert-proof for the property $\Box_3 B_3$ from the facts and rules in Section 2. Please prove this much(!) more systematically than in the informal introduction in class. Is there a contradiction because the first wise man would be able to conclude a fact like the third one did, after the second wise man announced $\neg\Box_2 B_2$? Discuss how knowledge would change if the task for the wise guys would be to deduce the answer of any wise man, rather than the color of their own hats. Would this different setting still make sense?*

Exercise 2 *In the definition of Kripke structures you will sometimes find that v is not mentioned and that the notation $s(A)$ is used instead of $v(s)(A)$. Hence, the truth-value of the propositional variables is associated with the state. Does this make a difference? If so, give an example where the difference can be seen and explain why. If not, prove that the the original and the new semantics are actually equivalent.*

Exercise 3 *Prove or disprove that the following formulas are modal tautologies. If you disprove it, also try to find a variation of the formula or a class of Kripke structures for which you can prove it.*

1. $\Box\phi \land \Box(\phi \to \psi) \to \Box\psi$
2. $\phi \to \Diamond\phi$
3. $\phi \to \Box\phi$
4. $\Box\phi \leftrightarrow \neg\Diamond\neg\phi$
5. $\Box(\phi \land \psi) \leftrightarrow (\Box\phi \land \Box\psi)$
6. $\Box(\phi \lor \psi) \leftrightarrow (\Box\phi \lor \Box\psi)$
7. $\Diamond(\phi \land \psi) \leftrightarrow (\Diamond\phi \land \Diamond\psi)$
8. $\Diamond(\phi \lor \psi) \leftrightarrow (\Diamond\phi \lor \Diamond\psi)$
9. $\Box\phi \to \Diamond\phi$

Exercise 4 *How does the following variation H of the Gödel translation affect the results*

$$\begin{aligned} H(a) &= \Box a && \text{if } a \in \Sigma \text{ is a propositional letter} \\ H(\phi \land \psi) &= H(\phi) \land H(\psi) \\ H(\phi \lor \psi) &= H(\phi) \lor H(\psi) \\ H(\phi \supset \psi) &= \Box\big(H(\phi) \to H(\psi)\big) \end{aligned}$$

Do $\mathcal{G}$ and H share the same properties or is there an important difference? Does it establish a different connection to intuitionistic logic or the same? Do we need the same axioms and rules or not? Prove or disprove each of these conjectures.

Exercise 5 *Prove or disprove both directions of the local deduction theorem Lemma 10.*

Exercise 6 *Prove or disprove both directions of the variation of deduction theorem Lemma 10 with $\vDash_l$ replaced by $\vDash_g$.*

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