

Assignment 7

15-816: Substructural Logics
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Due
Thursday, December 1, 2016

This assignment consists of several, somewhat open-ended problems. **You should pick one of them to do.** You may also pick any of the problems from [Assignment 6](#) that you have not yet done.

You may do these assignments **by yourself or in pairs**. They are somewhat open-ended, so you have to use your judgment as to when you consider the homework completed. Feel free to contact the instructor when you have questions about the extent of a problem.

As usual, you are allowed and encouraged to use *all* resources (papers, lecture notes, technical reports) that you can find, but you must properly cite and acknowledge any resources you use.

Please submit this assignment as a PDF by email. LaTeX templates and macros that may be helpful are available on the course web pages, but you are not required to use them.

Exercise 1 (Natural Deduction) Present a natural deduction version of the logic that combines structural and ordered logic using $\uparrow_o^u A_o$ and $\downarrow_o^u A_u$ and prove its relationship to the sequent calculus.

Exercise 2 (Linear Call-by-Push-Value) Explore a linear extension form of call-by-push-value with additional shifts to and from Levy's structural layer, including typing rules and a substructural operational semantics. Interpret the result: what, if anything, does linearity express? Use example programs to illustrate your conclusions.

Exercise 3 (Substructural Operational Semantics) Specify, in SSOS form, fragments of a functional language using call-by-push-value with the following features. Your semantics should *not* use substitution, except of parameters (destinations) for variables. This is different from the semantics

for call-by-push-value we have given in lecture which uses substitution of values for variables.

1. Call-by-push-value, including functions, lazy pairs, eager pairs, unit, sums, and shifts
2. Recursive types and computations
3. Mutable store (creating, reading, and updating cells)
4. Can we encode futures, where function body and argument are evaluated in parallel?

You are encouraged to implement and test your SSOS specification in CLF, but you are not required to do so.

Exercise 4 (Rule Permutations) An inference rule can be permuted up in a proof past another rule application with a different principal formula if the inferences can take place in either order. For example, $\otimes L$ can be permuted upwards past $\multimap R$:

$$\frac{\frac{\Gamma ; \Delta, A, B, C \vdash D}{\Gamma ; \Delta, A, B \vdash C \multimap D} \multimap R}{\Gamma ; \Delta, A \otimes B \vdash C \multimap D} \otimes L \quad \longleftrightarrow \quad \frac{\frac{\Gamma ; \Delta, A, B, C \vdash D}{\Gamma ; \Delta, A \otimes B, C \vdash D} \otimes L}{\Gamma ; \Delta, A \otimes B \vdash C \multimap D} \multimap R$$

It is important to consider this only for different formulas. For example, the right-to-left direction of the conversion above is deemed to hold in general, even though in the sequent $\vdash (A \otimes B) \multimap C$ an application of $\multimap R$ always has to precede $\otimes L$. For the case above, we write $\otimes L \setminus \multimap R$ and (because this one can be interpreted also from right to left), $\multimap R \setminus \otimes L$. In some cases, such as $\otimes L \setminus \top R$, the lower inference disappears altogether, or may be duplicated (as in $\otimes L \setminus \& R$).

In the cut-free sequent calculus, identify all pairs of rules $X \setminus Y$ such that X can *not* be permuted upward over Y in general and provide a counterexample. You do not need to prove that all other pairs commute.

Can you conjecture or establish a connection to the classification of connectives as positive or negative? To focusing?

Exercise 5 (Confluence of Inversion) In our (full) focusing system, the focusing rules can only be applied to *stable* sequents, where linear antecedents must be negative propositions or positive atoms and the succedent must be a positive proposition or a negative atom. This means that in an arbitrary

sequent without focus, all inversion steps must be applied before we can focus.

From a practical perspective it is important to observe that the order in which these inversion steps are applied is irrelevant: no matter how we decompose an arbitrary goal sequent, (a) inversion always terminates, and (b) we always arrive at the same collection of stable subgoal sequents.

Prove these two properties.

Exercise 6 (Connecting Adjoint with Linear Logic) The traditional presentation of (intuitionistic) linear logic does not use shifts, but $!A$. This amounts to the same as a polarized logic where the structural layer contains only the proposition $\uparrow_L^U A_L$. We can then define $!A_L = \downarrow_L^U \uparrow_L^U A_L$. Prove that this is sufficient to express all the propositions in the structural layer in some suitably equivalent form. Investigate if this is still true for $!A_O = \downarrow_O^L \uparrow_O^L A_O$ which embeds the ordered in the linear layer. What are the operational interpretations of your arguments in the two cases?