Lecture Notes on Resource Semantics

15-816: Substructural Logics
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In this lecture we explore a new presentation of substructural logics, one where resources are explicitly tracked in the judgments. It is a new form of semantics, given by intuitionistic means, while generally semantic investigations take a classical point of view even if the studied subject is intuitionistic.

In the present lecture we change the judgments, but we try to disturb the nature of proofs as little as possible. In the following lecture, we will take a less restrictive view, which leads to new ways to reason linearly.

One of the important reasons to investigate a resource semantics is that it allows us to express new properties and relations, beyond what is possible in linear logic itself. Further materials and properties on resource semantics are given by Reed [Ree09].

1 Resource-Aware Judgments

In order to give a Kripke-like resource semantics for ordered logic we label all the resources with unique labels representing that resource. In the succedent we record all the resources that may be used, which may be a subset of the resources listed in the antecedent. So a sequent has the form

$$A_1[\alpha_1], \ldots, A_n[\alpha_n] \vdash C[p]$$

where p is formed from $\alpha_1, \ldots, \alpha_n$ with a binary resource combinator '$\cdot$'. In addition we have the empty resource label ϵ, which is the unit of '$\cdot$'.

Resource combination is associative, so we have the laws

$$\begin{array}{rcl} p \cdot \epsilon & = & p \\ \epsilon \cdot q & = & q \\ (p \cdot q) \cdot r & = & p \cdot (q \cdot r) \end{array}$$

We will apply these equations silently, just as we, for example, silently reorder hypotheses.

By labeling resources we recover the property of weakening.

> **Weakening:** If $\Gamma \vdash C[p]$ then $\Gamma, A[\alpha] \vdash C[p]$. Here α must be new in order to maintain the invariant on seqents that all antecedents are labeled with distinct resource parameters.

Contraction in general cannot be quite formulated, since we cannot contract $A[\alpha], A[\beta]$ to $A[\alpha \cdot \beta]$ because, at least for the moment, hypotheses can only be labeled with resource parameters and not combinations of them. Nevertheless, we treat antecedents *persistently* by propagating them to all premises of the rules.

The identity is quite straightforward:

$$\frac{}{\Gamma, A[\alpha] \vdash A[\alpha]}\ \mathsf{id}$$

Cut is a bit more complicated, because resource labels in succedents are more general than in antecedents. But we have already seen this in substitution principles with proof terms, so we imitate this solution, substituting resources p for resource parameter α:

$$\frac{\Gamma \vdash A[p] \quad \Gamma, A[\alpha] \vdash C[l \cdot \alpha \cdot r]}{\Gamma \vdash C[l \cdot p \cdot r]}\ \mathsf{cut}$$

We now revisit each of the connectives so far in turn, deriving the appropriate rules. The goal is to achieve an exact isomorphism between the linear logic inference rules based on hypothetical judgments and the inference rules based on resources. Hidden behind the isomorphism is the equational reasoning in the resource algebra.

Fuse. The resources available to achieve the goals are split between the two premises. Previously, this was achieved by splitting the context itself.

Note that we use Γ here to stand for a context in which all assumptions are labeled with unique resource parameters.

$$\frac{\Gamma \vdash A[p] \quad \Gamma \vdash B[q]}{\Gamma \vdash A \bullet B[p \cdot q]} \bullet R$$

In order to apply a left rule to a given assumption $A[\alpha]$, the resource α must actually be available, which is recorded in the succedent. Upon application of the rule the resource is no longer available, but new resources may now be available (depending on the connective).

$$\frac{\Gamma, (A \bullet B)[\alpha], A[\beta], B[\gamma] \vdash C[l \cdot (\beta \cdot \gamma) \cdot r]}{\Gamma, (A \bullet B)[\alpha] \vdash C[l \cdot \alpha \cdot r]} \bullet L^{\beta,\gamma}$$

In this rule, α is consumed and new resources β and γ are introduced. By associativity, we could omit the parentheses in the resources of the premise.

Under and Over. The intuitions above give us enough information to write out these rules directly, modeling the linear sequent calculus.

$$\frac{\Gamma, A[\alpha] \vdash B[\alpha \cdot p]}{\Gamma \vdash (A \setminus B)[p]} \setminus R^{\alpha} \qquad \frac{\Gamma, A[\alpha] \vdash B[p \cdot \alpha]}{\Gamma \vdash (B \mathbin{/} A)[p]} / R^{\alpha}$$

In the elimination rule we see again how a split between the antecedents is represented as a split between the resources.

$$\frac{\Gamma, (A \setminus B)[\alpha] \vdash A[p] \quad \Gamma, (A \setminus B)[\alpha], B[\beta] \vdash C[l \cdot \beta \cdot r]}{\Gamma, (A \setminus B)[\alpha] \vdash C[l \cdot p \cdot \alpha \cdot r]} \setminus L^{\beta}$$

By strengthening, we can see that the antecedent $(A \setminus B)[\alpha]$ can not be used in either premise. At this point we can easily see how $B \mathbin{/} A$ should work, just reversion $\alpha \cdot p$ in the succedent of the conclusion.

$$\frac{\Gamma, (B \mathbin{/} A)[\alpha] \vdash A[p] \quad \Gamma, (B \mathbin{/} A)[\alpha], B[\beta] \vdash C[l \cdot \beta \cdot r]}{\Gamma, (B \mathbin{/} A)[\alpha] \vdash C[l \cdot \alpha \cdot p \cdot r]} / L^{\beta}$$

Unit. Here, we just have to enforce the emptiness of the resources.

$$\frac{}{\Gamma \vdash \mathbf{1}[\epsilon]} \mathbf{1}R \qquad \frac{\Gamma, \mathbf{1}[\alpha] \vdash C[l \cdot r]}{\Gamma, \mathbf{1}[\alpha] \vdash C[l \cdot \alpha \cdot r]} \mathbf{1}L$$

In the $\mathbf{1}L$ rule we replace α by ϵ and then use its unit property to obtain $l \cdot r$.

2 Exponentials

Our representation technique for the sequent calculus using explicit resources is already rich enough to handle persistence. We just allow antecedents $A_{\mathsf{U}}[\epsilon]$ together with resource-bound $A_{\mathsf{O}}[\alpha]$. In $\downarrow R$, we effectively "check" the emptiness of resource-bound assumptions. All propositions annotated here with resources are ordered.

$$\frac{\Gamma \vdash A_{\mathsf{U}}[\epsilon]}{\Gamma \vdash (\downarrow^{\mathsf{U}}_{\mathsf{O}} A_{\mathsf{U}})[\epsilon]}\ \downarrow R \qquad \frac{\Gamma, (\downarrow^{\mathsf{U}}_{\mathsf{O}} A_{\mathsf{U}})[\alpha], A_{\mathsf{U}}[\epsilon] \vdash C[l \cdot r]}{\Gamma, (\downarrow^{\mathsf{U}}_{\mathsf{O}} A_{\mathsf{U}})[\alpha] \vdash C[l \cdot \alpha \cdot r]}\ \downarrow L$$

$$\frac{\Gamma \vdash A[\epsilon]}{\Gamma \vdash (\uparrow^{\mathsf{U}}_{\mathsf{O}} A)[\epsilon]}\ \uparrow R \qquad \frac{\Gamma, (\uparrow^{\mathsf{U}}_{\mathsf{O}} A)[\epsilon], A[\alpha] \vdash C[l \cdot \alpha \cdot r]}{\Gamma, (\uparrow^{\mathsf{U}}_{\mathsf{O}} A)[\epsilon] \vdash C[l \cdot r]}\ \uparrow L^{\alpha}$$

The rules for the remaining connectives are easy to fill in, including the expected rules for structural proposition A_{U}.

One can also write A_{U} instead of the more verbose $A_{\mathsf{U}}[\epsilon]$.

3 Correspondence

It is now easy to establish that the resource calculus is in bijective correspondence with the ordered sequent calculus. Moroever, it satisfies the expected properties of cut and identity. Key is the crucial strengthening property.

For the remainder of this lecture we assume that a resource context has hypotheses of the form $A_{\mathsf{U}}[\epsilon]$ and $A_{\mathsf{O}}[\alpha]$, where all resource parameters α are distinct, and the succedent has the form $C[p]$, where p is a product of distinct resource parameters. We write $\alpha \notin p$ if α does not occur in p. The equational theory for resources remains associativity for '$\cdot$' with unit ϵ.

Theorem 1 (Strengthening for Resource Semantics) *If $\Gamma, A[\alpha] \vdash C[p]$ and $\alpha \notin p$ then $\Gamma \vdash C[p]$ with the same proof.*

Proof: By induction on the structure of the given proof. □

Theorem 2 (Identity) *In the system where identity is restricted to atomic propositions, general identity is admissible. That is, $A[\alpha] \vdash A[\alpha]$ for any proposition A.*

Proof: By induction on the structure of A. □

Theorem 3 (Cut) *In the system without the* cut *and* cut! *rules, they are admissible. That is, for $p = q = \epsilon$ or $q = \alpha$, we have*

$$\frac{\Gamma \vdash A[p] \quad \Gamma, A[q] \vdash C[l \cdot q \cdot r]}{\Gamma \vdash C[l \cdot p \cdot r]} \text{ cut}$$

Proof: By nested induction, first on the cut formula A, then on the structure of the proofs in the two premises (one must become smaller while the other remains the same). □

In order to formulate a correspondence theorem, we need to express relationships between assumptions. We write $(A_1, \ldots, A_n)[\epsilon] = A_1[\epsilon], \ldots, A_n[\epsilon]$ and $(A_1, \ldots, A_n)[\vec{\alpha}] = A_1[\alpha_1], \ldots, A_n[\alpha_n]$. Furthermore, we need to construct a pair of contexts $\Gamma_{\mathsf{U}}; \Omega_{\mathsf{O}}$ from given a resource context. For ease of definition, we do not require a separation of zones but generate a mixed context with linear and structural antecedents that can then be separated.

$$\begin{array}{lcl}
(\cdot)|_\epsilon & = & (\cdot) \\
(\Gamma, \Gamma')|_{p \cdot q} & = & \Gamma|_p, \Gamma'|_q \\
(A_{\mathsf{O}}[\alpha])|_\alpha & = & A_{\mathsf{O}} \\
(A_{\mathsf{O}}[\alpha])|_\epsilon & = & (\cdot) \\
(A_{\mathsf{U}}[\epsilon])|_\epsilon & = & A_{\mathsf{U}}
\end{array}$$

Because of the equational theory, this definition has some nondeterminism. Under the general assumptions of this section, $\Gamma|_p$ will be defined and unique.

Theorem 4 (Correspondence)

(i) *If $\Gamma_{\mathsf{U}}; \Omega_{\mathsf{O}} \vdash C$ then $\Gamma_{\mathsf{U}}[\epsilon], \Omega_{\mathsf{O}}[\vec{\alpha}] \vdash C[\alpha_1 \cdots \alpha_n]$.*

(ii) *If $\Gamma \vdash C[p]$ then $\Gamma|_p \vdash C$.*

Moreover, the correspondence between linear and resource proofs is a bijection.

Proof: By straightforward inductions, exploiting strengthening. □

4 Linear Resource Semantics

So far have presented the resource semantics for an ordered logic. How do we get one for linear logic? Actually, this is quite easy: we just add the equation

$$p \cdot q = q \cdot p$$

and we get linear logic! Under and over now collapse, because $\alpha \cdot p = p \cdot \alpha$. In other ways, the rules remain exactly the same.

What about structural logic? We get this by identifying *all* terms

$$p = \epsilon$$

which have already done implicitly by writing structural antecedents as $A[\epsilon]$.

An interesting intermediate point is affine logic. For this, it seems best to postulate a resource inequality, defined here (in the presence of symmetry, that is, linear logic) as

$$p \leq q \quad \text{iff} \quad \exists r. p \cdot r = q$$

and then changing the rule to allow subset at some critical junctures, such as

$$\frac{\alpha \leq p}{\Gamma, A[\alpha] \vdash A[p]}\ \mathsf{id}$$

References

[Ree09] Jason C. Reed. *A Hybrid Logical Framework*. PhD thesis, Carnegie Mellon University, September 2009. Available as Technical Report CMU-CS-09-155.