

Lecture Notes on Focusing

15-816: Substructural Logics
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As discussed in the last lecture, focusing [And92] is one of the major achievements of proof theory. It decomposes into inversion and chaining, which we presented last time. In this lecture we first complete the development of chaining by sketching its proof of cut elimination and then we introduce full focusing.

1 Summary of Chained Inference

We summarize chaining from last lecture. While we present it here for ordered logic, it applies as well to other structural and substructural logics; we will see examples in the next lecture.

Negative $A^-, B^- ::= p^- \mid A^+ \setminus B^- \mid B^- / A^+ \mid A^- \& B^- \mid \uparrow A^+$
Positive $A^+, B^+ ::= p^+ \mid A^+ \bullet B^+ \mid A^+ \circ B^+ \mid \mathbf{1} \mid A^+ \oplus B^+ \mid \downarrow A^-$

There are three judgments

$$\begin{array}{l} \Omega \Vdash A \\ \Omega \Vdash [C^+] \\ \Omega_L [A^-] \Omega_R \Vdash C \end{array}$$

We abbreviate

$$\begin{array}{l} \overline{\Omega} ::= \Omega \mid \Omega_L [A^-] \Omega_R \\ \overline{C} ::= C \mid [C^+] \end{array}$$

and globally presuppose for any judgment $\overline{\Omega} \Vdash \overline{C}$:

There is at most one proposition in focus in any sequent.

We provide the rules for a selection of the connectives.

$$\begin{array}{c}
\frac{}{p^+ \Vdash [p^+]} \text{id}^+ \qquad \frac{}{[p^-] \Vdash p^-} \text{id}^- \\
\frac{\Omega \Vdash [A^+]}{\Omega \Vdash A^+} \text{focus}^+ \qquad \frac{\Omega_L [A^-] \Omega_R \Vdash C}{\Omega_L A^- \Omega_R \Vdash C} \text{focus}^- \\
\frac{\bar{\Omega} \Vdash A^+}{\bar{\Omega} \Vdash \uparrow A^+} \uparrow R \qquad \frac{\Omega_L A^+ \Omega_R \Vdash C}{\Omega_L [\uparrow A^+] \Omega_R \Vdash C} \uparrow L \\
\frac{\Omega \Vdash A^-}{\Omega \Vdash [\downarrow A^-]} \downarrow R \qquad \frac{\bar{\Omega}_L A^- \bar{\Omega}_R \Vdash \bar{C}}{\bar{\Omega}_L (\downarrow A^-) \bar{\Omega}_R \Vdash \bar{C}} \downarrow L \\
\frac{A^+ \bar{\Omega} \Vdash B^-}{\bar{\Omega} \Vdash A^+ \setminus B^-} \setminus R \qquad \frac{\Omega \Vdash [A^+] \quad \Omega_L [B^-] \Omega_R \Vdash C}{\Omega_L \Omega [A^+ \setminus B^-] \Omega_R \Vdash C} \setminus L \\
\frac{\Omega_L \Vdash [A^+] \quad \Omega_R \Vdash [B^+]}{\Omega_L \Omega_R \Vdash [A^+ \bullet B^+]} \bullet R \qquad \frac{\bar{\Omega}_L A B \bar{\Omega}_R \Vdash \bar{C}}{\bar{\Omega}_L (A \bullet B) \bar{\Omega}_R \Vdash \bar{C}} \bullet L \\
\frac{}{\cdot \Vdash [1]} \mathbf{1}R \qquad \frac{\bar{\Omega}_L \bar{\Omega}_R \Vdash \bar{C}}{\bar{\Omega}_L \mathbf{1} \bar{\Omega}_R \Vdash \bar{C}} \mathbf{1}L \\
\frac{\bar{\Omega} \Vdash A \quad \bar{\Omega} \Vdash B}{\bar{\Omega} \Vdash A \& B} \&R \qquad \frac{\Omega_L [A^-] \Omega_R \Vdash C}{\Omega_L [A^- \& B^-] \Omega_R \Vdash C} \&L_1 \qquad \frac{\Omega_L [B^-] \Omega_R \Vdash C}{\Omega_L [A^- \& B^-] \Omega_R \Vdash C} \&L_2 \\
\frac{\Omega \Vdash [A^+]}{\Omega \Vdash [A^+ \oplus B^+]} \oplus R_1 \qquad \frac{\Omega \Vdash [B^+]}{\Omega \Vdash [A^+ \oplus B^+]} \oplus R_2 \qquad \frac{\bar{\Omega}_L A^+ \bar{\Omega}_R \Vdash \bar{C} \quad \bar{\Omega}_L B^+ \bar{\Omega}_R \Vdash \bar{C}}{\bar{\Omega}_L (A^+ \oplus B^+) \bar{\Omega}_R \Vdash \bar{C}} \oplus L
\end{array}$$

2 Admissibility of Cut in Chained Inference

Neither the chaining calculus nor the upcoming full focusing calculus allow the rule of cut. It would violate the basic goal of restricting proof search. However, as we have seen in the last lecture, admissibility of cut (together with admissibility of identity), is the key to completeness of focusing. Keeping in mind our central presupposition that no more than one

proposition can be focus, we obtain the following versions of cut. Note that in cut_A^* at most one of the overlined antecedents or succedents can contain a proposition in focus.

$$\frac{\overline{\Omega} \Vdash A \quad \overline{\Omega}_L A \overline{\Omega}_R \Vdash \overline{C}}{\overline{\Omega}_L \overline{\Omega} \overline{\Omega}_R \Vdash \overline{C}} \text{cut}_A^*$$

$$\frac{\Omega \Vdash [A^+] \quad \overline{\Omega}_L A^+ \overline{\Omega}_R \Vdash \overline{C}}{\overline{\Omega}_L \Omega \overline{\Omega}_R \Vdash \overline{C}} \text{cut}_A^+ \quad \frac{\overline{\Omega} \Vdash A^- \quad \Omega_L [A^-] \Omega_R \Vdash C}{\Omega_L \overline{\Omega} \Omega_R \Vdash C} \text{cut}_A^-$$

Theorem 1 (Admissibility of Cut in Chained Deduction)

The rules cut_A^* , cut_A^+ and cut_A^- are all admissible.¹

Proof: By nested induction, first on the structure of A and second simultaneously on the structure of the two given deductions.

The only significant change compared to the usual proof of admissibility of cut is that we restrict the forms of commuting reductions for cut_A^* to preserve the invariant in the conclusion. \square

3 Full Focusing

We obtain the full focusing system by forcing all possible inversion steps to be completed before allowing focus. This also means while a proposition is in focus, inferences can only be applied to the focus formula and no other rules are applicable.

This can be specified in two ways. One is to just restrict focus^+ and focus^- so that no inversion rule can apply. This means inversion steps can be applied arbitrarily, which entails some nondeterminism because there may be multiple invertible propositions in the antecedents or succedent. However, this is *don't-care nondeterminism* since the remaining subgoals after all inversion rules have been applied will always be the same, a property called *confluence*.

Alternatively (as advocated, for example, by Simmons [Sim14]) we can write the rules to force a particular order of application of these rules, say, left-to-right. This leads to a simpler proof of its completeness via admissibility of cut, since one does not have to prove confluence.

For the purpose of these notes, we use the don't-care nondeterministic version since it has less syntactic overhead. The following rules capture the

¹As of the time I am writing up these notes, not all of these have been checked carefully.

same connectives as before. We say $\Omega \Vdash C$ is *stable* if Ω consists only of negative propositions or positive atoms and C is either a positive proposition or negative atom. A focusing sequent is stable exactly if no inversion rule applies. In the rules below, inversion rules no longer allow other propositions to be in focus.

$$\begin{array}{c}
\frac{}{p^+ \Vdash [p^+]} \text{id}^+ \qquad \frac{}{[p^-] \Vdash p^-} \text{id}^- \\
\frac{(\Omega \Vdash A^+) \text{ stable} \quad \Omega \Vdash [A^+]}{\Omega \Vdash A^+} \text{focus}^+ \\
\frac{(\Omega_L A^- \Omega_R \Vdash C) \text{ stable} \quad \Omega_L [A^-] \Omega_R \Vdash C}{\Omega_L A^- \Omega_R \Vdash C} \text{focus}^- \\
\frac{\Omega \Vdash A^+}{\Omega \Vdash \uparrow A^+} \uparrow R \qquad \frac{\Omega_L A^+ \Omega_R \Vdash C}{\Omega_L [\uparrow A^+] \Omega_R \Vdash C} \uparrow L \\
\frac{\Omega \Vdash A^-}{\Omega \Vdash \downarrow A^-} \downarrow R \qquad \frac{\Omega_L A^- \Omega_R \Vdash C}{\Omega_L (\downarrow A^-) \Omega_R \Vdash C} \downarrow L \\
\frac{A^+ \Omega \Vdash B^-}{\Omega \Vdash A^+ \setminus B^-} \setminus R \qquad \frac{\Omega \Vdash [A^+] \quad \Omega_L [B^-] \Omega_R \Vdash C}{\Omega_L \Omega [A^+ \setminus B^-] \Omega_R \Vdash C} \setminus L \\
\frac{\Omega_L \Vdash [A^+] \quad \Omega_R \Vdash [B^+]}{\Omega_L \Omega_R \Vdash [A^+ \bullet B^+]} \bullet R \qquad \frac{\Omega_L A B \Omega_R \Vdash C}{\Omega_L (A \bullet B) \Omega_R \Vdash C} \bullet L \\
\frac{}{\cdot \Vdash [1]} \mathbf{1}R \qquad \frac{\Omega_L \Omega_R \Vdash C}{\Omega_L \mathbf{1} \Omega_R \Vdash C} \mathbf{1}L \\
\frac{\Omega \Vdash A \quad \Omega \Vdash B}{\Omega \Vdash A \& B} \&R \qquad \frac{\Omega_L [A^-] \Omega_R \Vdash C}{\Omega_L [A^- \& B^-] \Omega_R \Vdash C} \&L_1 \qquad \frac{\Omega_L [B^-] \Omega_R \Vdash C}{\Omega_L [A^- \& B^-] \Omega_R \Vdash C} \&L_2 \\
\frac{\Omega \Vdash [A^+]}{\Omega \Vdash [A^+ \oplus B^+]} \oplus R_1 \qquad \frac{\Omega \Vdash [B^+]}{\Omega \Vdash [A^+ \oplus B^+]} \oplus R_2 \qquad \frac{\Omega_L A^+ \Omega_R \Vdash C \quad \Omega_L B^+ \Omega_R \Vdash C}{\Omega_L (A^+ \oplus B^+) \Omega_R \Vdash C} \oplus L
\end{array}$$

We do not present here the proof of soundness, completeness or the admissibility of cut and identity for this calculus, which can be found in [Sim12, Sim14] for closely related calculi.

4 Application: Focused Parsing

As an example of focusing, we reconsider the example of parsing *Alice likes Bob here* from [Lecture 17, Section 3](#).

$$\begin{array}{cccc} Alice & likes & Bob & here \\ : & : & : & : \\ n & n \setminus (s / n) & n & s \setminus s \quad \vdash ? : s \end{array}$$

We have to decide on a polarity for the atoms n and s . We start by making both of them *positive*. This gives us the following theorem proving problem:

$$n^+ \quad (n^+ \setminus (\uparrow s^+ / n^+)) \quad n^+ \quad (s^+ \setminus \uparrow s^+) \Vdash s^+$$

Since we can focus only on negative antecedents and positive succedents, we can only focus on $n^+ \setminus (\uparrow s^+ / n^+)$, $s^+ \setminus \uparrow s^+$, or the succedent s^+ . Focusing on the succedent will immediately fail since the antecedents are not equal to just s^+ . We can try to focus on $s^+ \setminus \uparrow s^+$, but this will fail for a similar reason:

$$\frac{\text{no rule applicable}}{n^+ \quad (n^+ \setminus (\uparrow s^+ / n^+)) \quad n^+ \Vdash [s^+] \quad [\uparrow s^+] \Vdash s^+} \backslash L$$

So it only remains to focus on the transitive verb *likes*. Note in the focusing phase, all the steps are forced once we have decided to focus, leaving only one subgoal.

$$\frac{\frac{\frac{\frac{}{n^+ \Vdash [n^+]}}{id^+} \quad \frac{\frac{\frac{}{s^+ \setminus \uparrow s^+} \Vdash s^+}{id^+} \quad \frac{}{[s^+] \quad [\uparrow s^+] \Vdash s^+} \uparrow L}{[s^+] \quad [\uparrow s^+] \Vdash s^+} \uparrow L}{\frac{}{n^+ \setminus (\uparrow s^+ / n^+)} \quad \frac{}{[s^+] \quad [\uparrow s^+] \Vdash s^+} \uparrow L}}{n^+ \quad [n^+ \setminus (\uparrow s^+ / n^+)] \quad n^+ \quad (s^+ \setminus \uparrow s^+) \Vdash s^+} \backslash L$$

The remaining subgoal can be proved only by focusing on $s^+ \setminus \uparrow s^+$:

$$\frac{\frac{\frac{}{s^+ \Vdash [s^+]}}{id^+} \quad \frac{\frac{}{s^+ \setminus \uparrow s^+} \Vdash s^+}{id^+} \quad \frac{}{[s^+] \quad [\uparrow s^+] \Vdash s^+} \uparrow R}{\frac{}{s^+ \setminus \uparrow s^+} \Vdash s^+} \uparrow R$$

The final subgoal can now be proved by focusing on the succedent s^+ :

$$\frac{}{s^+ \Vdash [s^+]} \text{id}^+$$

Note that in all of these steps there was no choice: in every stable sequent, there was only one possibility to focus. In essence, focusing has reduced the number of proofs to just one, which is a highly significant restriction compared to the nondeterminism present if we proceed in small steps.

If we mark all atoms as negative, there is a small choice right at the beginning, because we could focus on either $\downarrow n^- \setminus (s^- / \downarrow n^-)$ or $\downarrow s^- \setminus s^-$. Only the latter will succeed, so we show that proof.

$$\frac{\frac{\frac{\vdots}{n^- (\downarrow n^- \setminus (s^- / \downarrow n^-))} n^- \Vdash s^-}{n^- (\downarrow n^- \setminus (s^- / \downarrow n^-))} \downarrow R \quad \frac{}{[s^-] \Vdash s^-} \text{id}^-}{n^- (\downarrow n^- \setminus (s^- / \downarrow n^-))} \downarrow R \quad \frac{}{[s^-] \Vdash s^-} \text{id}^-}{n^- (\downarrow n^- \setminus (s^- / \downarrow n^-))} \downarrow R \quad \frac{}{[s^-] \Vdash s^-} \text{id}^-} \setminus L$$

Only one focus is possible now.

$$\frac{\frac{\frac{\vdots}{n^- \Vdash n^-} \downarrow R \quad \frac{\frac{\vdots}{n^- \Vdash [n^-]} \downarrow R \quad \frac{}{[s^-] \Vdash s^-} \text{id}^-}{n^- \Vdash [n^-]} \downarrow R \quad \frac{}{[s^- / \downarrow n^-]} \text{id}^-}{n^- \Vdash [n^-]} \downarrow R \quad \frac{}{[s^- / \downarrow n^-]} \text{id}^-}{n^- \Vdash [n^-]} \downarrow R \quad \frac{}{[s^- / \downarrow n^-]} \text{id}^-} \setminus L$$

The remaining (identical) subgoals follow by focusing on the left.

$$\frac{}{[n^-] \Vdash n^-} \text{id}^-$$

5 Summary

Focusing [And92] is a tremendous simplification and restructuring of the search space provided by the cut-free sequent calculus. Instead of having to make individual decisions on inference rules, which are really tiny steps, focusing allows big steps of inference. It is also widely applicable,

for example, to ordered logic [Pol01], linear, intuitionistic and classical logics [LM09], with very elegant proofs of completeness based on cut and identity [Cha06, Sim12, Sim14]. Indeed, as we will see in the next lecture, deduction is so controlled that it can be seen as the foundation for *logic programming*, where computation is modeled as inference.

6 Synthetic Inference Rules

Intuitively, focusing proofs of arbitrary sequents start by breaking down all invertible connectives to obtain a stable sequent. From a stable sequent, we then focus on a particular proposition which will be broken down in a chained phase of inference. Once we have lost focus (in the $\downarrow R$ and $\uparrow L$ rules) when we enter a phase of inversion until we reach another stable sequent along each branch of the proof that has not yet been completed. The idea behind *synthetic rules of inference* [And02] is to replace the general rules of inference entirely by specialized ones that implement this strategy.

Let's see how this plays out in the parsing example, starting with the positive polarization.

$$n^+ (n^+ \setminus (\uparrow s^+ / n^+)) n^+ (s^+ \setminus \uparrow s^+) \Vdash s^+$$

The subformulas we might focus on in a potential proof of this sequent are antecedents $tv^- = n^+ \setminus (\uparrow s^+ / n^+)$, $adv^- = s^+ \setminus \uparrow s^+$ and the succedent s^+ . Let's see what would happen if we focused on each of these propositions in a general (stable) sequent. First, focusing on tv^- .

$$\frac{(\Omega_1 = \Omega_{11} \Omega_{12}) \quad \begin{array}{c} \vdots \\ \Omega_{12} \Vdash [n^+] \end{array} \quad \begin{array}{c} \vdots \\ \Omega_{11} [\uparrow s^+ / n^+] \end{array} \quad \Omega_2 \Vdash C}{\Omega_1 [n^+ \setminus (\uparrow s^+ / n^+)] \Omega_2 \Vdash C} \setminus R$$

Now the first subgoal $\Omega_{12} \Vdash [n^+]$ can only succeed if $\Omega_{12} = n^+$, so we can fill that in as a consequence of focusing.

$$\frac{(\Omega_1 = \Omega_{11} \Omega_{12}) \quad \frac{(\Omega_{12} = n^+)}{\Omega_{12} \Vdash [n^+]} \text{id}^+ \quad \begin{array}{c} \vdots \\ \Omega_{11} [\uparrow s^+ / n^+] \end{array} \quad \Omega_2 \Vdash C}{\Omega_1 [n^+ \setminus (\uparrow s^+ / n^+)] \Omega_2 \Vdash C} \setminus R$$

In the second subgoal we continue the focusing phase, this time splitting up Ω_2 .

$$\frac{(\Omega_1 = \Omega_{11} \ \Omega_{12}) \quad \frac{(\Omega_{12} = n^+)}{\Omega_{12} \Vdash [n^+]} \text{id}^+ \quad \frac{(\Omega_2 = \Omega_{21} \ \Omega_{22}) \quad \frac{\vdots \quad \Omega_{21} \Vdash [n^+]}{\Omega_{11} [\uparrow s^+ / n^+]} \quad \frac{\vdots \quad \Omega_{11} [\uparrow s^+]}{\Omega_{22} \Vdash C}}{\Omega_{11} [\uparrow s^+ / n^+] \ \Omega_2 \Vdash C} /L}{\Omega_1 [n^+ \setminus (\uparrow s^+ / n^+)] \ \Omega_2 \Vdash C} \setminus L$$

In the first remaining subgoal we succeed, but only if $\Omega_{21} = n^+$; in the second remaining subgoal we lose focus.

$$\frac{(\Omega_1 = \Omega_{11} \ \Omega_{12}) \quad \frac{(\Omega_{12} = n^+)}{\Omega_{12} \Vdash [n^+]} \text{id}^+ \quad \frac{(\Omega_2 = \Omega_{21} \ \Omega_{22}) \quad \frac{(\Omega_{21} = n^+)}{\Omega_{21} \Vdash [n^+]} \text{id}^+ \quad \frac{\Omega_{11} s^+ \ \Omega_{22} \Vdash C}{\Omega_{11} [\uparrow s^+]} \uparrow R}{\Omega_{11} [\uparrow s^+ / n^+] \ \Omega_2 \Vdash C} /L}{\Omega_1 [n^+ \setminus (\uparrow s^+ / n^+)] \ \Omega_2 \Vdash C} \setminus L$$

Summarizing all this into one synthetic rule, we obtain for $\text{tv}^- = n^+ \setminus (\uparrow s^+ / n^+)$

$$\frac{\Omega_{11} s^+ \ \Omega_{22} \Vdash C}{\Omega_{11} n^+ \ \text{tv}^- \ n^+ \ \Omega_{22} \Vdash C} \text{tv}$$

Similarly, if we focus on $\text{adv}^- = s^+ \setminus \uparrow s^+$ we obtain

$$\frac{(\Omega_1 = \Omega_{11} \ \Omega_{12}) \quad \frac{\Omega_{12} = s^+}{\Omega_{12} \Vdash [s^+]} \text{id}^+ \quad \frac{\Omega_{11} s^+ \ \Omega_2 \Vdash C}{\Omega_{11} [\uparrow s^+]} \uparrow R}{\Omega_1 [s^+ \setminus \uparrow s^+] \ \Omega_2 \Vdash C} \setminus L$$

which we can summarize as

$$\frac{\Omega_{11} s^+ \ \Omega_2 \Vdash C}{\Omega_{11} s^+ \ \text{adv}^- \ \Omega_2 \Vdash C} \text{adv}$$

Finally, we can focus on s^+ in the succedent:

$$\frac{(\Omega = s^+)}{\Omega \Vdash [s^+]} \text{id}^+$$

which we summarize as

$$\frac{}{s^+ \Vdash s^+} s$$

Writing all three rules down together:

$$\frac{\Omega_{11} s^+ \Omega_{22} \Vdash C}{\Omega_{11} n^+ tv^- n^+ \Omega_{22} \Vdash C} tv^-$$

$$\frac{\Omega_{11} s^+ \Omega_2 \Vdash C}{\Omega_{11} s^+ adv^- \Omega_2 \Vdash C} adv^-$$

$$\frac{}{s^+ \Vdash s^+} s^+$$

The remarkable property is that *any focused proof* of

$$n^+ (n^+ \setminus (\uparrow s^+ / n^+)) n^+ (s^+ \setminus \uparrow s^+) \Vdash s^+$$

or, in abbreviated form

$$n^+ tv^- n^+ adv^- \Vdash s^+$$

can be written with only these three derived rules of inference. This is because tv^- , adv^- and s^+ are the only propositions we could possibly focus on, and focused proofs are complete. Let's explore the qualities of this search space. Neither rules adv or s are applicable, so must start with tv .

$$\frac{\vdots}{s^+ adv^- \Vdash s^+} tv$$

At this point, only adv is applicable, which yields

$$\frac{\frac{\vdots}{s^+ \Vdash s^+} adv}{s^+ adv^- \Vdash s^+} tv$$

Now, only rule s applies, completing the proof.

$$\frac{\frac{\frac{}{s^+ \Vdash s^+} s}{s^+ adv^- \Vdash s^+} adv}{n^+ tv^- n^+ adv^- \Vdash s^+} tv$$

Note that there was no nondeterminism in this proof at all, and it proceeds in three simple steps. Compare this with the small-step proof in [Section 4](#).

We can also assign negative polarity to all atoms and derive synthetic rules as we have determined which propositions we might focus on. Note that our definitions of tv and adv need to change.

$$\begin{aligned}\text{tv}^- &= \downarrow n^- \setminus (s^- / \downarrow n^-) \\ \text{adv}^- &= \downarrow s^- \setminus s^-\end{aligned}$$

and our goal becomes

$$n^- \text{tv}^- n^- \text{adv}^- \Vdash s^-$$

We can focus on tv^- , adv^- and n^- .

$$\frac{(\Omega_1 = \Omega_{11} \ \Omega_{12}) \quad \frac{\Omega_{12} \Vdash n^-}{\Omega_{12} \Vdash [\downarrow n^-]} \downarrow R \quad \frac{(\Omega_2 = \Omega_{21} \ \Omega_{22}) \quad \frac{\Omega_{21} \Vdash n^-}{\Omega_{21} \Vdash [\downarrow n^-]} \downarrow R \quad \frac{(\Omega_{11} = \Omega_{22} = \cdot, C = s^-)}{\Omega_{11} [s^-] \ \Omega_{22} \Vdash C} \text{id}^-}{\Omega_{11} [s^- / \downarrow n^-] \ \Omega_2 \Vdash C} /L}{\Omega_1 [\downarrow n^- \setminus (s^- / \downarrow n^-)] \ \Omega_2 \Vdash C} \setminus L$$

Reading off the synthetic rule:

$$\frac{\Omega_{12} \Vdash n^- \quad \Omega_{21} \Vdash n^-}{\Omega_{12} \text{tv}^- \ \Omega_{21} \Vdash s^-} \text{tv}$$

Similarly, for adv^- :

$$\frac{(\Omega_1 = \Omega_{11} \ \Omega_{12}) \quad \frac{\Omega_{12} \Vdash s^-}{\Omega_{12} \Vdash [\downarrow s^-]} \downarrow R \quad \frac{(\Omega_{11} = \Omega_2 = \cdot, C = s^-)}{\Omega_{11} [s^-] \ \Omega_2 \Vdash C} \text{id}^-}{\Omega_1 [\downarrow s^- \setminus s^-] \ \Omega_2 \Vdash C} \setminus L$$

which yields the synthetic rule

$$\frac{\Omega_{12} \Vdash s^-}{\Omega_{12} \text{adv}^- \ \Omega_{12} \Vdash s^-} \text{adv}$$

Finally, focusing on n^-

$$\frac{(\Omega_1 = \Omega_2 = \cdot, C = n^-)}{\Omega_1 [n^-] \ \Omega_2 \Vdash C} \text{id}^-$$

Summarizing the synthetic rules

$$\frac{\Omega_{12} \Vdash n^- \quad \Omega_{21} \Vdash n^-}{\Omega_{12} \text{ tv}^- \quad \Omega_{21} \Vdash s^-} \text{ tv} \quad \frac{\Omega_{12} \Vdash s^-}{\Omega_{12} \text{ adv}^- \Vdash s^-} \text{ adv} \quad \frac{}{n^- \Vdash n^-} \text{ n}$$

Reconsidering our goal

$$n^- \text{ tv}^- \quad n^- \text{ adv}^- \Vdash s^-$$

two rules are applicable: tv, which fails in two synthetic steps,

$$\begin{array}{c} \text{no rule applicable} \\ \frac{n^- \Vdash s^-}{n^- \text{ adv}^- \Vdash s^-} \text{ adv} \\ \frac{n^- \Vdash n^- \quad n^- \text{ adv}^- \Vdash s^-}{n^- \text{ tv}^- \quad n^- \text{ adv}^- \Vdash s^-} \text{ tv} \end{array}$$

and adv, which succeeds:

$$\frac{\frac{\frac{}{n^- \Vdash n^-} \text{ n} \quad \frac{}{n^- \Vdash n^-} \text{ n}}{n^- \text{ tv}^- \quad n^- \Vdash s^-} \text{ tv}}{n^- \text{ tv}^- \quad n^- \text{ adv}^- \Vdash s^-} \text{ adv}}$$

In general, there do not appear to be clear heuristics for deciding which polarization of the atoms is better for the purpose of theorem proving [MP08, MP09]. We will see in a later lecture that bottom-up logic programming (in the style of Datalog) and top-down logic programming (in the style of Prolog) can be obtained from purely positive or purely negative atoms in a fragment of the logic [CPP08].

Exercises

Exercise 1 Show one principal, one identity, one left commutative, and one right commutative case in the proof of admissibility of cut on chained sequents.

Exercise 2 Derive synthetic rules of inference for the remaining two possible polarizations of the parsing example in [Section 6](#) (where n is positive and s is negative, and vice versa). Characterize the resulting search space.

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