Lecture Notes on Chaining

15-816: Substructural Logics
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In this lecture we return to the basics of logics: how do we organize efficient proof search? This question was the origin of proof theory and it remains a central one, even if the questions around proof reduction have obtained a comparable status.

What are the key papers in history of proof theory? I am not a historian, but Frege [Fre79] seems to have initiated the formal study of mathematical proof, and Hilbert [Hil05] the study of the structure of proofs to show that certain axiomatic systems are free from contradiction. A major milestone was Gentzen's dissertation [Gen35] in which he introduces natural deduction, the sequent calculus, and cut elimination (his *Hauptsatz*) from which consistency follows easily. The discovery of the Curry-Howard isomorphism between intuitionistic natural deduction and the typed λ-calculus [How69] was a central discovery by establishing a strong connection between programming notations and constructive logic. On the side of proof search, Andreoli [And92] introduced focusing for linear logic [Gir87], which provides a much deeper understanding of proof search than either natural deduction or the sequent calculus. Focusing has proven as universal as cut elimination and eventually reshaped our understanding of proof search. It has also provided us with significant insights into computation based on proof reduction.

We will cover focusing in this and the next lecture, although it will come up throughout the remainder of this course.

1 Inversion, Chaining, and Focusing

Focusing can be seen as arising from two tightly coupled but complementary observations about proof search. The first, perhaps most easily understood, is that certain rules in the sequent calculus are *invertible*, which means that the premises can be proved if and only if the conclusion can be proved. If we see an opportunity to apply such a rule when constructing a proof in a bottom-up way, we can always safely do so. Actually, Andreoli's notion of *asynchronous connective* [And92] is slightly more refined because it abstracts away from the particular way in which we write the rules. Because of an unfortunate clash of terminology with concurrency theory, we just refer to asynchronous connectives as *negative*. A connective is *negative* if we can always decompose it via its right rule, independently of the rest of the sequent. For example, $A \mathbin{\&} B$ is negative, because whenever we are confronted with the goal of proving $\Omega \vdash A \mathbin{\&} B$ we can break this down into the subgoals $\Omega \vdash A$ and $\Omega \vdash B$ without thinking or pausing. If there is a proof, there will be one ending with the $\& R$ rule. On the other hand, $A \bullet B$ is not negative because, for example, a cut-free proof of $A \bullet (B \bullet C) \vdash (A \bullet B) \bullet C$ must end in a $\bullet L$ rule. Remarkably, connectives that are not negative turn out to be *positive* in the sense that we can always decompose them with a left rule when they appear as antecedents, independently of the rest of the sequent. There are other ways to define positive and negative connectives (see, for example, Zeilberger [Zei09]).

Chaining is a somewhat less obvious concept. Let's call a sequent *stable* if it contains only negative antecedents and positive succedents, which means that none of its propositions can a priori be decomposed. When we have reached a stable sequent we have a choice between whether to apply a right rule to the succedent or a left rule to one of the antecedents. Chaining says we can make this decision and then continue to apply right or left rules on this particular proposition and its subformulas as long as they remain positive (on the right) or negative (on the left). For example, we may be trying to prove

$$\Omega \vdash (A \oplus B) \oplus (C \oplus D)$$

for negative propositions A, B, C, and D. Because $\oplus$ is positive, we may not be able to apply a right rule, but *if* we decide to do so we can decompose this all the way to prove one of $\Omega \vdash A$ or $\Omega \vdash B$ or $\Omega \vdash C$ or $\Omega \vdash D$. We do not have to pause and consider a left rule for an antecedent after applying $\oplus R$ once.

Chaining can be considered independently of inversion: during proof search we can decide to focus on a particular positive succedent or negative antecedent even if the sequent is not stable. We can then continue to focus on its subformulas as we continue in proof search.

Inversion and chaining can be applied independently during proof search. Proofs which satisfy both strategies are called *focused*.

2 Capturing Chaining

A pervasive theme in proof theory is to capture strategies of proof search as deductive calculi. In fact, the sequent calculus was devised by Gentzen as a way to capture proof search in natural deduction. This approach has many benefits, most importantly perhaps that theorems about proof search strategies can be stated and proven without reference to an explicit external language of strategies. Instead, it often turn out (as it will here) that key properties become *internal* properties of a deductive system and therefore become subject to the battery of techniques from proof theory.

In this lecture we are interested in calculi that are more restrictive than cut-free, identity-expanded proofs. This means, proofs contain no applications of cut and the identity rule is only applied to atomic propositions. Unless otherwise stated, you should assume this for all deductive systems in today's lecture.

At a high level of abstraction, natural deduction arises from a single judgment, that of truth, and a single judgmental concept, that of *hypothetical judgment*. Sequent calculus arises when we introduce a distinction between antecedent and succedent, leading to two judgments still connected by a hypothetical judgment. Chaining arises if we have three judgments: antecedents, succedents, and propositions in focus. We also need a further principle, namely unicity: there can be at most one proposition in focus. We have already seen unicity in singleton logic and (implicitly) in all other calculi since there can be at most one succedent in a sequent.

Before we write down the judgments, we commit to the polarized form of the logic we have already seen in Section 3 of Lecture 13 where we used it to characterize communication behavior. You might recall that processes of positive type send while processes of negative type receive. Moreover, any polarization $A^\pm$ of an ordinary proposition A is provable if and only if A is.

$$\begin{array}{lll} \text{Negative} & A^-, B^- & ::= \; p^- \mid A^+ \setminus B^- \mid B^- \,/\, A^+ \mid A^- \,\&\, B^- \mid {\uparrow}A^+ \\ \text{Positive} & A^+, B^+ & ::= \; p^+ \mid A^+ \bullet B^+ \mid A^+ \circ B^+ \mid \mathbf{1} \mid A^+ \oplus B^+ \mid {\downarrow}A^- \end{array}$$

The considerations above mean that only a positive proposition can be in focus as a succedent, and only a negative proposition can be in focus as an antecedent. We have three forms of judgment, where we write $[A]$ for a proposition in focus. When we do not indicate the polarity of a proposition it can be either positive or negative.

$$\begin{array}{rcl} \Omega & \Vdash & A \\ \Omega & \Vdash & [C^+] \\ \Omega_L\ [A^-]\ \Omega_R & \Vdash & C \end{array}$$

As a shorthand, we write $\overline{\Omega}$ for Ω or $\Omega_L\ [A^-]\ \Omega_R$ and $\overline{C}$ for C or $[C^+]$. We maintain the following presupposition for all judgments:

> *There is at most one proposition in focus in any sequent.*

The right rules for negative propositions or left rules for positive proposition can be applied at any time and are not subject to focus. For example:

$$\frac{A^+\ \overline{\Omega} \Vdash B^-}{\overline{\Omega} \Vdash A^+ \setminus B^-}\ {\setminus}R \qquad \frac{\overline{\Omega}_L\ A\ B\ \overline{\Omega}_R \Vdash \overline{C}}{\overline{\Omega}_L\ (A \bullet B)\ \overline{\Omega}_R \Vdash \overline{C}}\ {\bullet}L$$

$$\frac{\overline{\Omega} \Vdash A^+}{\overline{\Omega} \Vdash {\uparrow}A^+}\ {\uparrow}R \qquad \frac{\overline{\Omega}_L\ A^-\ \overline{\Omega}_R \Vdash \overline{C}}{\overline{\Omega}_L\ ({\downarrow}A^-)\ \overline{\Omega}_R \Vdash \overline{C}}\ {\downarrow}L$$

To enter a phase of chaining we pick an arbitrary proposition of the correct polarity and put it into focus. This is a judgmental rule in the sense that it does not depend on any particular logical connective.

$$\frac{\Omega \Vdash [A^+]}{\Omega \Vdash A^+}\ \mathsf{focus}^+ \qquad \frac{\Omega_L\ [A^-]\ \Omega_R \Vdash C}{\Omega_L\ A^-\ \Omega_R \Vdash C}\ \mathsf{focus}^-$$

Once a proposition has been focused on, we can apply right and left rules to them. We show some sample set of rules.

$$\frac{\Omega_L \Vdash [A^+] \quad \Omega_R \Vdash [B^+]}{\Omega_L\ \Omega_R \Vdash [A^+ \bullet B^+]}\ {\bullet}R \qquad \frac{\Omega \Vdash [A^+] \quad \Omega_L\ [B^-]\ \Omega_R \Vdash C}{\Omega_L\ \Omega\ [A^+ \setminus B^-]\ \Omega_R \Vdash C}\ {\setminus}L$$

In each rule, the subformulas remain in focus. This is one reason why in $A^+ \setminus B^-$, A is positive while B is negative. We must lose focus when we encounter a shift since it changes to a polarity which cannot be under focus in the given position.

$$\frac{\Omega \Vdash A^-}{\Omega \Vdash [{\downarrow}A^-]}\ {\downarrow}R \qquad \frac{\Omega_L\ A^+\ \Omega_R \Vdash C}{\Omega_L\ [{\uparrow}A^+]\ \Omega_R \Vdash C}\ {\uparrow}L$$

A few rules deserve special mention. $\mathbf{1}$ is positive, and therefore has following two rules.

$$\frac{}{\cdot \Vdash [\mathbf{1}]}\ \mathbf{1}R \qquad \frac{\overline{\Omega_L}\ \overline{\Omega_R} \Vdash \overline{C}}{\overline{\Omega_L}\ \mathbf{1}\ \overline{\Omega_R} \Vdash \overline{C}}\ \mathbf{1}L$$

Note that an attempt to prove $\Omega \Vdash [\mathbf{1}]$ simply fails if Ω is not empty.

Next we consider identity, which we a priori restricted to atomic propositions. Atoms may be either positive or negative, which means they can be in focus on the right or on the left respectively. From these considerations, we obtain:

$$\frac{}{p^+ \Vdash [p^+]}\ \mathsf{id}^+ \qquad \frac{}{[p^-] \Vdash p^-}\ \mathsf{id}^-$$

A remarkable property is that a proof attempt that is focused on a positive atom p^+ will simply fail unless the collection of antecedents consists of exactly p^-. Similarly, a proof focused on p^- will fail unless it is the only antecedent and the succedent is exactly p^-.

We also note that when applying rules like $\bullet R$ and $\backslash L$ one has to decide how to split the antecedents between the two premises. We will introduce a general mechanism called *resource management* for reducing this form of nondeterminism in a future lecture.

3 Example: Parsing Revisited

We will see that chaining can be remarkably restrictive when we consider how to perform proof search. Our example comes from the Lambek calculus, where proof search corresponds to parsing.

We will try to parse *Alice likes Bob here*. Recall from Section 5 of Lecture 1 that parsing requires us to prove s, which represents a sentence.

$$\begin{array}{ccccc} \textit{Alice} & \textit{likes} & \textit{Bob} & \textit{here} & \\ : & : & : & : & \\ n & n \mathbin{\backslash} (s \mathbin{/} n) & n & s \mathbin{\backslash} s & \vdash ? : s \end{array}$$

We could start at the beginning of the sentence to combine *Alice* with *likes* or at the end of the sentence to reduce to problem of parsing the whole sentence to parsing *Alice likes Bob* as a sentence. Let's see how these two first steps work out without chaining.

$$\dfrac{\dfrac{}{n \vdash n}\,\mathsf{id} \qquad \overset{\textit{(Alice likes) Bob here}}{\vdots \atop (s \,/\, n)\; n\; (s \setminus s) \vdash s}}{n\; (n \setminus (s \,/\, n))\; n\; (s \setminus s) \vdash s}\,\setminus L \quad \text{or} \quad \dfrac{\overset{\textit{Alice likes Bob}}{\vdots \atop n\; (n \setminus (s \,/\, n))\; n \;\vdash s} \qquad \dfrac{}{s \vdash s}\,\mathsf{id}}{n\; (n \setminus (s \,/\, n))\; n\; (s \setminus s) \vdash s}\,\setminus L$$

With chaining, we can restrict the proof search such that only one of these will be possible.

To start with, we have to polarize the proposition. We pursue two options: one is where every atomic proposition is positive, and one where every atomic proposition is negative. We can also hedge our bets and make some positive and some negative (see Exercise 2).

First, we label all atoms as positive and insert the minimal number of shifts to obtain a properly polarized proposition.

$$n^+\; (n^+ \setminus (\uparrow s^+ \,/\, n^+))\; n^+\; (s^+ \setminus \uparrow s^+) \Vdash s^+$$

At this point we can not, for example, focus on $s^+ \setminus \uparrow s^+$. If we try:

$$\dfrac{\dfrac{\overset{\text{fails: no rule applicable}}{n^+\; (n^+ \setminus (\uparrow s^+ \,/\, n^+))\; n^+ \Vdash [s^+]} \qquad \overset{\vdots}{[\uparrow s^+] \Vdash s^+}}{n^+\; (n^+ \setminus (\uparrow s^+ \,/\, n^+))\; n^+\; [s^+ \setminus \uparrow s^+] \Vdash s^+}}{n^+\; (n^+ \setminus (\uparrow s^+ \,/\, n^+))\; n^+\; (s^+ \setminus \uparrow s^+) \Vdash s^+}\,\mathsf{focus}^-$$

In the first premise, no rule is applicable since the atom s^+ is in focus, but the antecedent is not just s^+. The second premise would actually be provable after one more forced step.

We can try each possibility, but they all fail immediately, leaving only $n^+ \setminus (\uparrow s^+ \,/\, n^+)$. We show the full phase of focusing, until we close each branch or no proposition is in focus any longer. You should verify, that these are the only possibilities.

$$\dfrac{\dfrac{}{n^+ \Vdash [n^+]}\,\mathsf{id}^+ \quad \dfrac{\dfrac{}{n^+ \Vdash [n^+]}\,\mathsf{id}^+ \quad \dfrac{\dfrac{\vdots}{s^+\ (s^+ \setminus {\uparrow}s^+) \Vdash s^+}}{[{\uparrow}s^+]\ (s^+ \setminus {\uparrow}s^+) \Vdash s^+}\,{\uparrow}L}{[{\uparrow}s^+ / n^+]\ n^+\ (s^+ \setminus {\uparrow}s^+) \Vdash s^+}\,/L}{\dfrac{n^+\ [n^+ \setminus ({\uparrow}s^+ / n^+)]\ n^+\ (s^+ \setminus {\uparrow}s^+) \Vdash s^+}{n^+\ (n^+ \setminus ({\uparrow}s^+ / n^+))\ n^+\ (s^+ \setminus {\uparrow}s^+) \Vdash s^+}\,\mathsf{focus}^-}\,\setminus L$$

At this point we can once again only focus on one antecedent, namely $s^+ \setminus {\uparrow}s^+$.

$$\dfrac{\dfrac{}{s^+ \Vdash [s^+]}\,\mathsf{id}^+ \quad \dfrac{\dfrac{\vdots}{s^+ \Vdash s^+}}{[{\uparrow}s^+] \Vdash s^+}\,{\uparrow}L}{s^+\ (s^+ \setminus {\uparrow}s^+) \Vdash s^+}\,\mathsf{focus}^-$$

We can now complete the proof by finally focusing on s^+ on the right.

$$\dfrac{\dfrac{}{s^+ \Vdash [s^+]}\,\mathsf{id}^+ \quad \dfrac{\dfrac{\dfrac{}{s^+ \Vdash [s^+]}\,\mathsf{id}^+}{s^+ \Vdash s^+}\,\mathsf{focus}^+}{[{\uparrow}s^+] \Vdash s^+}\,{\uparrow}L}{s^+\ (s^+ \setminus {\uparrow}s^+) \Vdash s^+}\,\mathsf{focus}^-$$

Alternatively to this whole proof, we now make all atomic propositions *negative* and insert the minimum number of shifts to the result is well-polarized.

$$n^-\ ({\downarrow}n^- \setminus (s^- / {\downarrow}n^-))\ n^-\ ({\downarrow}s^- \setminus s^-) \Vdash s^-$$

As you should verify from the rules, the only possibility here is to focus on ${\downarrow}s^- \setminus s^-$; all other propositions we either cannot focus on at all (like s^- on the right), or will fail quickly without the focusing phase completing.

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ n^- \; (\downarrow n^- \setminus (s^- / \downarrow n^-)) \; n^- \Vdash s^-\end{array}}{n^- \; (\downarrow n^- \setminus (s^- / \downarrow n^-)) \; n^- \Vdash [\downarrow s^-]}\,{\downarrow}R \qquad \dfrac{}{[s^-] \Vdash s^-}\,\mathsf{id}^-}{n^- \; (\downarrow n^- \setminus (s^- / \downarrow n^-)) \; n^- \; [\downarrow s^- \setminus s^-] \Vdash s^-}\,{\setminus}L}{n^- \; (\downarrow n^- \setminus (s^- / \downarrow n^-)) \; n^- \; (\downarrow s^- \setminus s^-) \Vdash s^-}\,\mathsf{focus}^-$$

At this point, only the focus on $\downarrow n^- \setminus (s^- / n^-)$ will succeed. The focusing phase will build this partial proof:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ n^- \Vdash n^-\end{array}}{n^- \Vdash [\downarrow n^-]}\,{\downarrow}R \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\\ n^- \Vdash n^-\end{array}}{n^- \Vdash [\downarrow n^-]}\,{\downarrow}R \qquad \dfrac{}{[s^-] \Vdash s^-}\,\mathsf{id}^-}{[s^- / \downarrow n^-] \; n^- \Vdash s^-}\,/L}{n^- \; [\downarrow n^- \setminus (s^- / \downarrow n^-)] \; n^- \Vdash s^-}\,{\setminus}L}{n^- \; (\downarrow n^- \setminus (s^- / \downarrow n^-)) \; n^- \Vdash s^-}\,\mathsf{focus}^-$$

The remaining subproofs can be completed in two steps by focusing on n^- in the antecedent.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[n^-] \Vdash n^-}\,\mathsf{id}^-}{n^- \Vdash n^-}\,\mathsf{focus}^-}{n^- \Vdash [\downarrow n^-]}\,{\downarrow}R \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{[n^-] \Vdash n^-}\,\mathsf{id}^-}{n^- \Vdash n^-}\,\mathsf{focus}^-}{n^- \Vdash [\downarrow n^-]}\,{\downarrow}R \qquad \dfrac{}{[s^-] \Vdash s^-}\,\mathsf{id}^-}{[s^- / \downarrow n^-] \; n^- \Vdash s^-}\,/L}{n^- \; [\downarrow n^- \setminus (s^- / \downarrow n^-)] \; n^- \Vdash s^-}\,{\setminus}L}{n^- \; (\downarrow n^- \setminus (s^- / \downarrow n^-)) \; n^- \Vdash s^-}\,\mathsf{focus}^-$$

4 Soundness of Chaining

The next order of business is to convince ourselves (and everyone else who cares) that the system "works correctly". This means that (in polarized ordered logic) we prove exactly the same sequents in the chaining calculus as in the without the restrictions imposed by chaining. We need to show:

Soundness: If $\Omega \Vdash C$ then $\Omega \vdash C$.

Completeness: If $\Omega \vdash C$ then $\Omega \Vdash C$.

In general, proof search procedures restrict the allowed inferences in order to cut down on the search space, so soundness is usually straightforward while completeness is difficult.

Soundness here is very easy: take a chaining proof and erase all the focusing annotations, that is, remove the brackets $[-]$. The positive and negative focusing rules then disappear, since premise and conclusion become the same sequent, and all other rules become valid rules in the polarized, unfocused sequent calculus.

Formally, we would generalize the induction hypothesis over all three judgment forms and prove the theorem by mutual induction on the given derivation.

5 Completness of Chaining

How do we approach the completeness proof? Usually, we would just try the a straightforward structural induction over the sequent proof to see how it breaks down. This might provide some hints how to complete it.

Theorem 1 (Completeness of Chaining) *If* $\Omega \vdash A$ *then* $\Omega \Vdash A$.

Proof: (Attempt) By induction over the structure of $\overset{\mathcal{D}}{\Omega \vdash A}$. For a while, this goes well (depending on how we start).

Case:

$$\mathcal{D} = \dfrac{\begin{matrix}\mathcal{D}_2 \\ A_1^+\ \Omega \vdash A_2^-\end{matrix}}{\Omega \vdash A_1^+ \setminus A_2^-}\setminus R \qquad \text{Then} \qquad \dfrac{\begin{matrix}\text{i.h. on } \mathcal{D}_2 \\ A_1^+\ \Omega \Vdash A_2^-\end{matrix}}{\Omega \Vdash A_1^+ \setminus A_2^-}\setminus R$$

This pattern repeats for right rules on negative propositions and left rules on positive propositions. Clearly, our system was specifically engineered to make this possible. It breaks down, for example, in the case of a right rule on a positive proposition since the induction hypothesis will not give us anything in focus.

Case:

$$\mathcal{D} = \dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2 \\ \Omega_L \vdash A_1^+ & \Omega_R \vdash A_2^+\end{array}}{\Omega_L\ \Omega_R \vdash A_1^+ \bullet A_2^+}\ \bullet R$$

By applying the induction hypothesis we get something like

$$\begin{array}{c}\begin{array}{cc}\text{i.h. on } \mathcal{D}_1 & \text{i.h. on } \mathcal{D}_2 \\ \Omega_L \Vdash A_1^+ & \Omega_R \Vdash A_2^+\end{array} \\ \vdots \\ \Omega_L\ \Omega_R \Vdash A_1^+ \bullet A_2^+\end{array}$$

We might try focusing on $A_1^+ \bullet A_2^+$, but this fails:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\text{i.h. on } \mathcal{D}_1 \\ \Omega_L \Vdash A_1^+\end{array}}{\Omega_L \Vdash [A_1^+]}\ ?? \quad \dfrac{\begin{array}{c}\text{i.h. on } \mathcal{D}_2 \\ \Omega_R \Vdash A_2^+\end{array}}{\Omega_L \Vdash [A_2^+]}\ ??}{\Omega_L\ \Omega_R \Vdash [A_1^+ \bullet A_2^+]}\ \bullet R}{\Omega_L\ \Omega_R \Vdash A_1^+ \bullet A_2^+}\ \mathsf{focus}^+$$

The problem here is that it is *not* the case that, in general, $\Omega \Vdash A^+$ implies $\Omega \Vdash [A^+]$! As a simple counterexample, consider $\mathbf{1} \Vdash \mathbf{1}$: we have to apply $\mathbf{1}L$ before we can focus on $\mathbf{1}$ on the right. No amount of generalization of the induction hypothesis will make a counterexample go away. However, there is a different way forward: we can close the gap using cut and identity for the chaining calculus! We may have some hope that these will be provable.

$$\dfrac{\begin{array}{c}\text{i.h. on } \mathcal{D}_2 \\ \Omega_R \Vdash A_2^+\end{array} \quad \dfrac{\begin{array}{c}\text{i.h. on } \mathcal{D}_1 \\ \Omega_L \Vdash A_1^+\end{array} \quad \dfrac{\dfrac{\dfrac{}{A_1^+ \Vdash [A_1^+]}\ \mathsf{id}^+ \quad \dfrac{}{A_2^+ \Vdash [A_2^+]}\ \mathsf{id}^+}{A_1^+\ A_2^+ \Vdash [A_1^+ \bullet A_2^+]}\ \bullet R}{A_1^+\ A_2^+ \Vdash A_1^+ \bullet A_2^+}\ \mathsf{focus}^+}{\Omega_L\ A_2^+ \Vdash A_1^+ \bullet A_2^+}\ \mathsf{cut}_{A_1^+}}{\Omega_L\ \Omega_R \Vdash A_1^+ \bullet A_2^+}\ \mathsf{cut}_{A_2^+}$$

All other cases will follow a similar pattern, so if cut and identity are admissible in the chaining calculus, then it will be complete. □

It takes tremendous experience to find this particular elegant proof. Andreoli's original proof and many thereafter (for example, Howe [How98]) were much more complicated and less scalable. The idea to use admissibility of cut and identity in this manner originates with Chaudhuri [Cha06] for linear logic. Some remaining infelicities with identity expansion were finally solved by Simmons for ordered logic [Sim12] and intuitionistic structural logic [Sim14].

6 Admissibility of Identity in the Chaining Calculus

In the next lecture we will sketch the admissibility of cut in the chaining calculus; here we show admissibility of identity.

As usual, admissibility of identity follows by induction on the structure of the the proposition. We need three forms.

Theorem 2 (Admissibility of Identity for Chaining) *The following are admissible:*

$$\frac{}{[A^-] \Vdash A^-}\ \mathsf{id}_A^- \qquad \frac{}{A^+ \Vdash [A^+]}\ \mathsf{id}_A^+ \qquad \frac{}{A \Vdash A}\ \mathsf{id}_A$$

Proof: By induction on the structure of A, where id_A can call on the induction hypothesis for id_A^- or id_A^+, depending on the polarity of A. For example:

$$\dfrac{\dfrac{\begin{array}{c}\text{i.h. on } A_2^+ \\ A_2^+ \vdash [A_2^+]\end{array} \quad \begin{array}{c}\text{i.h. on } A_1^- \\ [A_1^-] \vdash A_1^-\end{array}}{[A_1^- \ / \ A_2^+]\ A_2^+ \vdash A_1^-}\ /L}{[A_1^- \ / \ A_2^+] \vdash A_1^- \ / \ A_2^+}\ /R$$

□

Exercises

Exercise 1 We might drop the focus$^-$ and focus$^+$ rules if instead the $\uparrow R$ and $\downarrow L$ rule put the positive or negative proposition, respectively, in focus. Try to discover and perhaps prove if this would yield an equivalent calculus. If yes, discuss its merits and demerits.

Exercise 2 Investigate how the set of proofs are restricted in the parsing example from Section 3 if

1. n is negative and s is positive, and
2. n is positive and s is negative.

Exercise 3 In the completeness proof for chaining, show the cases for

1. $\backslash L$,
2. $\mathbf{1}R$,
3. id^-,
4. $\uparrow R$, and
5. $\uparrow L$.

You should assume that identity and cut are admissible in the chaining calculus.

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