Types and Programming Languages (15-814), Fall 2018
Assignment 8: Proofs and Stages

Contact: 15-814 Course Staff

Due Tuesday, November 20, 2018, 11:59pm

This assignment is due by 11:59pm on the above date and it must be submitted electronically as a PDF file on Canvas. Please use the attached template to typeset your assignment and make sure to include your full name and Andrew ID. As before, problems marked "WB" are subject to the whiteboard policy; all other problems must be done individually.

Task 0 (0 points)**.** How long did you spend on this assignment? Please list the questions that you discussed with classmates using the whiteboard policy.

1 Staging Computation

Recall we defined binary numbers to be $\rho(\alpha.\,(\epsilon : \mathbf{1}) + (b0 : \alpha) + (b1 : \alpha))$, or equivalently in concrete syntax:

```
data Bin = Eps | B0 Bin | B1 Bin
```

We also implemented multiplication and addition functions:

```
mult : Bin -> Bin -> Bin
plus : Bin -> Bin -> Bin
```

You can treat these functions as built-in and use them freely below. You are welcome to use the following concrete syntax for $\mathbf{box}\ e$ and $\mathbf{case}\ e\ \{\mathbf{box}\ u \Rightarrow e'\}$:

```
box e
case e { box u => e' }
```

You should assume our language supports lazy and eager products, sums, recursive types, general recursion, etc.

Task 1 (10 points, WB). Give an exponentiation function

$$\mathsf{exp} : \mathsf{bin} \to \Box(\mathsf{bin} \to \mathsf{bin})$$

where the first argument of type bin is the exponent and the other argument is the base.

Task 2 (10 points, WB). Give a multiplication function

$$\mathsf{mult} : \mathsf{bin} \to \Box(\mathsf{bin} \to \mathsf{bin}).$$

If this cannot be done, briefly explain why.

Recall we defined a function $\mathsf{eval} : \forall\alpha.\Box\alpha \to \alpha$ by

$$\mathsf{eval} = \lambda x.\mathbf{case}\ x\ \{\mathbf{box}\ u \Rightarrow u\}.$$

Task 3 (10 points, WB). For each of the following types τ, define a closed term $\mathsf{lift}_\tau : \tau \to \Box\tau$ satisfying $v \simeq (\mathsf{eval} \circ \mathsf{lift}_\tau)(v)$ for all values $v : \tau$. If this cannot be done for a particular τ, briefly and informally explain why.

1. $\tau = \mathsf{bin}$,
2. $\tau = \mathsf{bin} \otimes \mathsf{bin}$, and
3. $\tau = \mathsf{bin}\ \&\ \mathsf{bin}$.

Hint. Do not forget what you have learned from past assignments.

2 Proofs and Programs

Task 4 (10 points, WB). Annotate the following proof with proof terms:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\boxed{} : B \supset C}\,f \qquad \dfrac{\dfrac{}{\boxed{} : A \supset B}\,g \quad \dfrac{}{\boxed{} : A}\,a}{\boxed{} : B}\,{\supset}E}{\boxed{} : C}\,{\supset}E}{\boxed{} : A \supset C}\,{\supset}I^a}{\boxed{} : (A \supset B) \supset (A \supset C)}\,{\supset}I^g}{\boxed{} : (B \supset C) \supset (A \supset B) \supset (A \supset C)}\,{\supset}I^f$$

Hint. To save time, copy-paste the task statement into the solution environment and fill in the `[]` in the `\fillmein[]` occurrences in the template!

Task 5 (10 points, WB)**.** For each of the following proof terms witnessing the proposition $(\top \wedge \top) \supset (\bot \vee \top) \supset (\top \wedge \top)$, give the corresponding natural deduction proof.

1. $\lambda x.\lambda y.\langle\!| x \cdot l, \langle\!|\,|\!\rangle |\!\rangle$,
2. $\lambda x.\lambda y.$**case** y $\{l \cdot w \Rightarrow$ **case** w $\{\,\} \mid r \cdot v \Rightarrow \langle\!| v, v |\!\rangle\}$.

Make sure to label all of your inferences with the names of the rule you used ($\supset I^x$, x, $\supset E$, $\wedge I$, $\wedge E_1$, $\wedge E_2$, $\vee I_1$, etc.).

The *most general proposition* witnessed by a proof term x is the proposition G such that

- x is a proof term of G, i.e., $x : G$, and
- for any other proposition P, if $x : P$, then there exists a substitution σ mapping proposition variables in G to propositions such that $P = \sigma G$.

For example, the most general proposition for the proof term $\lambda x.x$ is $A \supset A$, even though $\lambda x.x$ is also a proof term for $\bot \supset \bot$ and $(B \vee C \wedge \top) \supset (B \vee C \wedge \top)$. Similarly, the most general proposition for $\lambda x.\lambda y.\langle\!| x, y |\!\rangle$ is $A \supset B \supset A \wedge B$. The concept of the most general proposition witnessed by a proof term is analogous to the concept of the most general type for a term.

Task 6 (10 points, WB)**.** For each of the following proof terms, give the most general proposition it witnesses:

1. $\lambda x.\lambda y.\langle\!| x \cdot l, \langle\!|\,|\!\rangle |\!\rangle$,
2. $\lambda x.\lambda y.$**case** y $\{l \cdot w \Rightarrow$ **case** w $\{\,\} \mid r \cdot v \Rightarrow \langle\!| v, v |\!\rangle\}$.