Lecture Notes on
Sequent Calculus

15-814: Types and Programming Languages
Frank Pfenning

Lecture 20
November 13, 2018

1 Introduction

So far, we have presented logical inference in the style of *natural deduction*. Propositions corresponded to types, proofs to programs, and proof reduction to computation.

In this lecture we develop an alternative presentation of logical inference using the *sequent calculus*, also due to Gentzen [Gen35]. From a logical perspective, we change the direction of proof construction, without changing what can be proved. From a computational perspective, this opens up new avenues for capturing computational phenomena, namely *message-passing concurrency* (as we will see in the next lecture).

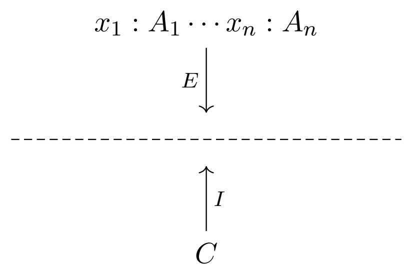
2 Sequent Calculus Constructs Natural Deductions

As we have seen in the last lecture, during proof construction we

1. Use *introduction rules* from the bottom up. For example, to prove $A \land B$ *true* we reduce it to the subgoals of proving A *true* and B *true*, using $\land I$
2. Use *elimination rules* from the top down. For example, if we know $A \land B$ *true* we may conclude A *true* using $\land E_1$.

The two directions of inference "meet in the middle", when something we have inferred by eliminations matches the conclusion we are trying to prove.

Schematically (and somewhat oversimplified), proving conclusion C from assumptions $x_1 : A_1, \ldots, x_n : A_n$ labeled with variables looks like

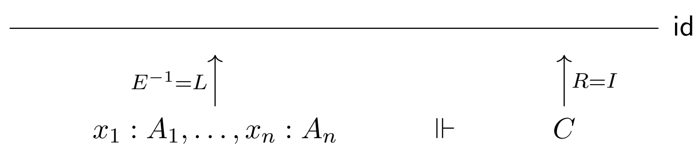


where I indicates introduction rules, E indicates elimination rules, and the dashed line is where proof construction meets in the middle.

This bidirectional reasoning can be awkward, especially if we are trying to establish metatheoretic properties such as consistency of a logical system, that is, that it cannot prove a contradiction $\bot$. Gentzen's idea was to write down the current state of proof construction in a *sequent*

$$x_1 : A_1, \ldots, x_n : A_n \Vdash C$$

and have *right rules* decomposing the *succedent* C while *left rules* decompose the *antecedents* A_i. In this transformation, the *right rules* correspond very directly to the *introduction rules* of natural deduction, because they proceed in the same direction (bottom-up). On the other hand, the left rules correspond to the *inverted elimination rules* because we have to changes their direction from top-down to bottom-up. Schematically:



Rather then meeting in the middle, we now complete the proof construction when we have inferred an antecedent that exactly matches the succedent with the *identity rule*.

$$\frac{}{\Gamma, x : A \Vdash A}\ \mathsf{id}$$

For this and the following rules to make sense, we assume the antecedents are unordered (can be freely exchanged) and all variables x_i are distinct.

Let's use our basic intuition to derive some rules, starting with conjunction.

$$\frac{\Gamma \Vdash A \quad \Gamma \Vdash B}{\Gamma \Vdash A \wedge B} \wedge R \qquad \frac{\Gamma, x : A \wedge B, y : A \Vdash C}{\Gamma, x : A \wedge B \Vdash C} \wedge L_1 \qquad \frac{\Gamma, x : A \wedge B, z : B \Vdash C}{\Gamma, x : A \wedge B \Vdash C} \wedge L_2$$

The right rule corresponds direction to the introduction rule and the two left rules to the two elimination rules (read upside down) with the twist that the antecedent $x : A \wedge B$ persists in the premise. All of our left rules in this lecture will preserve the antecedent to which we apply the rule so we can use it again, even though it some cases that may seem redundant. As usual, we assume that all antecedent labels x_i are distinct, so that y (in $\wedge L_1$) and z (in $\wedge L_2$) are not aready declared in Γ and different from x.

The right rule for implication is also straightforward.

$$\frac{\Gamma, x : A \Vdash B}{\Gamma \Vdash A \supset B} \supset R$$

How do we use the knowledge of $A \supset B$ in a proof of C? If we can also supply a proof of A we are allowed to assume B in the proof of C.

$$\frac{\Gamma, x : A \supset B \Vdash A \quad \Gamma, x : A \supset B, y : B \Vdash C}{\Gamma, x : A \supset B \Vdash C} \supset L$$

This rule looks a little clunky because we repeat x in both premises. If we leave this implicit

$$\frac{\Gamma \Vdash A \quad \Gamma, y : B \Vdash C}{\Gamma, x : A \supset B \Vdash C} \supset L^*$$

it looks better, but only if we understand that $x : A \supset B$ actually persists in both premises.

In lecture, a student asked the excellent question why we only extract A or B from $A \wedge B$ with the two left rules in the antecedent, but not both together? One answer that we want to faithfully model proof construction in natural deduction, and there happen to be two separate rules to extract the two components. Another answer is: yes, let's do this! What we obtain is actually a *different* logical connective!

$$\frac{\Gamma, x : A \otimes B, y : A, z : B \Vdash C}{\Gamma, x : A \otimes B \Vdash C} \otimes L$$

The corresponding right rule is actually familiar:

$$\frac{\Gamma \Vdash A \quad \Gamma \Vdash B}{\Gamma \Vdash A \otimes B} \otimes R$$

When we reverse-engineer the corresponding natural deduction rules we have

$$\frac{A\ \mathit{true} \quad B\ \mathit{true}}{A \otimes B\ \mathit{true}}\ \otimes I \qquad \frac{A \otimes B\ \mathit{true} \quad \begin{array}{c}\overline{A\ \mathit{true}}^{\ y} \quad \overline{B\ \mathit{true}}^{\ z} \\ \vdots \\ C\ \mathit{true}\end{array}}{C\ \mathit{true}}\ \otimes E^{y,z}$$

When looking at this from the lense of proof terms, we realize that $A \wedge B$ corresponds to *lazy pairs* $\tau \mathbin{\&} \sigma$, while $A \otimes B$ corresponds to *eager pairs* $\tau \otimes \sigma$. So even though, purely logically, $A \wedge B \equiv A \otimes B$, they have a different computational meaning. This meaning will diverge even further in the next lecture when we refine the logic and the two connectives are no longer equivalent.

We have left out disjunction, truth, and falsehood, but the rules for them are easy to complete.

However, there is still one rule we need, which is the *converse* of the identity rule. Identity

$$\frac{}{\Gamma, x : A \Vdash A}\ \mathsf{id}$$

expresses that if we assume A we can conclude A. The converse would say if we conclude A we can assume A. Expressed as a rule this is called *cut*:

$$\frac{\Gamma \Vdash A \quad \Gamma, x : A \Vdash C}{\Gamma \Vdash C}\ \mathsf{cut}$$

Mathematically, this corresponds to introducing the lemma A into a proof of C. We have to prove the lemma (first premise) but then we can use it to prove our succedent (second premise). Generally, in mathematics, finding the right lemma (such as: a generalization of the induction hypothesis) is a critical part of finding proofs. Here, in pure logic with only the usual connective, this rule turns out to be *redundant*. That is, any sequent $\Gamma \Vdash C$ we can derive *with* the rule of cut we can also derive *without* the rule of cut. This is of fundamental *logical* significance because it allows us to establish easily that the system is consistent. All other rules break down either a succedent or an antecedent, and there is no rule to break down falsehood $\bot$, and therefore the cannot be a cut-free proof of $\cdot \Vdash \bot$.

3 Soundness of the Sequent Calculus

By soundness we mean: whenever $\Gamma \Vdash A$ in the sequent calculus then also $\Gamma \vdash A$ in natural deduction. In other words, if we view natural deduction as defining the meaning of the logical connectives, then the sequent calculus let's us draw only correct conclusions. In the next section we prove that the other direction also holds.

Theorem 1 (Soundness of the Sequent Calculus) *If* $\Gamma \Vdash A$ *then* $\Gamma \vdash A$.

Proof: The proof is by rule induction over the given sequent calculus derivation. In constructing the natural deduction proof we write all the hypothesis as $x : A$ to the left of the turnstile instead of the assumption $\overline{A}^{\,x}$ in the usual two-dimensional form. We show only two cases.

Case:

$$\frac{\Gamma, x : A \Vdash B}{\Gamma \Vdash A \supset B} \supset\! R$$

Then

$\Gamma, x : A \vdash B$	By i.h.
$\Gamma \vdash A \supset B$	By rule $\supset\! I^x$

Case:

$$\frac{\Gamma, x : A \supset B \Vdash A \quad \Gamma, x : A \supset B, y : B \Vdash C}{\Gamma, x : A \supset B \Vdash C} \supset\! L$$

Then

$\Gamma, x : A \supset B \vdash A \supset B$	By rule var
$\Gamma, x : A \supset B \vdash A$	By i.h. on first premise
$\Gamma, x : A \supset B \vdash B$	By rule $\supset\! E$
$\Gamma, x : A \supset B, y : B \vdash C$	By i.h. on second premise
$\Gamma, x : A \supset B \vdash C$	By substitution

In the last step we use the substitution property on the two lines just above, substituting the proof of B for the hypothesis $y : B$ in the proof of C.

□

A perhaps more insightful way to present this proof is to annotate the sequent derivation with proof terms drawn from natural deduction. We want to synthesize

$$\Gamma \Vdash M : A$$

such that

$$\Gamma \vdash M : A$$

that is, M is a well-typed (natural deduction) proof term of A. If we can annotate each sequent derivation in this manner, then it will be sound. Fortunately, this is not very difficult. We just have to call upon substitution in the right places. Consider identity and cut.

$$\frac{x : A \in \Gamma}{\Gamma \Vdash x : A}\ \mathsf{id} \qquad \frac{\Gamma \Vdash M : A \quad \Gamma, x : A \Vdash N : C}{\Gamma \Vdash [M/x]N : C}\ \mathsf{cut}$$

Identity just uses a variable, while cut corresponds to substitution. Note that if $M : A$ we can substitute it for the variable $x : A$ appearing in N.

Next consider implication. The right rule (as usual) just mirrors the introduction rule. Intuitively, we obtain M from the induction hypothesis (for an induction we are not spelling out in detail).

$$\frac{\Gamma, x : A \Vdash M : B}{\Gamma \Vdash \lambda x.\, M : A \supset B}\ {\supset}R$$

The left rule is trickier (also as usual!)

$$\frac{\Gamma, x : A \supset B \Vdash M : A \quad \Gamma, x : A \supset B, y : B \Vdash N : C}{\Gamma, x : A \supset B \Vdash ?? : C}\ {\supset}L$$

We assume we can annotate the premises, so we have M and N. But how to we construct a proof term for C that does not depend on y? The explicit proof that we have done before tells is it has to be by substitution for $y : B$ and the term will be x (of type $A \supset B$) applied to M (of type A):

$$\frac{\Gamma, x : A \supset B \Vdash M : A \quad \Gamma, x : A \supset B, y : B \Vdash N : C}{\Gamma, x : A \supset B \Vdash [(x\, M)/y]N : C}\ {\supset}L$$

The rules for conjunction are even simpler: in the left rule the additional antecedent y or z is justified by the first and second projection of x.

$$\frac{\Gamma \Vdash M : A \quad \Gamma \Vdash N : B}{\Gamma \Vdash \langle\!| M, N |\!\rangle : A \wedge B}\ {\wedge}R$$

$$\frac{\Gamma, x : A \wedge B, y : A \Vdash N : C}{\Gamma, x : A \wedge B \Vdash [(x \cdot l)/y]N : C}\ {\wedge}L_1 \qquad \frac{\Gamma, x : A \wedge B, z : B \Vdash N : C}{\Gamma, x : A \wedge B \Vdash [(x \cdot r)/z]N : C}\ {\wedge}L_2$$

Finally, the other (eager) form of conjunction. No substitution is required here because the case-like elimination construct already matches the sequent calculus rule.

$$\frac{\Gamma \Vdash M : A \quad \Gamma \Vdash N : B}{\Gamma \Vdash \langle M, N\rangle : A \otimes B} \otimes R \qquad \frac{\Gamma, x : A \otimes B, y : A, z : B \Vdash N : C}{\Gamma, x : A \otimes B \Vdash \textbf{case}\ x\ \{\langle y, z\rangle \Rightarrow N\} : C} \otimes L$$

4 Completeness of the Sequent Calculus

Now we would like to go the other direction: anything we can prove with natural deduction we can also prove in the sequent calculus.

Theorem 2 (Completeness of the Sequent Calculus) *If* $\Gamma \vdash A$ *then* $\Gamma \Vdash A$.

Proof: By rule induction on the deduction of $\Gamma \vdash A$. We show only two representative cases.

Case:

$$\frac{\Gamma, x : A \vdash B}{\Gamma \vdash A \supset B} \supset I$$

Then we construct

$$\frac{\overset{\text{i.h.}}{\Gamma, x : A \Vdash B}}{\Gamma \Vdash A \supset B} \supset R$$

Case:

$$\frac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B} \supset E$$

This case requires some thought. From the induction hypothesis we obtain $\Gamma \Vdash A \supset B$ and $\Gamma \Vdash A$ and we need to conclude $\Gamma \Vdash B$. The left rules of the sequent calculus, however, go in the wrong direction, so we cannot easily use the knowledge of $A \supset B$.

In order to create an implication on the left-hand side, we can use the rule of cut, which says that if we know A we can assume A for any proposition A. That is,

$$\frac{\overset{\text{i.h.}}{\Gamma \Vdash A \supset B} \quad \Gamma, x : A \supset B \Vdash ??}{\Gamma \Vdash ??} \text{cut}$$

Since we are trying to prove $\Gamma \Vdash B$, using B for ?? appears to be the obvious choice.

$$\dfrac{\overset{\text{i.h.}}{\Gamma \Vdash A \supset B} \quad \Gamma, x : A \supset B \Vdash B}{\Gamma \Vdash B}\ \mathsf{cut}$$

Now we can use the $\supset L$ rules as intended and use the proof of A we have by induction hypothesis.

$$\dfrac{\overset{\text{i.h.}}{\Gamma \Vdash A \supset B} \quad \dfrac{\overset{\text{i.h.}}{\Gamma \Vdash A} \quad \Gamma, y : B \Vdash B}{\Gamma, x : A \supset B \Vdash B}\ \supset L}{\Gamma \Vdash B}\ \mathsf{cut}$$

The final unproved goal now just follows by the identity.

$$\dfrac{\overset{\text{i.h.}}{\Gamma \Vdash A \supset B} \quad \dfrac{\overset{\text{i.h.}}{\Gamma \Vdash A} \quad \dfrac{}{\Gamma, y : B \Vdash B}\ \mathsf{id}}{\Gamma, x : A \supset B \Vdash B}\ \supset L}{\Gamma \Vdash B}\ \mathsf{cut}$$

Here, we have omitted some unneeded antecedents, particularly $x : A \supset B$ in the premises of $\supset L$. They easily be restored by adding them to the antecedents of every sequent in the deduction. We do not prove this obvious property called *weakening*.

□

Before we investigate what this translation means on proof terms, we revise our language of proof terms for the sequent calculus.

5 Proof Terms for Sequent Calculus

In the soundness proof, we have simply assigned natural deduction proof terms to sequent deductions. This served the purpose perfectly, but such terms do not contain sufficient information to actually reconstruct a sequent proof. For example, in

$$\dfrac{\Gamma \Vdash M : A \quad \Gamma, x : A \Vdash N : C}{\Gamma \Vdash [M/x]N : C}\ \mathsf{cut}$$

we would know only the result of substituting M for x in N, which is clearly not enough information to extract M, N, or even A. We restate the rules, this time giving informative proof terms.

$$\frac{x : A \in \Gamma}{\Gamma \Vdash x : A}\ \mathsf{id} \qquad \frac{\Gamma \Vdash M : A \quad \Gamma, x : A \Vdash N : C}{\Gamma \Vdash \mathbf{let}\ x : A = M\ \mathbf{in}\ N : C}\ \mathsf{cut}$$

$$\frac{\Gamma, x : A \Vdash M : B}{\Gamma \Vdash \lambda x.\, M : A \supset B}\ {\supset}R \qquad \frac{\Gamma, x : A \supset B \Vdash M : A \quad \Gamma, x : A \supset B, y : B \Vdash N : C}{\Gamma, x : A \supset B \Vdash \mathbf{let}\ y = x\, M\ \mathbf{in}\ N : C}\ {\supset}L$$

$$\frac{\Gamma \Vdash M : A \quad \Gamma \Vdash N : B}{\Gamma \Vdash \langle\!| M, N |\!\rangle : A \wedge B}\ {\wedge}R$$

$$\frac{\Gamma, x : A \wedge B, y : A \Vdash N : C}{\Gamma, x : A \wedge B \Vdash \mathbf{let}\ y = x \cdot l\ \mathbf{in}\ N : C}\ {\wedge}L_1 \qquad \frac{\Gamma, x : A \wedge B, z : B \Vdash N : C}{\Gamma, x : A \wedge B \Vdash \mathbf{let}\ z = x \cdot r\ \mathbf{in}\ N : C}\ {\wedge}L_2$$

$$\frac{\Gamma \Vdash M : A \quad \Gamma \Vdash N : B}{\Gamma \Vdash \langle M, N \rangle : A \otimes B}\ {\otimes}R \qquad \frac{\Gamma, x : A \otimes B, y : A, z : B \Vdash N : C}{\Gamma, x : A \otimes B \Vdash \mathbf{case}\ x\ \{\langle y, z\rangle \Rightarrow N\} : C}\ {\otimes}L$$

Just like continuation-passing style, this form of proof term names intermediate values, but it does not make a continuation explicit. We could now rewrite our dynamics on these terms and the rules would be more streamlined since they already anticipate the order in which expressions are evaluated. We can also easily translate from this form to natural deduction terms by replacing all constructs **let** $x = M$ **in** N by $[M/x]N$. More formally, we write $M^\dagger$:

$$\begin{array}{lcl}
(x)^\dagger & = & x \\
(\mathbf{let}\ x : A = M\ \mathbf{in}\ N)^\dagger & = & [M^\dagger/x]N^\dagger \\[4pt]
(\lambda x.\, M)^\dagger & = & \lambda x.\, M^\dagger \\
(\mathbf{let}\ y = x\, M\ \mathbf{in}\ N)^\dagger & = & [x\, M^\dagger/y]N^\dagger \\[4pt]
\langle\!| M, N |\!\rangle^\dagger & = & \langle\!| M^\dagger, N^\dagger |\!\rangle \\
(\mathbf{let}\ y = x \cdot l\ \mathbf{in}\ N)^\dagger & = & [x \cdot l/y]N^\dagger \\
(\mathbf{let}\ z = x \cdot r\ \mathbf{in}\ N)^\dagger & = & [x \cdot r/z]N^\dagger \\[4pt]
\langle M, N \rangle^\dagger & = & \langle M^\dagger, N^\dagger \rangle \\
(\mathbf{case}\ x\ \{\langle y, z\rangle \Rightarrow N\})^\dagger & = & \mathbf{case}\ x\ \{\langle y, z\rangle \Rightarrow N^\dagger\}
\end{array}$$

One question is how we translate in the other direction, from natural deduction to these new forms of terms. We write this as M^*. Our proof of the completeness of the sequent calculus holds the key. We read off:

$$\begin{array}{lcl}(x)^* & = & x \\ (\lambda x.\, M)^* & = & \lambda x.\, M^* \\ (M\, N)^* & = & \textbf{let } x = M^* \textbf{ in let } y = x\, N^* \textbf{ in } y\end{array}$$

Here, we have omitted the type of x (that is, the type of M) in the last line since, computationally, we are not interested in this type. We only tracked it in order to be able to reconstruct the sequent derivation uniquely. Completing this translation is straightforward, keep in mind the proof term language we assigned to the sequent calculus.

$$\begin{array}{lcl}\langle\!| M, N |\!\rangle^* & = & \langle\!| M^*, N^* |\!\rangle \\ (M \cdot l)^* & = & \textbf{let } x = M^* \textbf{ in let } y = x \cdot l \textbf{ in } y \\ (M \cdot r)^* & = & \textbf{let } x = M^* \textbf{ in let } z = x \cdot r \textbf{ in } z \\ \\ \langle M, N\rangle^* & = & \langle M^*, N^*\rangle \\ (\textbf{case } M\ \{\langle y, z\rangle \Rightarrow N\})^* & = & \textbf{let } x = M^* \textbf{ in case } x\ \{\langle y, z\rangle \Rightarrow N^*\}\end{array}$$

A remarkable property of these translations is that if we translate from natural deduction to sequent calculus and then back we obtain the original term. This does not immediately entail the operational correctness of these translations in the presence of recursion and recursive types, but it does show that the sequent calculus really is a calculus of proof search for natural deduction. If there is a natural deduction proof term M we can find a sequent proof term M' that translates back to M—we have "found" M by construction M'. In general, there will be many different sequent terms M' which all map to the same natural deduction term M, because M' tracks some details on the order which rules were applied that are not visible in natural deduction.

6 Cut Elimination

Gentzen's goal was to prove the consistency of logic as captured in natural deduction. One step in his proof was to show that it is equivalent to the sequent calculus. Now we can ask if the sequent calculus is enough to show that we cannot prove a contradiction. For that purpose we give the rules for $\bot$:

$$\text{no } \bot R \text{ rule} \qquad \frac{}{\Gamma, x : \bot \Vdash C}\ \bot L$$

Ideally, we would like to show that there is there **cannot** be a proof of

$$\cdot \Vdash \bot$$

This, however, is not immediately apparent, because we may just need to find the right "lemma" A and prove

$$\frac{\cdot \Vdash A \quad x : A \Vdash \bot}{\cdot \Vdash \bot}\ \mathsf{cut}$$

Then Gentzen showed a remarkable property: the rule of cut, so essentially in everyday mathematics (Which proof gets by without needing a lemma?) is redundant here in pure logic. That is:

Theorem 3 (Cut Elimination [Gen35]) *If* $\Gamma \Vdash A$ *then there is a proof of* $\Gamma \Vdash A$ *without using the rule of cut.*

This immediately implies consistency by inversion: there is no rule with a conclusion matching $\cdot \Vdash \bot$.

The proof of cut elimination is deep and interesting, and there are many resources to understand it.[1] From a computational perspective, however, it is only the so-called *cut reductions* we will discuss in the next lecture that are relevant. This is because in programming languages we impose a particular strategy of evaluation, and, moreover, one that does not evaluate underneath λ-abstractions or inside lazy pairs. In cut elimination, we obey no such restrictions. Plus, in realistic languages we have recursion and recursive types and cut elimination either no longer holds, or holds only for some restricted fragments.

In the next lecture we explore the computational consequences of the sequent calculus from the programming language perspective.

References

[Gen35] Gerhard Gentzen. Untersuchungen über das logische Schließen. *Mathematische Zeitschrift*, 39:176–210, 405–431, 1935. English translation in M. E. Szabo, editor, *The Collected Papers of Gerhard Gentzen*, pages 68–131, North-Holland, 1969.

[1] For example, Lecture Notes on Cut Elimination