Midterm 1

15-317: Constructive Logic

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Instructions

- This exam is closed-book, but one two-sided sheet of notes is permitted. The last page of the exam recaps some rules you may find useful.
- There are four problems, each with several parts. Not all problems are the same size or difficulty. You have 80 minutes to complete the exam.
- When writing proofs, remember to label each inference with the rule used and any variables or parameters discharged (e.g., $\supset I^u$).
- You may find it helpful to construct your proofs on scratch paper (such as the back of a page) before writing it clearly in the space provided.
- Most important,

DON'T PANIC

Good luck!

	Problem 1	Problem 2	Problem 3	Problem 4	**Total**
Score					
Max	55	55	20	20	150
Grader					

1 Natural Deduction and Harmony (55 points)

This problem is inspired by a suggestion from a student during the first lecture on the sequent calculus. Consider the following alternative definition of conjunction:

$$\dfrac{A\ \mathit{true} \quad \begin{array}{c}\overline{A\ \mathit{true}}^{\,u} \\ \vdots \\ B\ \mathit{true}\end{array}}{A \star B\ \mathit{true}}\ \star I^u \qquad \dfrac{A \star B\ \mathit{true}}{A\ \mathit{true}}\ \star E_L \qquad \dfrac{A \star B\ \mathit{true}}{B\ \mathit{true}}\ \star E_R$$

The introduction rule has a new, hypothetical premise, while the elimination rules are the standard ones. We would like to show that the elimination rules are still in harmony with the new introduction rule.

Task 1 (10 pts)**.** Prove the elimination rules locally sound by giving local reductions.

Solution. Since there are two elimination rules and one introduction rule, there are two local reductions. For the second, we must make use of the substitution principle.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D} \\ A\ \mathit{true}\end{array} \quad \begin{array}{c}\overline{A\ \mathit{true}}^{\,u} \\ \mathcal{E} \\ B\ \mathit{true}\end{array}}{A \star B\ \mathit{true}}\ \star I^u}{A\ \mathit{true}}\ \star E_L \quad \Longrightarrow_R \quad \begin{array}{c}\mathcal{D} \\ A\ \mathit{true}\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D} \\ A\ \mathit{true}\end{array} \quad \begin{array}{c}\overline{A\ \mathit{true}}^{\,u} \\ \mathcal{E} \\ B\ \mathit{true}\end{array}}{A \star B\ \mathit{true}}\ \star I^u}{B\ \mathit{true}}\ \star E_R \quad \Longrightarrow_R \quad \begin{array}{c}\mathcal{D} \\ \overline{A\ \mathit{true}}\ u \\ \mathcal{E} \\ B\ \mathit{true}\end{array}$$

■

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Task 2 (10 pts). Prove the elimination rules locally complete by giving a local expansion.

Solution. The local expansion is just the usual one for conjunction; the hypothesis u is not used.

$$\begin{array}{c}\mathcal{D}\\ A \star B\ \mathit{true}\end{array} \Longrightarrow_E \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A \star B\ \mathit{true}\end{array}}{A\ \mathit{true}}\star E_L \quad \dfrac{\begin{array}{c}\mathcal{D}\\ A \star B\ \mathit{true}\end{array}}{B\ \mathit{true}}\star E_R}{A \star B\ \mathit{true}}\star I^u$$

■

Task 3 (10 pts). Give rules for verifications and uses of $A \star B$.

Solution.

$$\dfrac{A\uparrow \quad \begin{array}{c}\overline{A\downarrow}\,u\\ \vdots\\ B\uparrow\end{array}}{A \star B\uparrow}\star I^u \qquad \dfrac{A \star B\downarrow}{A\downarrow}\star E_L \qquad \dfrac{A \star B\downarrow}{B\downarrow}\star E_R$$

■

Task 4 (10 pts). Propose sequent calculus left and right rules for $A \star B$ that correspond to the introduction and elimination rules.

Solution.

$$\dfrac{\Gamma \Longrightarrow A \quad \Gamma, A \Longrightarrow B}{\Gamma \Longrightarrow A \star B}\star R \qquad \dfrac{\Gamma, A \star B, A \Longrightarrow C}{\Gamma, A \star B \Longrightarrow C}\star L_L \qquad \dfrac{\Gamma, A \star B, B \Longrightarrow C}{\Gamma, A \star B \Longrightarrow C}\star L_R$$

■

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Task 5 (5 pts)**.** Thinking of the sequent calculus as a method for performing proof search, why might we prefer this formulation of conjunction over the standard one?

Solution. Searching for the proof of A may be expensive. During the course of searching for a proof of B, we might find that we need a proof of A, and remembering that we have one might save time. ■

Task 6 (10 pts)**.** Here is a proof term assignment for $\star I^u$:

$$\dfrac{M : A \qquad \begin{array}{c}\overline{u : A}^{\,u} \\ \vdots \\ N : B\end{array}}{\langle M, u.\, N\rangle : A \star B}\ \star I^u$$

Propose a proof term assignment for the elimination rules and write your local reductions using only proof terms.

Solution. The standard proof term assignment for conjunction suffices:

$$\dfrac{M : A \star B}{\mathbf{fst}\, M : A}\ \star E_L \qquad\qquad \dfrac{M : A \star B\ \mathit{true}}{\mathbf{snd}\, M : B\ \mathit{true}}\ \star E_R$$

The first reduction is the usual one, while the second uses substitution.

$$\begin{aligned} \mathbf{fst}\,\langle M, u.\, N\rangle &\Longrightarrow_R M \\ \mathbf{snd}\,\langle M, u.\, N\rangle &\Longrightarrow_R [M/u]\, N \end{aligned}$$ ■

2 Natural Numbers and Induction (55 points)

Recall the rules for natural number arithmetic and induction (recapped in Figure 1). Consider extending arithmetic with predicates for even and odd defined by the following introduction and elimination rules.

$$\dfrac{}{\mathsf{even}(0)}\ \mathsf{ev}I_0 \qquad \dfrac{\mathsf{odd}(n)}{\mathsf{even}(\mathsf{s}\ n)}\ \mathsf{ev}I_{\mathsf{s}} \qquad \dfrac{\mathsf{even}(n)}{\mathsf{odd}(\mathsf{s}\ n)}\ \mathsf{od}I_{\mathsf{s}}$$

$$\dfrac{\mathsf{odd}(0)}{J}\ \mathsf{od}E_0 \qquad \dfrac{\mathsf{even}(\mathsf{s}\ n)}{\mathsf{odd}(n)}\ \mathsf{ev}E_{\mathsf{s}} \qquad \dfrac{\mathsf{odd}(\mathsf{s}\ n)}{\mathsf{even}(n)}\ \mathsf{od}E_{\mathsf{s}}$$

Task 1 (10 pts)**.** Show the following rule derivable:

$$\dfrac{\mathsf{even}(n) \lor \mathsf{odd}(n)}{\mathsf{even}(\mathsf{s}\ n) \lor \mathsf{odd}(\mathsf{s}\ n)}\ \mathsf{eo}I_{\lor}$$

Solution.

$$\dfrac{\mathsf{even}(n) \lor \mathsf{odd}(n) \qquad \dfrac{\dfrac{\dfrac{}{\mathsf{even}(\mathsf{s}\ n)}\ u}{\mathsf{odd}(\mathsf{s}\ n)}\ \mathsf{od}I_s}{\mathsf{even}(\mathsf{s}\ n) \lor \mathsf{odd}(\mathsf{s}\ n)}\ {\lor}I_R \qquad \dfrac{\dfrac{\dfrac{}{\mathsf{odd}(\mathsf{s}\ n)}\ v}{\mathsf{even}(\mathsf{s}\ n)}\ \mathsf{ev}I_s}{\mathsf{even}(\mathsf{s}\ n) \lor \mathsf{odd}(\mathsf{s}\ n)}\ {\lor}I_L}{\mathsf{even}(\mathsf{s}\ n) \lor \mathsf{odd}(\mathsf{s}\ n)}\ {\lor}E^{u,v}$$

■

Task 2 (10 pts)**.** Show the following rule derivable:

$$\dfrac{\mathsf{even}(\mathsf{s}\ n) \land \mathsf{odd}(\mathsf{s}\ n)}{\mathsf{even}(n) \land \mathsf{odd}(n)}\ \mathsf{eo}E_{\land}$$

Solution.

$$\dfrac{\dfrac{\dfrac{\mathsf{even}(\mathsf{s}\ n) \land \mathsf{odd}(\mathsf{s}\ n)}{\mathsf{odd}(\mathsf{s}\ n)}\ {\land}E_R}{\mathsf{even}(n)}\ \mathsf{od}E_s \qquad \dfrac{\dfrac{\mathsf{even}(\mathsf{s}\ n) \land \mathsf{odd}(\mathsf{s}\ n)}{\mathsf{even}(\mathsf{s}\ n)}\ {\land}E_L}{\mathsf{odd}(n)}\ \mathsf{ev}E_s}{\mathsf{even}(n) \land \mathsf{odd}(n)}\ {\land}I$$

■

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Task 3 (10 pts)**.** Translate the following assertions into first-order logic:

- *Every natural number is even or odd.* **(*)**

Solution. $\forall x{:}\mathsf{nat}.\, \mathsf{even}(x) \lor \mathsf{odd}(x)$ ■

- *No natural number is both even and odd.*

Solution. $\neg\exists x{:}\mathsf{nat}.\, \mathsf{even}(x) \land \mathsf{odd}(x)$ ■

Task 4 (15 pts)**.** Give a natural deduction proof of your translation of the assertion **(*)**, "*Every natural number is even or odd.*" You may use the rules you derived above.

Solution. The proof is by induction, and we can make use of the derived rule $\mathsf{eo}I_\lor$ in the inductive case.

$$\dfrac{\dfrac{\dfrac{}{a : nat} \qquad \dfrac{\dfrac{}{\mathsf{even}(0)}\,\mathsf{ev}I_0}{\mathsf{even}(0) \lor \mathsf{odd}(0)}\,{\lor}I_L \qquad \dfrac{\dfrac{}{\mathsf{even}(b) \lor \mathsf{odd}(b)}\,u}{\mathsf{even}(\mathsf{s}\ b) \lor \mathsf{odd}(\mathsf{s}\ b)}\,\mathsf{eo}I_\lor}{\mathsf{even}(a) \lor \mathsf{odd}(a)}\,\mathsf{nat}E^{b,u}}{\forall x{:}\mathsf{nat}.\, \mathsf{even}(x) \lor \mathsf{odd}(x)}\,\forall I^a$$

■

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Task 5 (10 pts)**.** We now consider the computational content of your proof. Assume we are not interested in the evidence that a number is even or odd, just in whether it is even or odd. The type of the function extracted from your proof then is

$$\mathsf{decide} : \mathsf{nat} \to 1 + 1$$

where 1 is the unit type inhabited by the unit element $\langle\,\rangle$. Give the definition of decide that corresponds to your proof. You may use the schema of primitive recursion, the primitive recursion operator R, or Tutch syntax, whichever you prefer.

Solution. Using the schema of primitive recursion:

$$\begin{aligned} \mathsf{decide}(0) &= \mathbf{inl}\,\langle\,\rangle \\ \mathsf{decide}(\mathsf{s}\; x) &= \mathbf{case}\;\mathsf{decide}(x)\;\mathbf{of}\;\mathbf{inl}\,u \Rightarrow \mathbf{inr}\,\langle\,\rangle \mid \mathbf{inr}\,v \Rightarrow \mathbf{inl}\,\langle\,\rangle \end{aligned}$$

Using the recursor R:

$$\mathsf{decide} = \lambda n{:}\mathsf{nat}.\, R(n, \mathbf{inl}\,\langle\,\rangle, x.\,r.\,\mathbf{case}\;r\;\mathbf{of}\;\mathbf{inl}\,u \Rightarrow \mathbf{inr}\,\langle\,\rangle \mid \mathbf{inr}\,v \Rightarrow \mathbf{inl}\,\langle\,\rangle)$$

Using Tutch syntax:

```
val decide : nat -> 1 + 1 =
    fn n =>
        rec n of d 0 => inl ()
               | d (s x) => case d x of
                                inl u => inr ()
                              | inr v => inl ()
                            end
        end;
```

■

3 Classical Logic (20 points)

Recall the rules for classical logic (recapped in Figure 2). In classical logic, implication may be defined in terms of negation and disjunction: $A \supset B := \neg A \lor B$.

Task 1 (15 pts)**.** Using classical natural deduction, prove $(A \supset B) \supset (\neg A \lor B)$. You may use any classical reasoning principles we've shown in lecture or in homework, including proof by contradiction (*PBC*), the law of the excluded middle (*LEM*), and double-negation elimination (*DNE*).

Solution. A simple proof uses the law of the excluded middle:

$$\dfrac{\dfrac{\dfrac{}{A \lor \neg A \ \mathit{true}}\,\mathit{LEM} \qquad \dfrac{\dfrac{\dfrac{}{A \supset B \ \mathit{true}}\,u \quad \dfrac{}{A \ \mathit{true}}\,v}{B \ \mathit{true}}\,{\supset}E}{\neg A \lor B \ \mathit{true}}\,{\lor}I_R \qquad \dfrac{\dfrac{}{\neg A \ \mathit{true}}\,w}{\neg A \lor B \ \mathit{true}}\,{\lor}I_L}{\neg A \lor B \ \mathit{true}}\,{\lor}E^{v,w}}{(A \supset B) \supset \neg A \lor B \ \mathit{true}}\,{\supset}I^u$$

■

Task 2 (5 pts)**.** Explain informally why this theorem cannot be proven intuitionistically.

Solution. Intuitionistic disjunction is too strong: we would have to provide a direct proof of either $\neg A$ *true* or B *true*, and we can do neither.

More formally, if this theorem were intuitionistically provable, we could easily use it to prove $\neg A \lor A$ *true* by implication elimination on a proof of $A \supset A$ *true*. Since we know we cannot prove the law of the excluded middle, we must not be able to prove this theorem either. ■

4 Mistakes Were Made (20 points)

Consider the following purported proof:

$$
\dfrac{
  \dfrac{}{\exists x{:}\tau.\,A(x)\ \mathit{true}}\,u_1
  \qquad
  \dfrac{
    \dfrac{}{\exists x{:}\tau.\,B(x)\ \mathit{true}}\,u_2
    \qquad
    \dfrac{
      \dfrac{}{a : \tau}
      \qquad
      \dfrac{
        \dfrac{}{A(a)\ \mathit{true}}\,w_1
        \qquad
        \dfrac{}{B(a)\ \mathit{true}}\,w_2
      }{A(a) \wedge B(a)\ \mathit{true}}\,{\wedge}I
    }{\exists x{:}\tau.\,A(x) \wedge B(x)\ \mathit{true}}\,\exists I
  }{\exists x{:}\tau.\,A(x) \wedge B(x)\ \mathit{true}}\,\exists E^{a,w_2}
}{
  \dfrac{
    \dfrac{\exists x{:}\tau.\,A(x) \wedge B(x)\ \mathit{true}}{(\exists x{:}\tau.\,B(x)) \supset \exists x{:}\tau.\,A(x) \wedge B(x)\ \mathit{true}}\,{\supset}I^{u_2}
  }{(\exists x{:}\tau.\,A(x)) \supset (\exists x{:}\tau.\,B(x)) \supset \exists x{:}\tau.\,A(x) \wedge B(x)\ \mathit{true}}\,{\supset}I^{u_1}
}\,\exists E^{a,w_1}
$$

Task 1 (15 pts)**.** This proof is incorrect. Circle the label(s) of the rule(s) that are applied incorrectly. Explain what is wrong with each.

Solution. The highlighted rule is applied incorrectly: existential elimination must introduce a fresh parameter, but this rule is reusing the parameter introduced by the other existential elimination. ■

Task 2 (5 pts)**.** Explain informally why the purported theorem could not possibly be true.

Solution. The fact that there happens to be a τ for which A holds and a τ for which B holds says nothing about there being a τ for which *both* hold—the sets of τs with property A may be disjoint from the set of τs with property B.

More formally, we could use such a theorem to prove, for instance, that there is a natural number which is both even and odd. Then using induction and the elimination rules for even and odd, we could prove that 0 is odd and from that, we could conclude any judgement, making our logic inconsistent. Since we know the logic we've defined is consistent, the purported theorem must be false. ■

A Useful Rules

$$\frac{}{0 : \mathsf{nat}}\ \mathsf{nat}I_0 \qquad \frac{n : \mathsf{nat}}{\mathsf{s}\ n : \mathsf{nat}}\ \mathsf{nat}I_{\mathsf{s}} \qquad \frac{n : \mathit{nat} \quad \mathsf{C}(0)\ \mathit{true} \quad \begin{array}{c}\overline{x : \mathit{nat}}\ ,\ \overline{\mathsf{C}(x)\ \mathit{true}}^{\,u} \\ \vdots \\ \mathsf{C}(\mathsf{s}\ x)\ \mathit{true}\end{array}}{C(n)\ \mathit{true}}\ \mathsf{nat}E^{x,u}$$

Figure 1: Rules for natural numbers and induction.

$$A\ \mathit{false} := \begin{array}{c} A\ \mathit{true} \\ \vdots \\ \# \end{array} \qquad \frac{A\ \mathit{false} \quad A\ \mathit{true}}{J}\ \mathit{contra} \qquad \frac{\begin{array}{c}\overline{A\ \mathit{false}}^{\,k} \\ \vdots \\ \# \end{array}}{A\ \mathit{true}}\ \mathit{PBC}^k$$

$$\frac{\begin{array}{c}\overline{A\ \mathit{true}}^{\,u} \\ \vdots \\ \# \end{array}}{\neg A\ \mathit{true}}\ \neg I^u \qquad \frac{\neg A\ \mathit{true} \quad \begin{array}{c}\overline{A\ \mathit{false}}^{\,k} \\ \vdots \\ J \end{array}}{J}\ \neg E^k$$

Figure 2: Rules for classical natural deduction.