

# Assignment 4: Curves and Surfaces

15-462 Graphics I  
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Sample Solution  
Based on the homework by  
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## 1 Bezier Curves (20 pts)

1. Under which conditions do we have  $C^1$  continuity for two joined Bezier curves? Write out the condition explicitly as a test on the control points  $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$  and  $\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$  of the two curves.

*In order to have  $C^0$  continuity we need  $p_3 = q_0$ . To get  $C^1$  continuity, we need the two tangent vectors to be the same magnitude and opposite direction. That is,  $p_3 - p_2 = q_1 - q_0$ .*

2. Under which conditions do we have  $G^1$  continuity for two joined Bezier curves? Again, write out the condition explicitly as in part 1.

*Just as before, we need  $C^0$  continuity, so  $p_3 = q_0$ . To get  $G^1$  continuity, we still need the two tangent vectors to be in opposite directions, but the magnitudes may be different. That is, the condition is the same as one for  $C^1$  continuity except that the vectors can be positive multiples of each other:  $p_3 - p_2 = k(q_1 - q_0)$  for some real  $k > 0$ . ( $C^1$  continuity is  $G^1$  continuity with  $k = 1$ )*

3. It is possible for a single segment Bezier curve to intersect itself. Give four control points with all coordinates between 0 and 1 that yield a self-intersecting Bezier curve.

*The following 4 points all between (0,0) and (1,1) will do it:*

$$\begin{aligned} p_0 &= (0.9, 1) \\ p_1 &= (0.9, 0) \\ p_2 &= (0, 0.5) \\ p_3 &= (1, 0.5) \end{aligned}$$

4. Include a printed image of a self-intersecting Bezier curve with your assignment. You can capture an X window with `xwd -out bezier.xwd` and convert it to JPEG format with `convert bezier.xwd bezier.jpg`.

*Omitted.*

## 2 Bezier Surfaces (15 pts)

1. Compute the normal vector of a Bezier surface patch at the four corners and at the center ( $u = v = 0.5$ ) for a given set of control points.

The normal vector of the Bezier surface is given by the normalized cross product of its partial derivatives. We write  $\text{norm}(\mathbf{n}) = \frac{\mathbf{n}}{\|\mathbf{n}\|}$ . Then

$$\mathbf{n} = \text{norm}\left(\frac{\partial \mathbf{B}}{\partial u} \times \frac{\partial \mathbf{B}}{\partial v}\right)$$

- (a) Normal at  $(0, 0)$ . We know that  $\frac{\partial \mathbf{B}}{\partial u}(0, 0) = 3(\mathbf{p}_{10} - \mathbf{p}_{00})$  and  $\frac{\partial \mathbf{B}}{\partial v}(0, 0) = 3(\mathbf{p}_{01} - \mathbf{p}_{00})$ . Therefore

$$\mathbf{n} = \text{norm}(3(\mathbf{p}_{10} - \mathbf{p}_{00}) \times 3(\mathbf{p}_{01} - \mathbf{p}_{00})).$$

- (b) Normal at  $(0, 1)$ . We know that  $\frac{\partial \mathbf{B}}{\partial u}(0, 1) = 3(\mathbf{p}_{13} - \mathbf{p}_{03})$  and  $\frac{\partial \mathbf{B}}{\partial v}(1, 0) = 3(\mathbf{p}_{02} - \mathbf{p}_{03})$ . Therefore

$$\mathbf{n} = \text{norm}(3(\mathbf{p}_{02} - \mathbf{p}_{03}) \times 3(\mathbf{p}_{13} - \mathbf{p}_{03})).$$

Note the order of the cross product in order to obtain the same directions for all normals.

- (c) Normal at  $(1, 1)$ . We know that  $\frac{\partial \mathbf{B}}{\partial u}(1, 1) = 3(\mathbf{p}_{23} - \mathbf{p}_{33})$  and  $\frac{\partial \mathbf{B}}{\partial v}(1, 1) = 3(\mathbf{p}_{32} - \mathbf{p}_{33})$ . Therefore

$$\mathbf{n} = \text{norm}(3(\mathbf{p}_{23} - \mathbf{p}_{33}) \times 3(\mathbf{p}_{32} - \mathbf{p}_{33})).$$

- (d) Normal at  $(1, 0)$ . We know that  $\frac{\partial \mathbf{B}}{\partial u}(1, 0) = 3(\mathbf{p}_{20} - \mathbf{p}_{30})$  and  $\frac{\partial \mathbf{B}}{\partial v}(1, 0) = 3(\mathbf{p}_{31} - \mathbf{p}_{30})$ . Therefore

$$\mathbf{n} = \text{norm}(3(\mathbf{p}_{31} - \mathbf{p}_{30}) \times 3(\mathbf{p}_{20} - \mathbf{p}_{30})).$$

Note again the order of the product.

- (e) Normal at  $(\frac{1}{2}, \frac{1}{2})$ . We have  $\mathbf{B}(u, v) = \sum_j \sum_i b_i(u) b_j(v) \mathbf{p}_{ij}$ . We can calculate the partial derivatives as

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial u}(u, v) &= -3(1-u)^2 \sum_j b_j(v) \mathbf{p}_{0j} \\ &\quad + 3(-2(1-u)u + (1-u)^2) \sum_j b_j(v) \mathbf{p}_{1j} \\ &\quad + 3(u^2 + (1-u)2u) \sum_j b_j(v) \mathbf{p}_{2j} \\ &\quad + 3u^2 \sum_j b_j(v) \mathbf{p}_{3j} \end{aligned}$$

and evaluate at  $u = v = \frac{1}{2}$ . We obtain the partial derivative  $\frac{\partial \mathbf{B}}{\partial v}(u, v)$  symmetrically. Then

$$\mathbf{n} = \text{norm}\left(\frac{\partial \mathbf{B}}{\partial u}\left(\frac{1}{2}, \frac{1}{2}\right) \times \frac{\partial \mathbf{B}}{\partial v}\left(\frac{1}{2}, \frac{1}{2}\right)\right)$$

2. Discuss how you would define the normals for a surface created from joined Bezier patches using Gouraud shading.

When drawing a surface composed from joined Bezier patches we would apply subdivision to obtain smaller Bezier patches (see [Angel, Ch. 10.9.4]). These smaller patches would have  $C^1$  continuity where they meet and we can just analytically compute the exact surface normals at the vertices as indicated above. With  $C^1$  continuity, any of the adjoining patches should give the same answer.

If we have Bezier patches that are not further subdivided and may meet without  $C^1$  continuity, the idea of Gouraud shading would be to take the surface normals at the center of the four adjoining patches ( $u = v = \frac{1}{2}$ ) and average them. Unless the patches are nearly flat, this is not likely to produce very good results.

### 3 Cubic B-Splines (15 pts)

1. Analyze the effect of four collinear control points on a cubic B-spline.

Clearly, if the four control points are collinear, the entire spline will simply be a line, because the spline must be wholly within the convex hull of the control points. However, the spline will not necessarily be exactly the line defined by the control points. For example, the first and last points will not be hit, in general. This follows from the blending functions. When  $u = 0$ , the first and third points contribute  $\frac{1}{6}$ , while the second point contributes  $\frac{4}{6}$ . Thus, if the control points are not only collinear but also evenly spaced, the spline will start on the second point. (By symmetry, it will also end on the third point.) You can pull the first or fourth control points outward to lengthen the spline, but you won't ever hit the first point (unless you start stacking control points on top of each other).

2. Verify the  $C^2$  continuity of the cubic spline at the join points.

Consider 5 control points  $p_0 \dots p_4$  and two B-spline segments  $p$  and  $q$ .  $p$  is based on  $p_0 \dots p_3$  and  $q$  is based on  $p_1 \dots p_4$ . We will show that  $p''(1) = q''(0)$ .

Using the blending functions we can see that the general form of  $p(u)$  is

$$p(u) = \frac{1}{6} \left[ (1-u)^3 p_0 + (4-6u^2+3u^3) p_1 + (1+3u+3u^2-3u^3) p_2 + u^3 p_3 \right]$$

What we want is the second derivative so

$$\begin{aligned} p''(u) &= \frac{d^2}{du^2} \left[ \frac{1}{6} \left[ (1-u)^3 p_0 + (4-6u^2+3u^3) p_1 + (1+3u+3u^2-3u^3) p_2 + u^3 p_3 \right] \right] \\ &= \frac{1}{6} \left[ \frac{d}{du} \left[ -3(1-u)^2 p_0 + (-12u+9u^2) p_1 + (3+6u-9u^2) p_2 + 3u^2 p_3 \right] \right] \\ &= \frac{1}{6} [6(1-u) p_0 + (-12+18u) p_1 + (6-18u) p_2 + 6u p_3] \\ &= (1-u) p_0 + (-2+3u) p_1 + (1-3u) p_2 + u p_3 \end{aligned}$$

Similarly, we can shift the control points over by one to get an expression for  $q''(u)$

$$q''(u) = (1-u) p_1 + (-2+3u) p_2 + (1-3u) p_3 + u p_4$$

Now, by inspection we see that  $p''(1) = p_1 - 2p_2 + p_3 = q''(0)$ . Thus,  $C^2$  continuity is preserved for arbitrary B-splines.