# Analysis of Algorithms: Solutions 6

```
            X        X
            X        X
            X        X
            X     X  X
            X  X  X  X
            X  X  X  X      X
      X     X  X  X  X   X  X
X     X  X  X  X  X  X   X  X
-----------------------------
2  3  4  5  6  7  8  9  10 11
           grades
```

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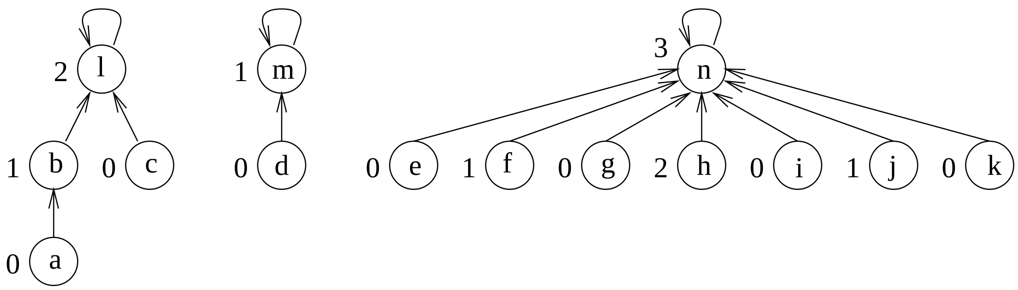
**Problem 1**

Suppose we apply the Connected-Components algorithm to an undirected graph $G$, with vertices $G[V] = \{a, b, c, d, e, f, g, h, i, j, k\}$, and its edges $E[G]$ are processed in the following order: $(e, g), (a, d), (i, k), (c, g), (b, f), (b, h), (f, k), (a, k), (f, h), (d, i)$. Using Figure 22.1 in the textbook as a model, illustrate the steps of Connected-Components on this graph.

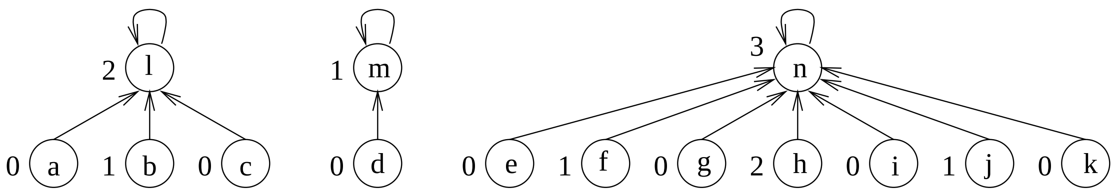
| Edge | Disjoint sets | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| initial sets | $\{a\}$ | $\{b\}$ | $\{c\}$ | $\{d\}$ | $\{e\}$ | $\{f\}$ | $\{g\}$ | $\{h\}$ | $\{i\}$ | $\{j\}$ | $\{k\}$ |
| $(e, g)$ | $\{a\}$ | $\{b\}$ | $\{c\}$ | $\{d\}$ | $\{e, g\}$ | $\{f\}$ | | $\{h\}$ | $\{i\}$ | $\{j\}$ | $\{k\}$ |
| $(a, d)$ | $\{a, d\}$ | $\{b\}$ | $\{c\}$ | | $\{e, g\}$ | $\{f\}$ | | $\{h\}$ | $\{i\}$ | $\{j\}$ | $\{k\}$ |
| $(i, k)$ | $\{a, d\}$ | $\{b\}$ | $\{c\}$ | | $\{e, g\}$ | $\{f\}$ | | $\{h\}$ | $\{i, k\}$ | $\{j\}$ | |
| $(c, g)$ | $\{a, d\}$ | $\{b\}$ | $\{c, e, g\}$ | | | $\{f\}$ | | $\{h\}$ | $\{i, k\}$ | $\{j\}$ | |
| $(b, f)$ | $\{a, d\}$ | $\{b, f\}$ | $\{c, e, g\}$ | | | | | $\{h\}$ | $\{i, k\}$ | $\{j\}$ | |
| $(b, h)$ | $\{a, d\}$ | $\{b, f, h\}$ | $\{c, e, g\}$ | | | | | | $\{i, k\}$ | $\{j\}$ | |
| $(f, k)$ | $\{a, d\}$ | $\{b, f, h, i, k\}$ | $\{c, e, g\}$ | | | | | | | $\{j\}$ | |
| $(a, k)$ | $\{a, b, d, f, h, i, k\}$ | | $\{c, e, g\}$ | | | | | | | $\{j\}$ | |
| $(f, h)$ | $\{a, b, d, f, h, i, k\}$ | | $\{c, e, g\}$ | | | | | | | $\{j\}$ | |
| $(d, i)$ | $\{a, b, d, f, h, i, k\}$ | | $\{c, e, g\}$ | | | | | | | $\{j\}$ | |

**Problem 2**

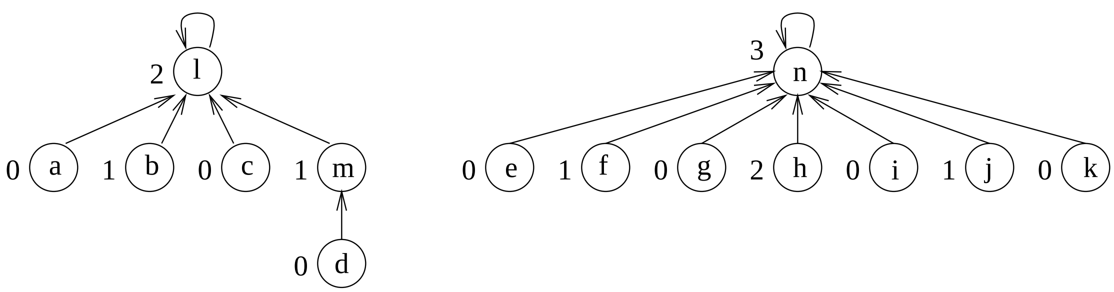
Consider the disjoint-set forest shown below, where numbers are the ranks of elements, and suppose that you apply three successive operations to this forest: FIND-SET($a$), UNION($l, d$), and UNION($d, e$). Give a picture of the disjoint forest after each of these operations.



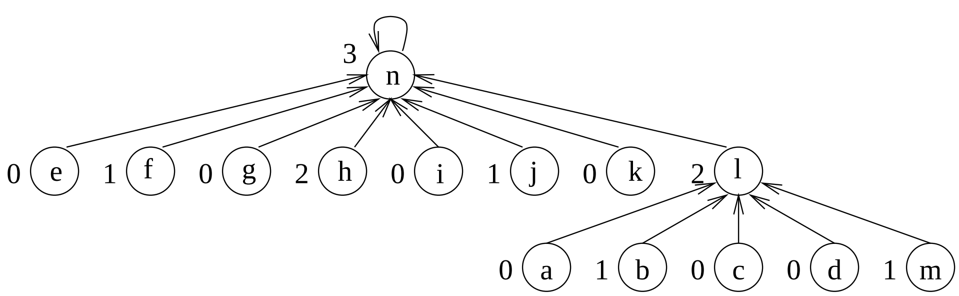

FIND-SET($a$):




UNION($l, d$):




UNION($d, e$):



**Problem 3**
The *transpose* of a graph $G$ is the result of reversing all edges in $G$. Write an algorithm that computes the transpose of a given graph.

We denote the array of initial adjacency lists by *Adj-Initial* and the array of transposed adjacency lists by *Adj-Transpose*. The time complexity of the algorithm is $\Theta(V+E)$.

```
TRANSPOSE(G)
for each u ∈ V[G]
    do initialize an empty list Adj-Transpose[u]
for each u ∈ V[G]
    do for each v ∈ Adj-Initial[u]
            do add u to Adj-Transpose[v]
```

**Problem 4**
Suppose that the rank of each node in a disjoint-set forest must be the exact height of the node, rather than an upper bound on the height. Then, FIND-SET has to change the ranks of the nodes on the compressed path.

Describe a modified representation of the disjoint-set forest, which supports this operation, and the corresponding modifications of MAKE-SET, UNION, and FIND-SET. What is the time complexity of the resulting implementation, in terms of $m$ and $n$?

The modified representation is similar to the standard disjoint-set forest. The difference is that each node $x$ includes a list *children*[$x$], which contains all children of $x$. The running time of this implementation is $O(m \cdot n)$; it is much slower than the standard implementation.

```
MAKE-SET(x)
parent[x] ← x
rank[x] ← 0
initialize an empty list children[x]

UNION(x, y)
LINK(FIND-SET(x), FIND-SET(y))

LINK(x, y)
if rank[x] > rank[y]
   then parent[y] ← x
        add y to children[x]
   else parent[x] ← y
        add x to children[y]
if rank[x] = rank[y]
   then rank[y] ← rank[y] + 1
```

```
FIND-SET(x)
y ← parent[x]
if y ≠ parent[y]
   ▷ neither x nor parent[x] is the root
   then remove x from children[y]
        RECOMPUTE-RANK[y]
        parent[x] ← FIND-SET(y)
        add x to children[parent[x]]
        if rank[parent[x]] < rank[x] + 1
           then rank[parent[x]] = rank[x] + 1
return parent[x]

RECOMPUTE-RANK(y)
rank[y] ← 0
for each z ∈ children[y]
    do if rank[y] < rank[z] + 1
          then rank[y] ← rank[z] + 1
```