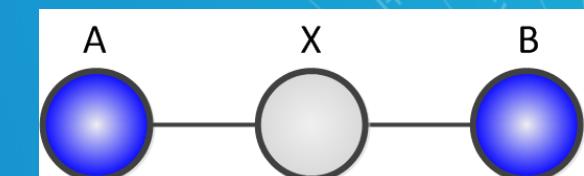


Probabilistic Graphical Models

Spectral Learning for Graphical Models

Eric Xing

Lecture 25, April 20, 2020

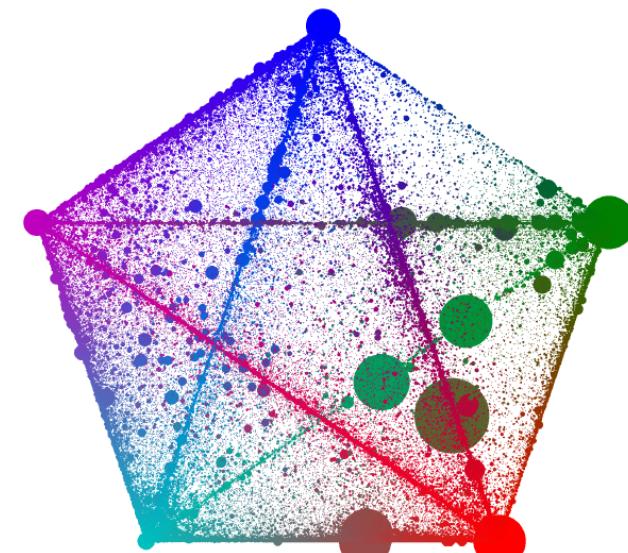
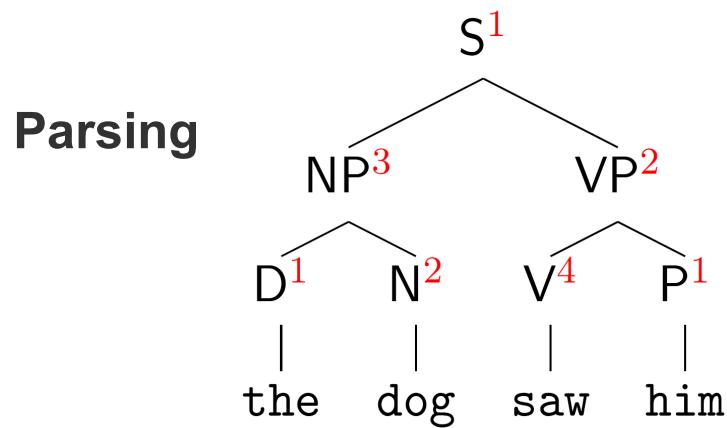
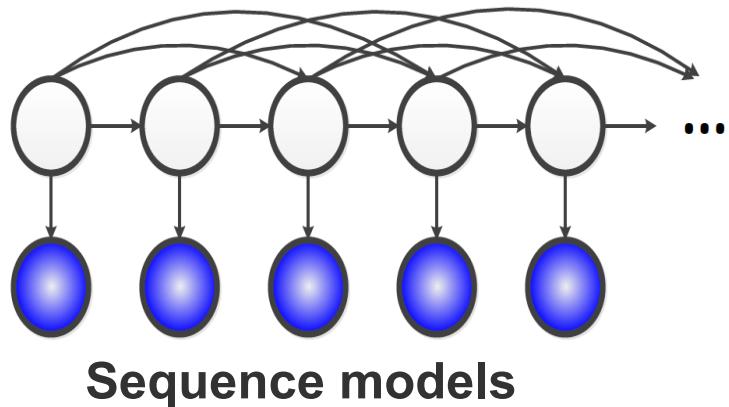


Reading: see class homepage

Acknowledgement: slides drafted by Ankur Parikh



Latent Variable Models

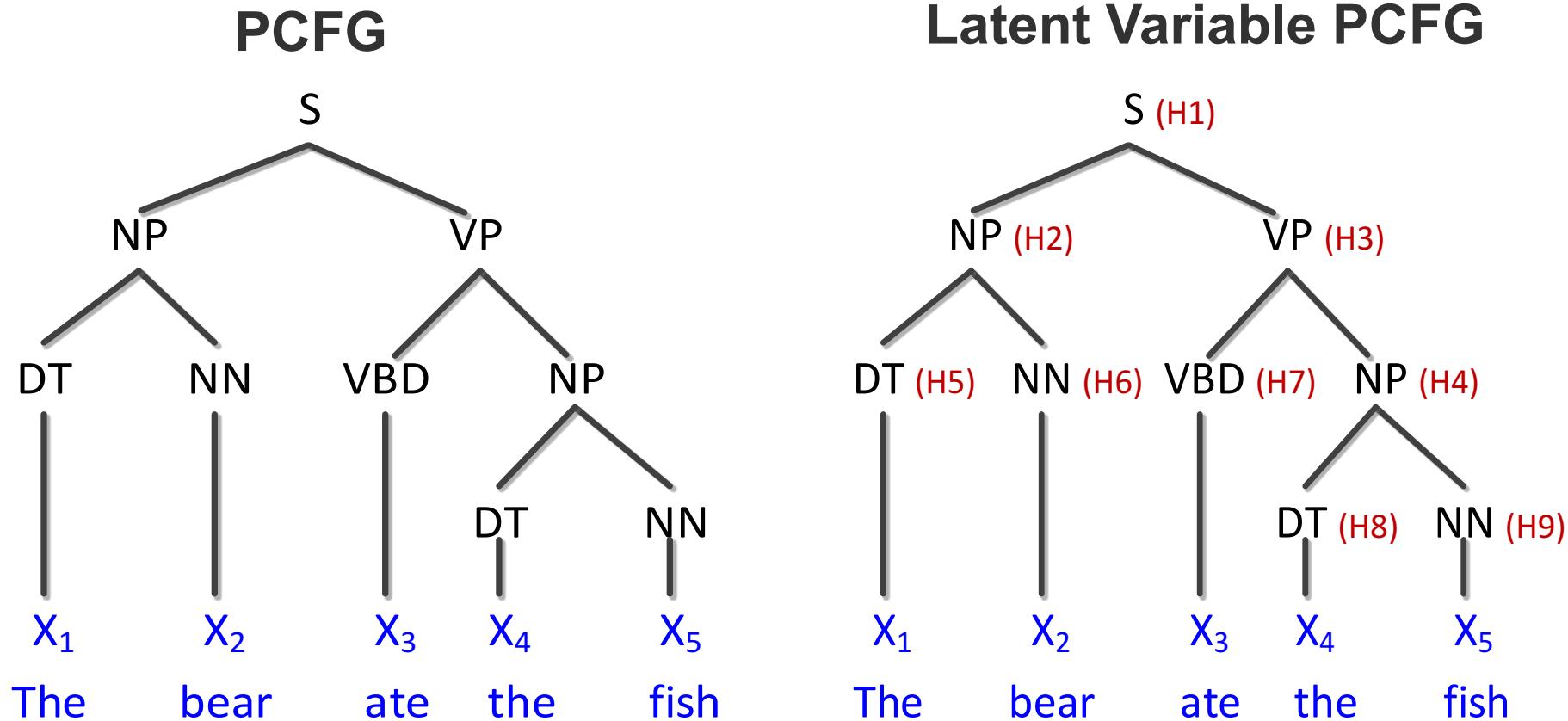


Mixed membership models



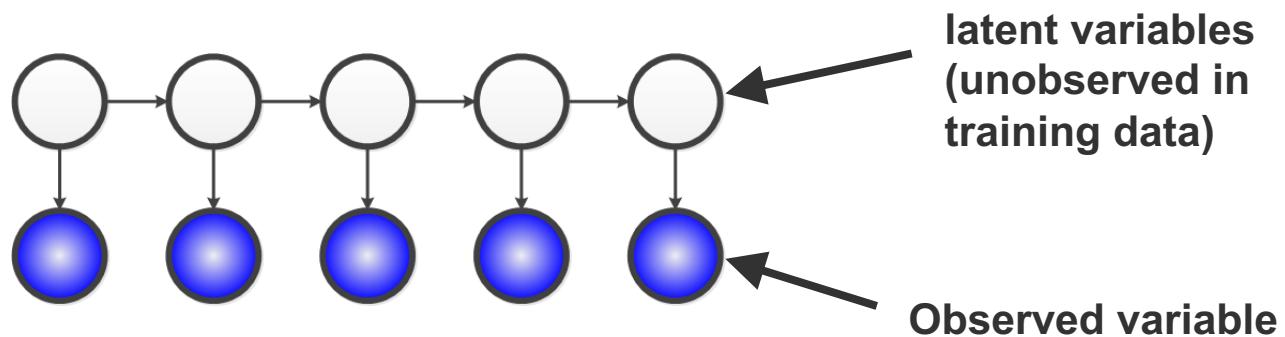


Latent Variable PCFG [Matsuzaki et al., 2005, Petrov et al. 2006]





Learning Parameters (EM)



$$\mathbb{P}[X_1, \dots, X_5, H_1, \dots, H_5] = \mathbb{P}[H_1] \prod_{i=2}^5 \mathbb{P}[H_i | H_{i-1}] \prod_{i=1}^5 \mathbb{P}[X_i | H_i]$$

Since latent variables are not observed in the data, we have to use Expectation Maximization (EM) to learn parameters

- Slow
- Local Minima





Spectral Learning

- ❑ Different paradigm of learning in latent variable models based on linear algebra
- ❑ Theoretically,
 - ❑ Provably consistent
 - ❑ Can offer deeper insight into the identifiability
- ❑ Practically,
 - ❑ Local minima free
 - ❑ As of now, performs comparably to EM with 10-100x speed-up
 - ❑ Can also model non-Gaussian continuous data using kernels (usually performs much better than EM in this case)





Related References

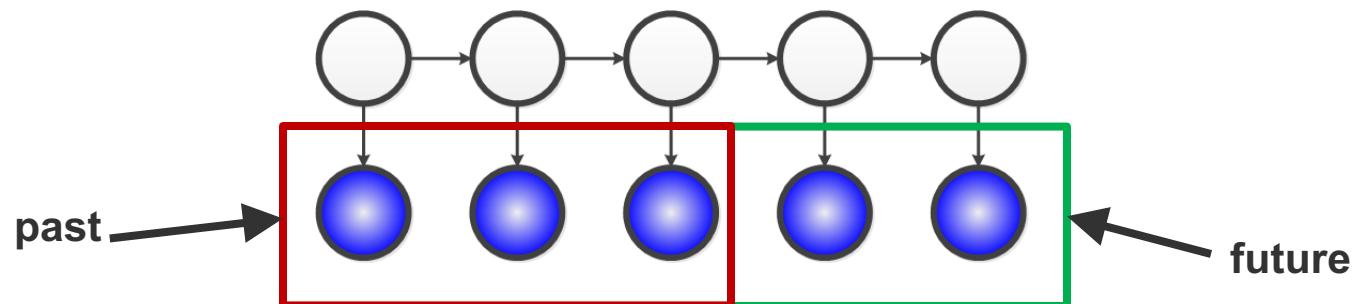
- Relevant works
 - **Hsu et al. 2009** – Spectral HMMs (also Bailly 2009)
 - **Siddiqi et al. 2009** – Features in Spectral Learning
 - **Parikh et al. 2011/2012** – Tensors to Generalize to Trees/Low Treewidth Graphs
 - **Cohen et al. 2012 / 2013** – Spectral Learning of latent PCFGs
- Will present it from “matrix factorization” view:
 - **Balle et al. 2012** – Connection between Spectral Learning / Hankel Matrix Factorization
 - **Song et al. 2013** – Spectral Learning as Hierarchical Tensor Decomposition





Focusing on Prediction

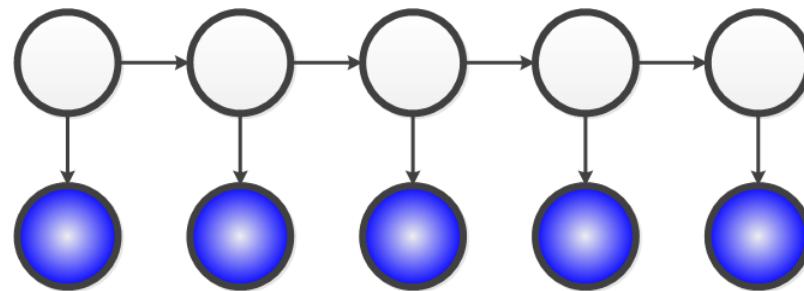
- ❑ In many applications that use latent variable models, the end task is not to recover the latent states, but rather to use the model for prediction among observed variables.
- ❑ Dynamical Systems – Predict future given past





Focusing on Prediction

- We will only be concerned with quantities related to the observed variables: $P[X_1, X_2, X_3, X_4, X_5]$
- We do not care about the latent variables explicitly.



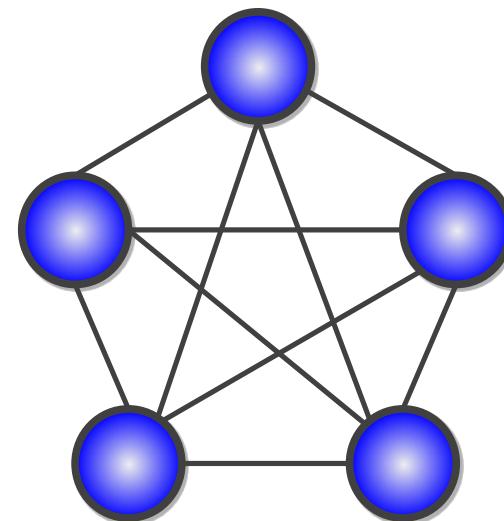
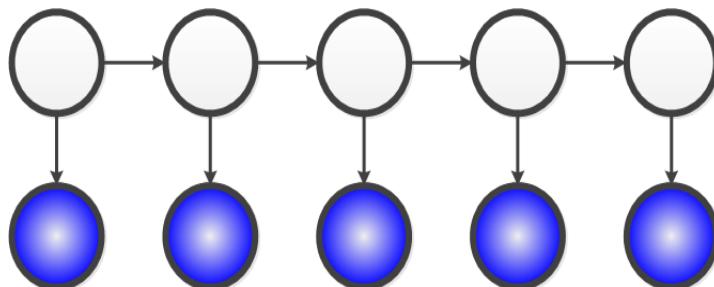
- **Do we still need EM to learn the parameters?**





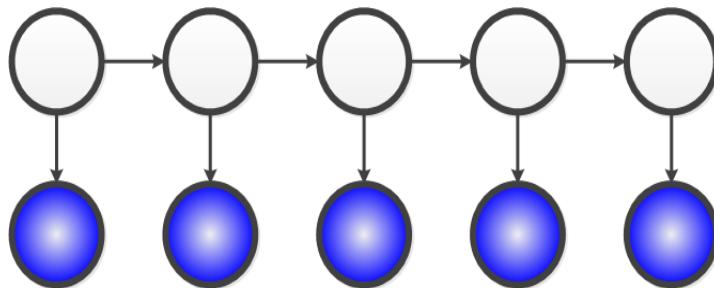
But if we don't care about the latent variables....

- ❑ Why don't we just integrate them out?
- ❑ Because integrating them out results in a clique ☹





Marginal Does Not Factorize



$$\mathbb{P}[X_1, X_2, X_3, X_4, X_5] = \sum_{H_1, \dots, H_5} \mathbb{P}[H_1] \mathbb{P}[H_1] \prod_{i=2}^5 \mathbb{P}[H_i | H_{i-1}] \prod_{i=1}^5 \mathbb{P}[X_i | H_i]$$

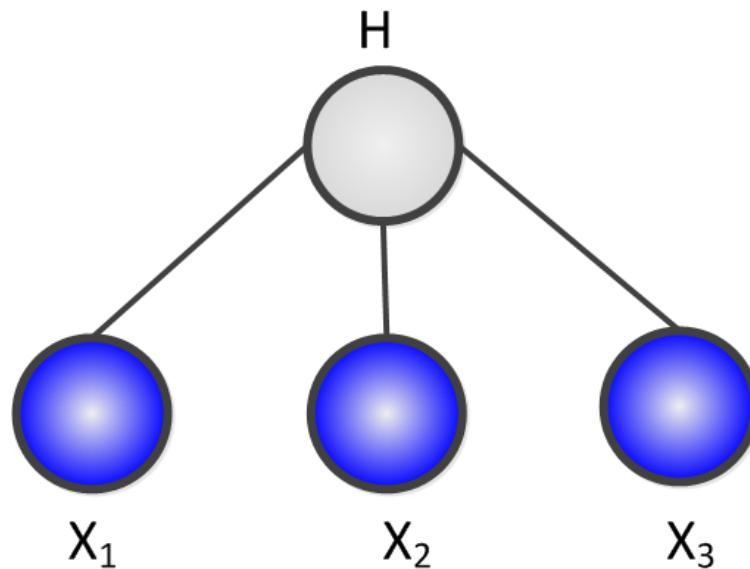
Does not factorize due to the outer sum (Can somewhat distribute the sum, but doesn't solve problem)





But isn't an HMM different from a clique?

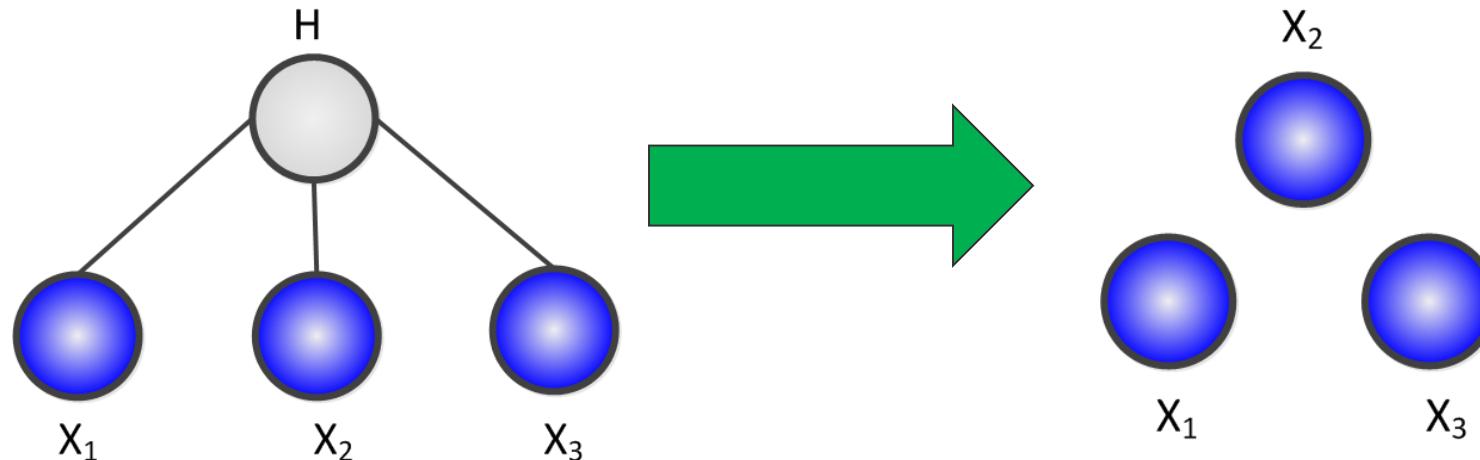
- ❑ It depends on the number of latent states.
- ❑ Consider the following model.





If H has only one state.....

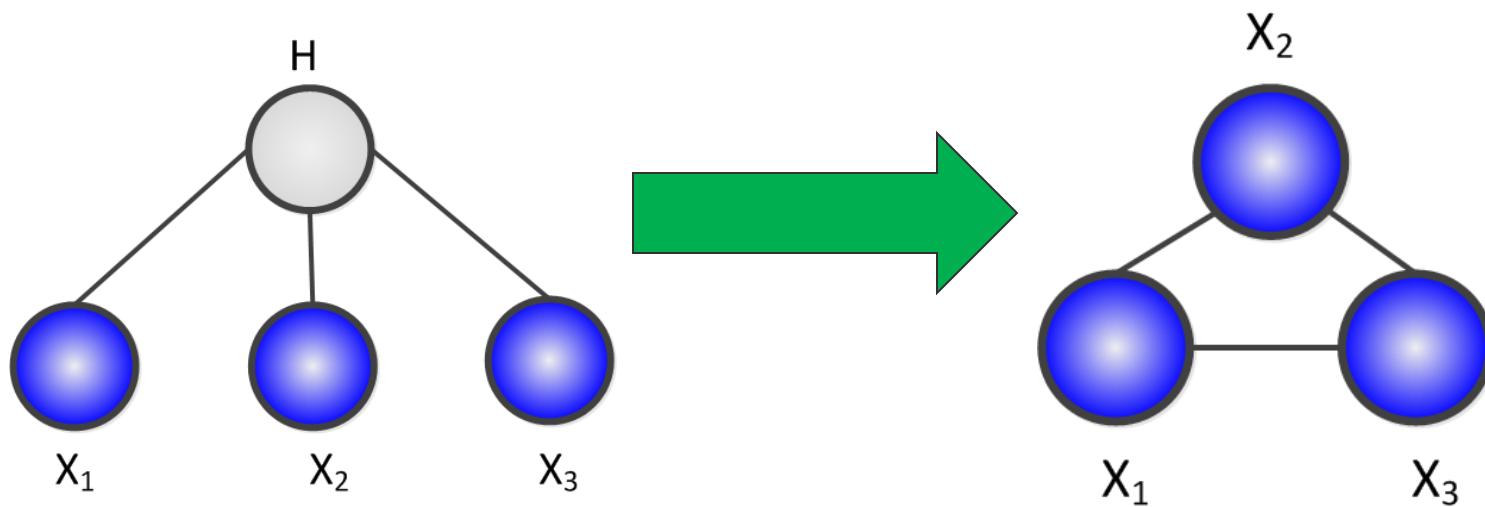
- Then the observed variables are independent!





What if H has many states?

- Let us say the observed variables each have m states.
- Then if H has m^3 states then the latent model can be exactly equivalent to a clique (depending on how parameters are set).



- But what about all the other cases?





The Question

- ❑ Under existing methods, latent models all require EM to learn regardless of the number of hidden states.
- ❑ However, is there a formulation of latent variable models where the difficulty of learning is a function of the number of latent states?
- ❑ This is the question that the *spectral view* will answer.





Sum Rule (Matrix Form)

- Sum Rule

$$\mathbb{P}[X] = \sum_Y \mathbb{P}[X|Y]\mathbb{P}[Y]$$

- Equivalent view using Matrix Algebra

$$\mathcal{P}[X] = \mathcal{P}[X|Y] \times \mathcal{P}[Y]$$

$$\begin{pmatrix} \mathbb{P}[X=0] \\ \mathbb{P}[X=1] \end{pmatrix} = \begin{pmatrix} \mathbb{P}[X=0|Y=0] & \mathbb{P}[X=0|Y=1] \\ \mathbb{P}[X=1|Y=0] & \mathbb{P}[X=1|Y=1] \end{pmatrix} \times \begin{pmatrix} \mathbb{P}[Y=0] \\ \mathbb{P}[Y=1] \end{pmatrix}$$





Chain Rule (Matrix Form)

- Chain Rule

$$\mathbb{P}[X, Y] = \mathbb{P}[X|Y]\mathbb{P}[Y] = \mathbb{P}[Y|X]\mathbb{P}[Y]$$

Means on diagonal

- Equivalent view using Matrix Algebra

$$\mathcal{P}[X, Y] = \mathcal{P}[X|Y] \times \mathcal{P}[\cancel{Y}]$$

$$\begin{pmatrix} \mathbb{P}[X = 0, Y = 0] & \mathbb{P}[X = 0, Y = 1] \\ \mathbb{P}[X = 1, Y = 0] & \mathbb{P}[X = 1, Y = 1] \end{pmatrix} =$$

$$\begin{pmatrix} \mathbb{P}[X = 0|Y = 0] & \mathbb{P}[X = 0|Y = 1] \\ \mathbb{P}[X = 1|Y = 0] & \mathbb{P}[X = 1|Y = 1] \end{pmatrix} \times \begin{pmatrix} \mathbb{P}[Y = 0] & 0 \\ 0 & \mathbb{P}[Y = 1] \end{pmatrix}$$

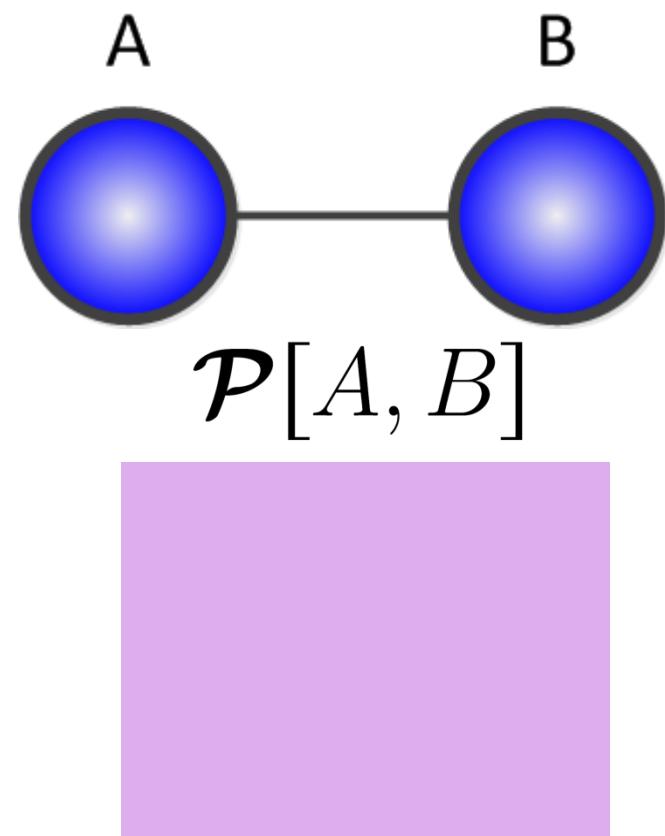
- Note how diagonal is used to keep Y from being marginalized out.





Graphical Models: The Linear Algebra View

- In general, nothing we can say about the nature of this matrix.



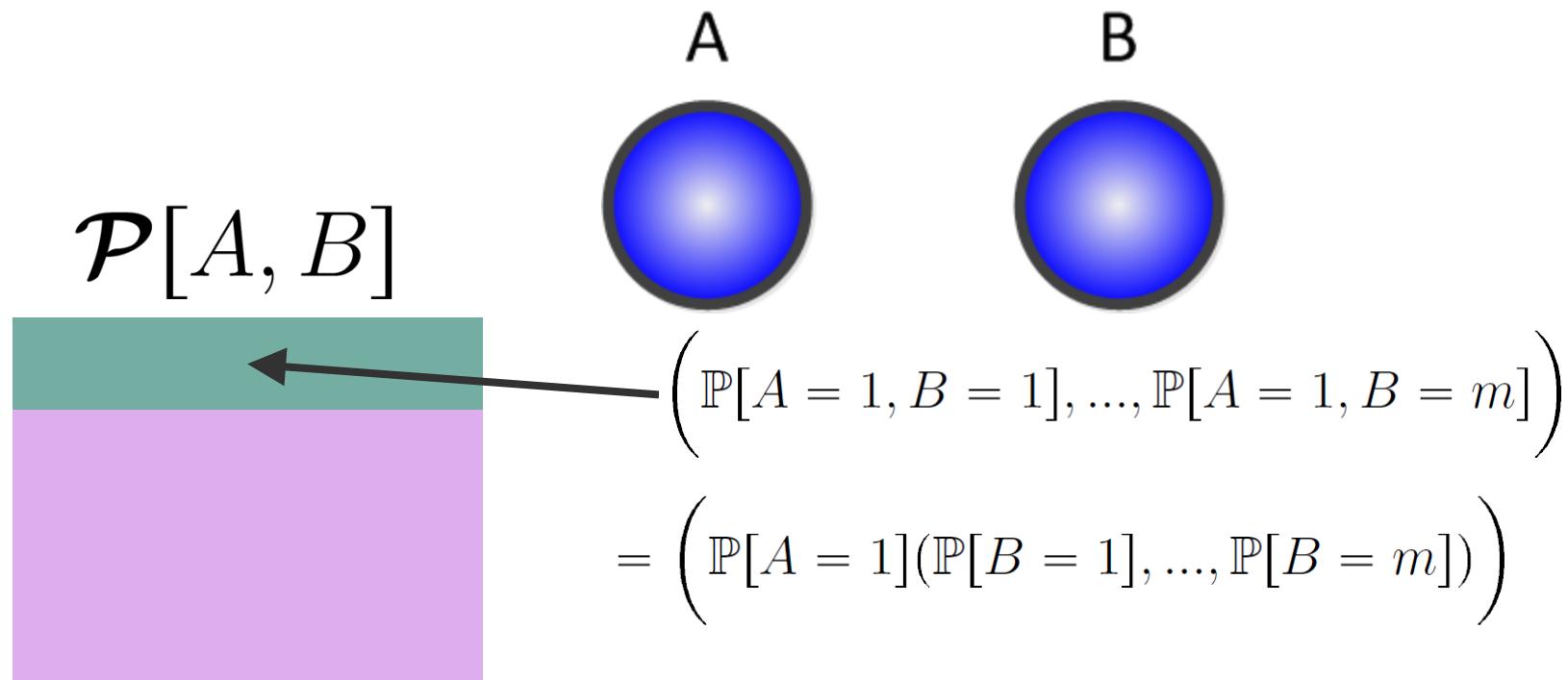
A and B have m states each.





Independence: The Linear Algebra View

- What if we know A and B are independent?



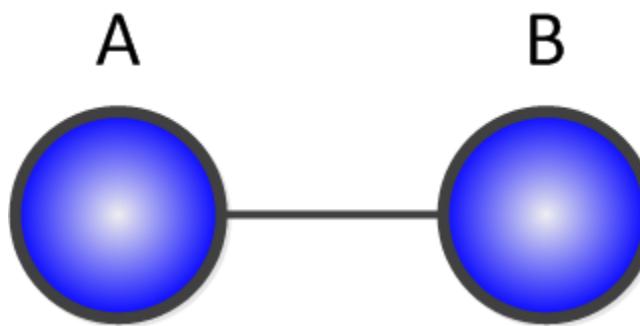
- Joint probability matrix is rank one, since all rows are multiples of one another!!



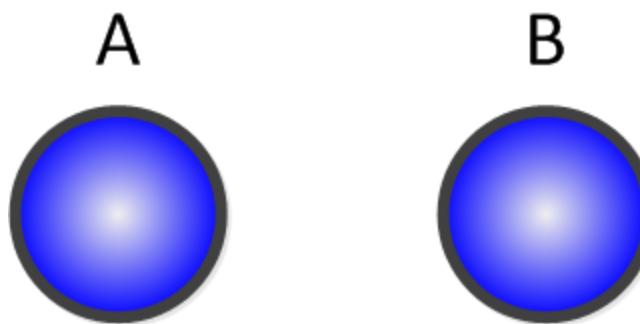


Independence and Rank

- What about rank in between 1 and m ?



$\mathcal{P}[A, B]$ has rank m (at most)



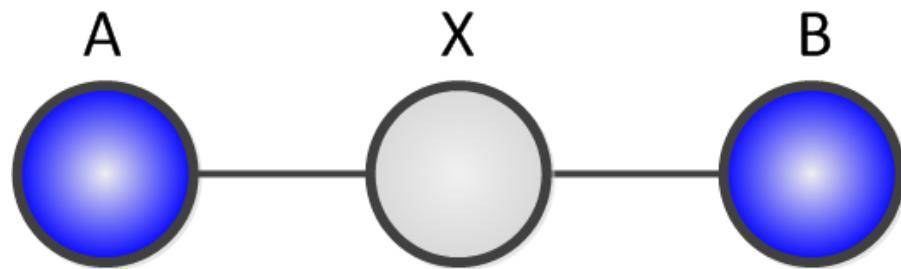
$\mathcal{P}[A, B]$ has rank 1





Low Rank Structure

- ❑ \mathbf{A} and \mathbf{B} are not marginally independent (They are only conditionally independent given \mathbf{X}).



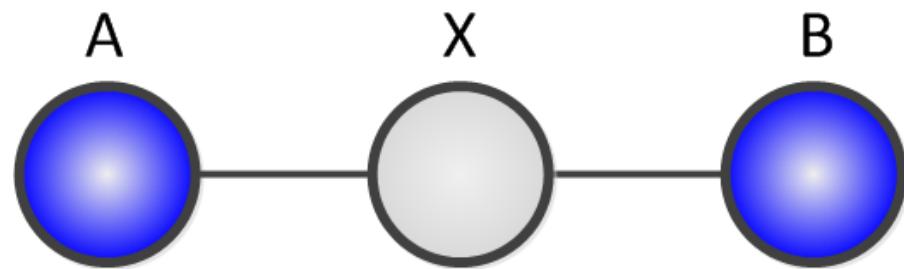
- ❑ Assume \mathbf{X} has k states (while \mathbf{A} and \mathbf{B} have m states).
- ❑ Then,
- ❑ Why?

$$\text{rank}(\mathcal{P}[A, B]) \leq k$$





Low Rank Structure



$$\mathcal{P}[A, B] = \mathcal{P}[A|X] \mathcal{P}(\emptyset|X) \mathcal{P}[B|X]^\top$$





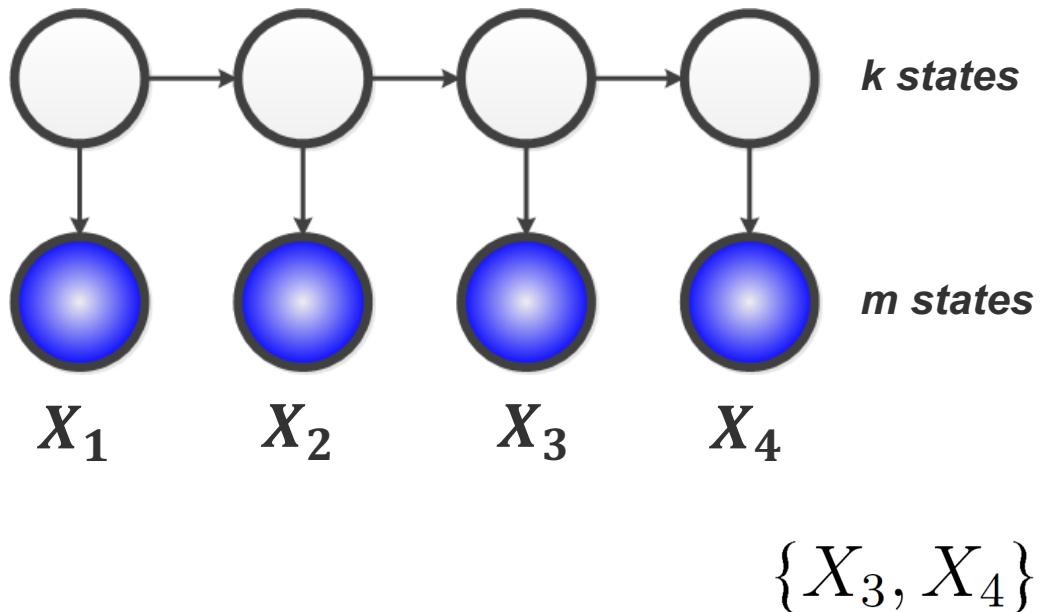
The Spectral View

- ❑ Latent variable models encode **low rank dependencies** among variables (*both marginal and conditional*)
- ❑ Use tools from linear algebra to exploit this structure.
 - ❑ Rank
 - ❑ Eigenvalues
 - ❑ SVD
 - ❑ Tensors





A More Interesting Example



$$\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}]$$

$$\{X_1, X_2\}$$





Low Rank Matrices “Factorize”

$$M = LR \quad \text{If } M \text{ has rank } k$$

m by n m by k k by n

We already know one factorization!!!

$$\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}] = \mathcal{P}[X_{\{1,2\}} | H_2] \mathcal{P}[\bigcirc H_2] \mathcal{P}[X_{\{3,4\}} | H_2]^\top$$

Factor of 4 variables Factor of 3 variables Factor of 3 variables

↑

Factor of 1 variable





Alternate Factorizations

- The key insight is that this factorization is not unique.
- Consider Matrix Factorization. Can add any invertible transformation:

$$\begin{aligned}M &= LR \\M &= LSS^{-1}R\end{aligned}$$

- **The magic of spectral learning is that there exists an alternative factorization that only depends on observed variables!**





An Alternate Factorization

- Let us say we only want to factorize this matrix of 4 variables

$$\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}]$$

such that it is product of matrices that contain at most three ***observed*** variables e.g.

$$\mathcal{P}[X_{\{1,2\}}, X_3]$$

$$\mathcal{P}[X_2, X_{\{3,4\}}]$$





An Alternate Factorization

- Note that

$$\mathcal{P}[X_{\{1,2\}}, X_3] = \underline{\mathcal{P}[X_{\{1,2\}}|H_2]} \underline{\mathcal{P}[\emptyset|H_2]} \underline{\mathcal{P}[X_3|H_2]}^\top$$

$$\mathcal{P}[X_2, X_{\{3,4\}}] = \underline{\mathcal{P}[X_2|H_2]} \underline{\mathcal{P}[\emptyset|H_2]} \underline{\mathcal{P}[X_{\{3,4\}}|H_2]}^\top$$

- Product of green terms (in some order) is

$$\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}]$$

- Product of red terms (in some order) is

$$\mathcal{P}[X_2, X_3]$$





An Alternate Factorization

$$\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}] = \mathcal{P}[X_{\{1,2\}}, X_3] \mathcal{P}[X_2, X_3]^{-1} \mathcal{P}[X_2, X_{\{3,4\}}]$$

factor of 4 variablesfactor of 3 variablesfactor of 3 variables

Advantage: Factors are only functions of observed variables! Can be directly computed from data without EM!!!!

Caveat: some factors are no longer probability tables (do not have to be non-negative)

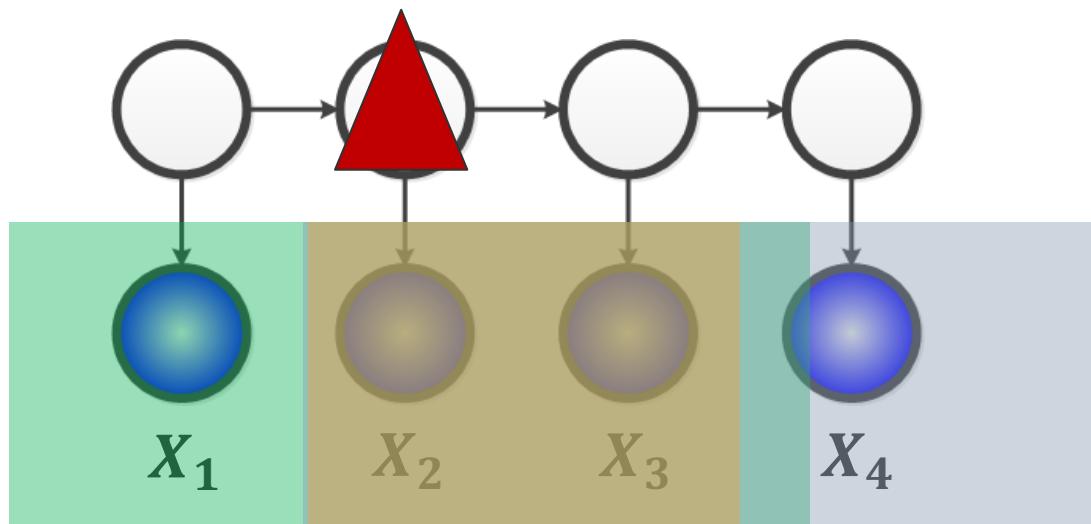
We will call this factorization the **observable factorization**.





Graphical Relationship

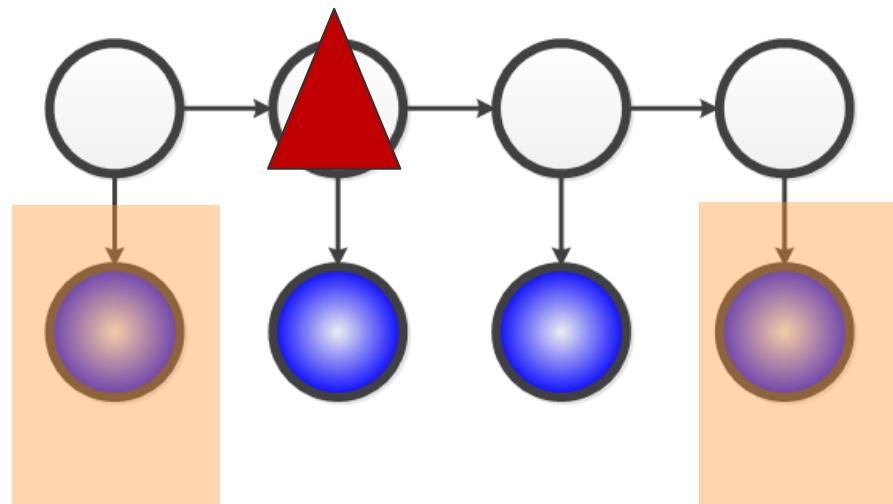
$$\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}] = \mathcal{P}[X_{\{1,2\}}, X_3] \mathcal{P}[X_2, X_3]^{-1} \mathcal{P}[X_2, X_{\{3,4\}}]$$





Another Factorization

$$\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}] = \mathcal{P}[X_{\{1,2\}}, X_4] \mathcal{P}[X_1, X_4]^{-1} \mathcal{P}[X_1, X_{\{3,4\}}]$$



- Seems we would do better empirically if you could “combine” both factorizations. Will come back to this later.





Relationship to Original Factorization

- What is the relationship between the original factorization and the new factorization?

$$\underline{\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}]} = \underline{\mathcal{P}[X_{\{1,2\}}|H_2]} \underline{\mathcal{P}[\emptyset|H_2]} \underline{\mathcal{P}[X_{\{3,4\}}|H_2]}^\top$$

$$M = LR$$

$$M = LSS^{-1}R$$

Can I choose S to get the observable factorization?





Relationship to Original Factorization

- Let

$$S := \mathcal{P}[X_3 | H_2]$$

$$\begin{aligned}\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}] &= \underline{\mathcal{P}[X_{\{1,2\}}, X_3]} \underline{\mathcal{P}[X_2, X_3]^{-1} \mathcal{P}[X_2, X_{\{3,4\}}]} \\ &= LS \quad \quad \quad = S^{-1}R\end{aligned}$$

$$\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}] = \mathcal{P}[X_{\{1,2\}} | H_2] \mathcal{P}[\emptyset | H_2] \mathcal{P}[X_{\{3,4\}} | H_2]^\top$$





Our Alternative Factorization

$$\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}] = \mathcal{P}[X_{\{1,2\}}, X_3] \mathcal{P}[X_2, X_3]^{-1} \mathcal{P}[X_2, X_{\{3,4\}}]$$

factor of 4 variables factor of 3 variables factor of 3 variables

- It may not seem very amazing at the moment (we have only reduced the size of the factor by 1)
- What is cool is that every latent tree of V variables has such a factorization where:
 - All factors are of size 3
 - All factors are only functions of observed variables

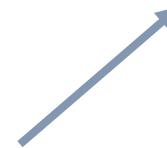




Generalizing To More Variables

- Consider HMM with 5 observations. Using similar arguments as before we will get that:

$$\mathcal{P}[X_{\{1,2\}}, X_{\{3,4,5\}}] = \mathcal{P}[X_{\{1,2\}}, X_3] \mathcal{P}[X_2, X_3]^{-1} \mathcal{P}[X_2, X_{\{3,4,5\}}]$$



reshape and decompose
recursively

$$\mathcal{P}[X_{\{2,3\}}, X_{\{4,5\}}] = \mathcal{P}[X_{\{2,3\}}, X_4] \mathcal{P}[X_3, X_4]^{-1} \mathcal{P}[X_3, X_{\{4,5\}}]$$





Training / Testing with Spectral Learning

- We have that

$$\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}] = \mathcal{P}[X_{\{1,2\}}, X_3] \mathcal{P}[X_2, X_3]^{-1} \mathcal{P}[X_2, X_{\{3,4\}}]$$

- In training, we compute estimates:

$$\mathcal{P}_{MLE}[X_{\{1,2\}}, X_3] \quad \mathcal{P}_{MLE}[X_2, X_3]^{-1} \quad \mathcal{P}_{MLE}[X_2, X_{\{3,4\}}]$$

- In test time, we can compute probability estimates (let lowercase letters denote fixed evidence values):

$$\hat{\mathbb{P}}_{spec}[x_1, x_2, x_3, x_4] = \mathcal{P}_{MLE}[x_{\{1,2\}}, X_3] \mathcal{P}_{MLE}[X_2, X_3]^{-1} \mathcal{P}_{MLE}[X_2, x_{\{3,4\}}]^T$$





Consistency

- ❑ A trivial consistent estimator is to simply attempt to estimate the “big” probability table from the data without making any conditional independence assumptions

$$\mathcal{P}_{MLE}[X_1, X_2; X_3, X_4] \rightarrow \mathcal{P}[X_1, X_2; X_3, X_4] \quad \text{as number of samples increases}$$

- ❑ While this is consistent, it is not very statistically efficient





Unsupervised Parsing

Training Set – Given sentences and **part-of-speech tags**

DT NN VB NN

The bear likes fish

DT NN VB DT NN

The llama eats the grass

Test Set – Find (unlabeled) parse tree for each sentence



Lions quickly chase deer and antelope

Parse tree structure is a *latent* variable



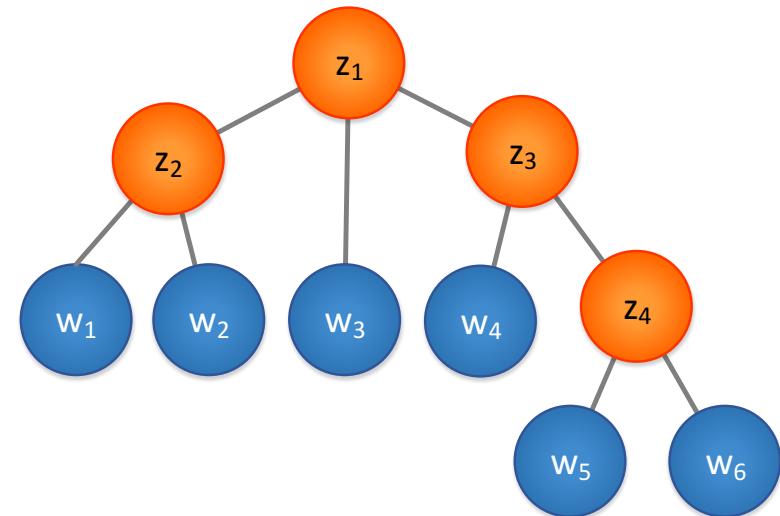


Conditional Latent Tree Model

- Each tag sequence \mathbf{x} associated with a latent tree

$$\mathbf{x}_2 = (DT, NN, VBD, DT, ADJ, NN)$$

$$p(\mathbf{w}, \mathbf{z} | \mathbf{x}) = \prod_{i=1}^H p(z_i | \pi_{\mathbf{x}}(z_i)) \\ \times \prod_{i=1}^{\ell(x)} p(w_i | \pi_{\mathbf{x}}(w_i))$$



- Traditional Approach

Training

(Given the latent tree) Estimate parameters using *nonconvex optimization*:

$$\hat{P}(H_1) \quad \hat{P}(X_1|H_2) \quad \hat{P}(X_5|H_4) \quad \dots$$

Test

To query probabilities: $\mathbb{P}(X_1 = 0, \dots, X_6 = 1)$
multiply learned parameters

The bear ate the big fish

The moose ran the tiring race





The Spectral Approach

Latent Tree *Observable* Factorization

$$\mathcal{P}(X_1, X_2, X_3, X_4, X_5, X_6) = \mathcal{F}(X_1, X_3, X_5) \times \mathcal{F}(X_1, X_2, X_3) \times \mathcal{F}(X_3, X_4, X_5) \times \mathcal{F}(X_4, X_5, X_6)$$

Training

Estimate alternate parameters:

$$\mathcal{F}(X_1, X_2, X_3)$$

$$\mathcal{F}(X_1, X_3, X_5)$$

$$\mathcal{F}(X_3, X_4, X_5)$$

$$\mathcal{F}(X_4, X_5, X_6)$$

Test

To query probabilities: $\mathcal{P}(X_1 = 0, \dots, X_6 = 1)$
tensor multiply parameters





Consistency

- ❑ A better estimate is to compute likelihood estimates of the factorization:

$$\begin{aligned}\mathcal{P}_{MLE}[X_{\{1,2\}}|H_2] \mathcal{P}_{MLE}[\emptyset|H_2] \mathcal{P}_{MLE}[X_{\{3,4\}}|H_2]^\top \\ \rightarrow \mathcal{P}[X_1, X_2; X_3, X_4]\end{aligned}$$

- ❑ But this requires running EM, which will get stuck in local optima and is not guaranteed to obtain the MLE of the factorized model





Consistency

- In spectral learning, we estimate the alternate factorization from the data

$$\begin{aligned} \mathcal{P}_{MLE}[X_{\{1,2\}}, X_3] \mathcal{P}_{MLE}[X_2, X_3]^{-1} \mathcal{P}_{MLE}[X_2, X_{\{3,4\}}] \\ \rightarrow \mathcal{P}[X_1, X_2; X_3, X_4] \end{aligned}$$

- This is consistent and computationally tractable (at some loss of statistical efficiency due to the dependence on the inverse)





Where's the Catch?

- ❑ Before we said that if the number of latent states was very large then the model was equivalent to a clique.
- ❑ Where does that scenario enter in our factorization?

$$\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}] = \mathcal{P}[X_{\{1,2\}}, X_3] \mathcal{P}[X_2, X_3]^{-1} \mathcal{P}[X_2, X_{\{3,4\}}]$$



When does this inverse exist?





When Does the Inverse Exist

$$\mathcal{P}[X_2, X_3] = \mathcal{P}[X_2 | H_2] \mathcal{P}[\emptyset | H_2] \mathcal{P}[X_3 | H_2]^\top$$

- All the matrices on the right hand side must have full rank. (This is in general a requirement of spectral learning, although it can be somewhat relaxed)





When $m > k$

- The inverse cannot exist, but this situation is easily fixable (project onto lower dimensional space)

$$\begin{aligned}\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}] &= \\ \mathcal{P}[X_{\{1,2\}}, X_3] \mathbf{V} (\mathbf{U}^\top \mathcal{P}[X_2, X_3] \mathbf{V})^{-1} \mathbf{U}^\top \mathcal{P}[X_2, X_{\{3,4\}}]\end{aligned}$$

- Where \mathbf{U} , \mathbf{V} are the top left/right k singular vectors of $\mathcal{P}[X_2, X_3]$





When $k > m$

- The inverse does exist. But it no longer satisfies the following property, which we used to derive the factorization

$$\mathcal{P}[X_2, X_3]^{-1} = (\mathcal{P}[X_3 | H_2]^\top)^{-1} \mathcal{P}[\emptyset | H_2]^{-1} \mathcal{P}[X_2 | H_2]^{-1}$$

- This is much more difficult to fix, and intuitively corresponds to how the problem becomes intractable if $k \gg m$.





What does $k>m$ mean?

- Intuitively, large k , small m means long range dependencies
- Consider following generative process:
 - (1) With probability 0.5, let $S=X$, and with probability 0.5 let $S=Y$.
 - (2) Print A n times.
 - (3) Print S
 - (4) Go back to step (2)

With $n=1$ we either generate:

AXAXAXA..... or AYAYAYA.....

With $n=2$ we either generate:

AAXAAXAA..... or AAYAAYAA.....





How many hidden states does HMM need?

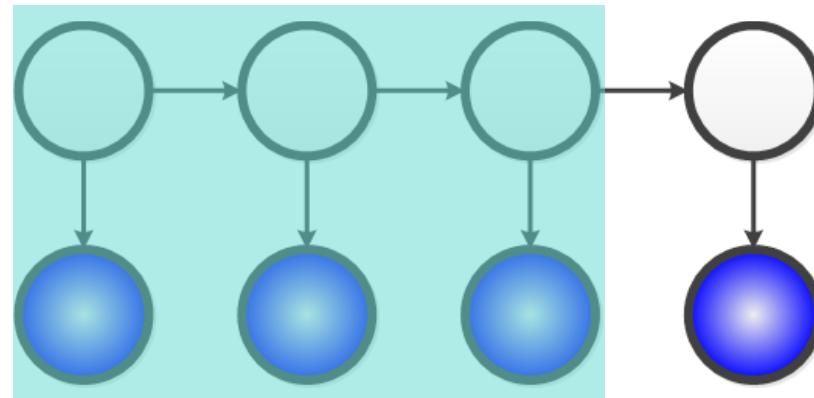
- ❑ HMM needs **2n** states.
- ❑ Needs to remember count as well as whether we picked **S=X** or **S=Y**
- ❑ However, number of observed states **m** does not change, so our previous spectral algorithm will break for **n > 2**.
- ❑ How to deal with this in spectral framework?





Making Spectral Learning Work In Practice

- We are only using marginals of pairs/triples of variables to construct the full marginal among the observed variables.
- Only works when $k < m$.



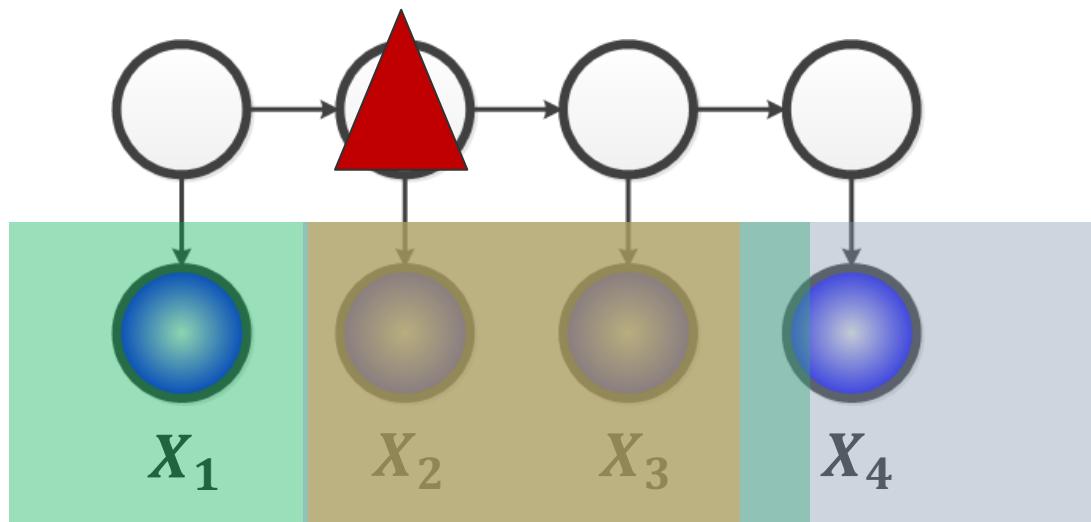
- However, in real problems we need to capture longer range dependencies.





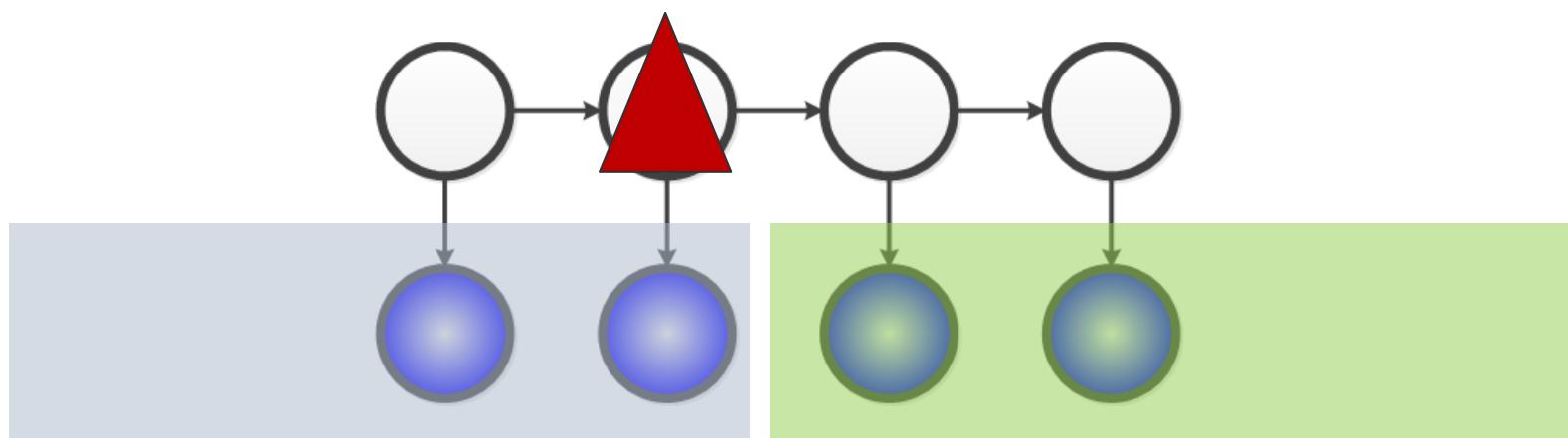
Recall our factorization

$$\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}] = \underline{\mathcal{P}[X_{\{1,2\}}, X_3]} \underline{\mathcal{P}[X_2, X_3]^{-1}} \underline{\mathcal{P}[X_2, X_{\{3,4\}}]}$$





Key Idea: Use Long-Range Features



Construct feature
vector of left side

$$\phi_L$$

Construct feature
vector of right side

$$\phi_R$$




Spectral Learning With Features

$$\mathcal{P}[X_2, X_3] = \mathbb{E}[\boldsymbol{\delta}_2 \otimes \boldsymbol{\delta}_3] := \mathbb{E}[\boldsymbol{\delta}_2 \boldsymbol{\delta}_3^\top]$$



Use more complex feature instead:

$$\mathbb{E}[\boldsymbol{\phi}_L \otimes \boldsymbol{\phi}_R]$$

$$\begin{aligned}\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}] &= \mathbb{E}[\boldsymbol{\delta}_{1\otimes 2}, \boldsymbol{\delta}_{3\otimes 4}] \\ &= \mathbb{E}[\boldsymbol{\delta}_{1\otimes 2}, \boldsymbol{\phi}_R] \mathbf{V} (\mathbf{U}^\top \mathbb{E}[\boldsymbol{\phi}_L \otimes \boldsymbol{\phi}_R] \mathbf{V})^{-1} \mathbf{U}^\top \mathcal{P}[\boldsymbol{\phi}_L, X_{\{3,4\}}]\end{aligned}$$





Experimentally,

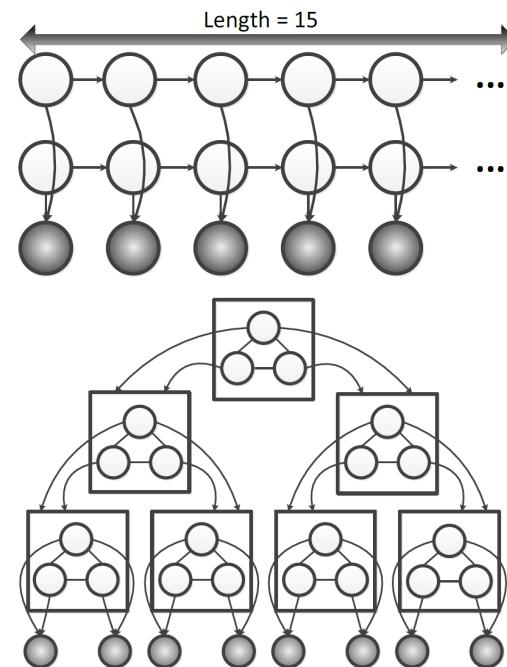
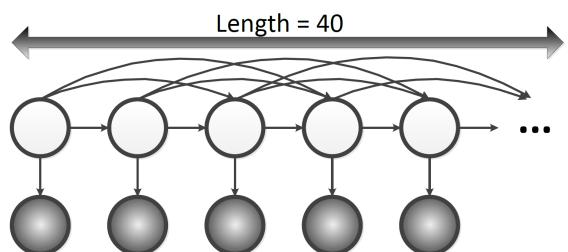
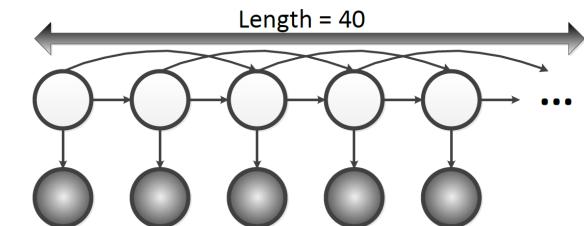
- ❑ Has been shown by many authors that (with some work) spectral methods achieve comparable results to EM but are 10-50x faster
 - ❑ Parikh et al. 2011 / 2012
 - ❑ Balle et al. 2012
 - ❑ Cohen et al. 2012 / 2013
- ❑ The following are some synthetic and real data results demonstrating the comparison between EM and spectral methods.





Synthetic Data [Parikh et al. 2012]

- Different latent variable models



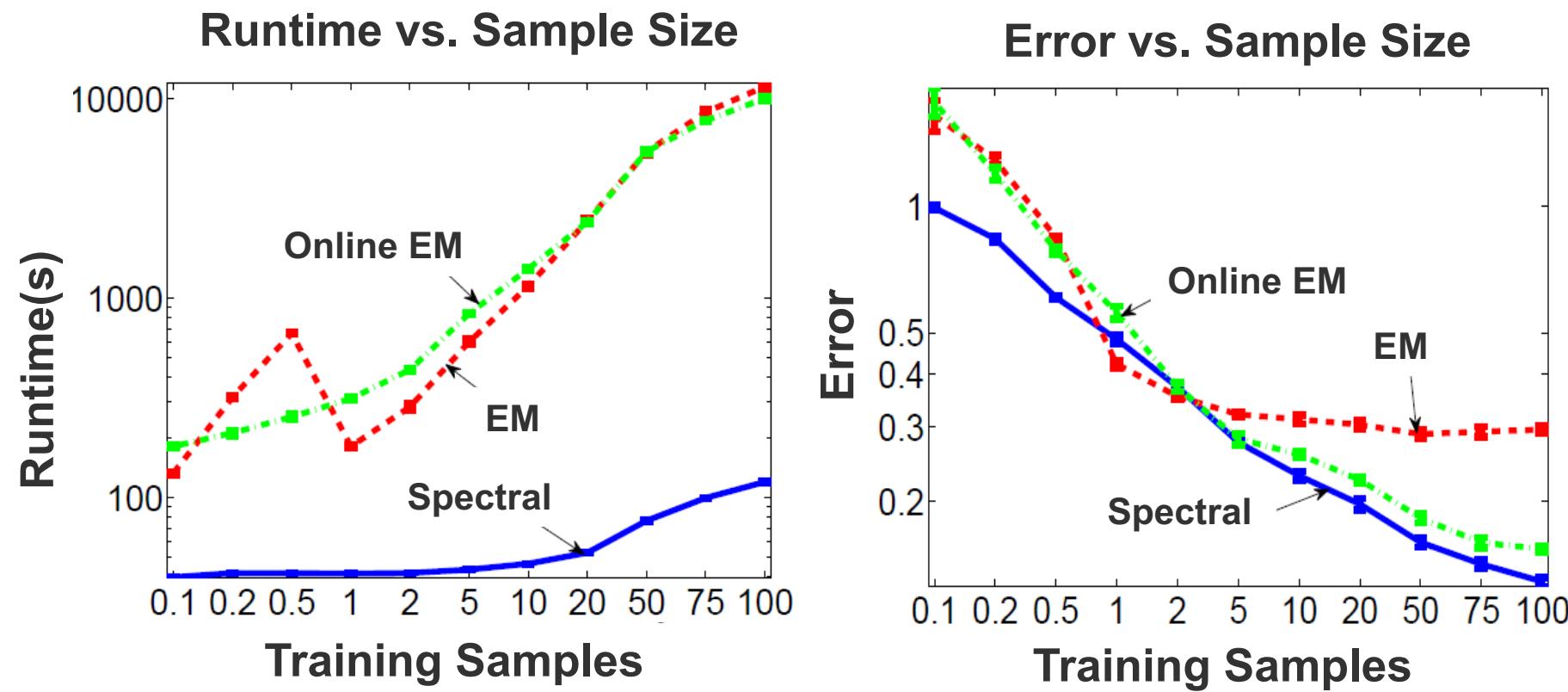
- **Train:** Learn parameters for a given model given samples of observed variables
- **Test:** Evaluate likelihood of random samples drawn from model and compare to the true likelihood





Synthetic Data [Parikh et al. 2012]

- Synthetic 3rd order HMM Example (Spectral/EM/Online EM):



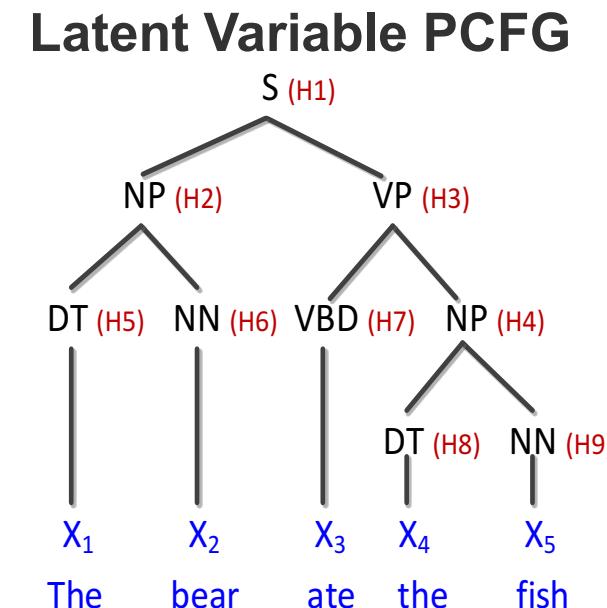
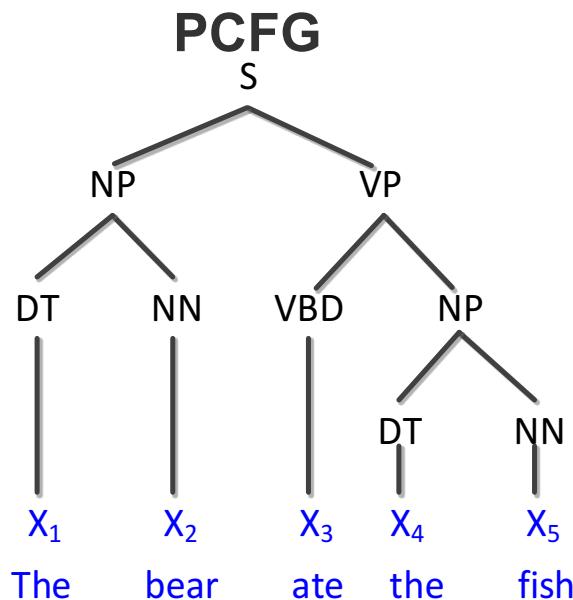
- Results for other structures look similar





Supervised Parsing [Cohen et al. 2012/2013]

- Learn a latent variable Probabilistic Context Free Grammar model (latent PCFG) which is a PCFG augmented with additional latent states



- Train:** Learn parameters given parse trees on training examples.
- Test:** Estimate most likely parse structure on test sentences





Empirical Results for Latent PCFGs [Cohen et al. 2013]

	section 22		section 23	
	EM	spectral	EM	spectral
$m = 8$	86.87	85.60	—	—
$m = 16$	88.32	87.77	—	—
$m = 24$	88.35	88.53	—	—
$m = 32$	88.56	88.82	87.76	88.05

Evaluation Measure: *F1 bracketing score*





Timing Results on Latent PCFGs [Cohen et al. 2013]

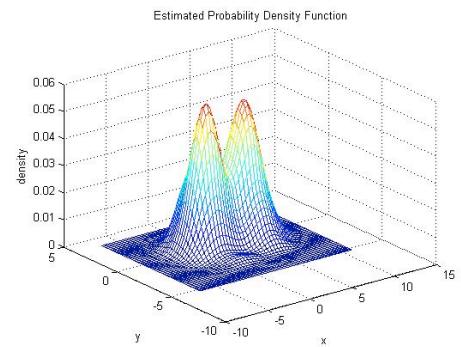
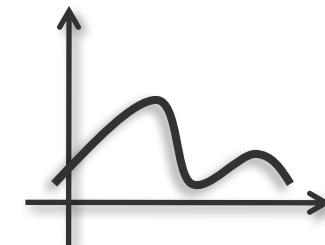
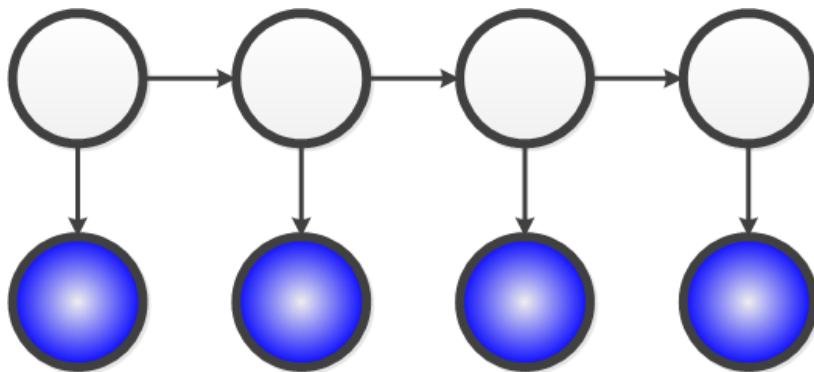
	single EM iter.	EM best model	total	spectral algorithm				
				feature	transfer + scaling	SVD	$a \rightarrow b$	c
$m = 8$	6m	3h	3h32m			36m	1h34m	10m
$m = 16$	52m	26h6m	5h19m			34m	3h13m	19m
$m = 24$	3h7m	93h36m	7h15m	22m		36m	4h54m	28m
$m = 32$	9h21m	187h12m	9h52m			35m	7h16m	41m





Dealing with Nonparametric, Continuous Variables

- It is difficult to run EM if the conditional/marginal distributions are continuous and do not easily fit into a parametric family.



- However, we will see that Hilbert Space Embeddings can easily be combined with spectral methods for learning nonparametric latent models.





Connection to Hilbert Space Embeddings

- Recall that we could substitute features for variables

$$\mathcal{P}[X_2, X_3] = \mathbb{E}[\delta_2 \otimes \delta_3] := \mathbb{E}[\delta_2 \delta_3^\top]$$



Use more complex feature instead:

$$\mathbb{E}[\phi_L \otimes \phi_R]$$





Can Also Use Infinite Dimensional Features

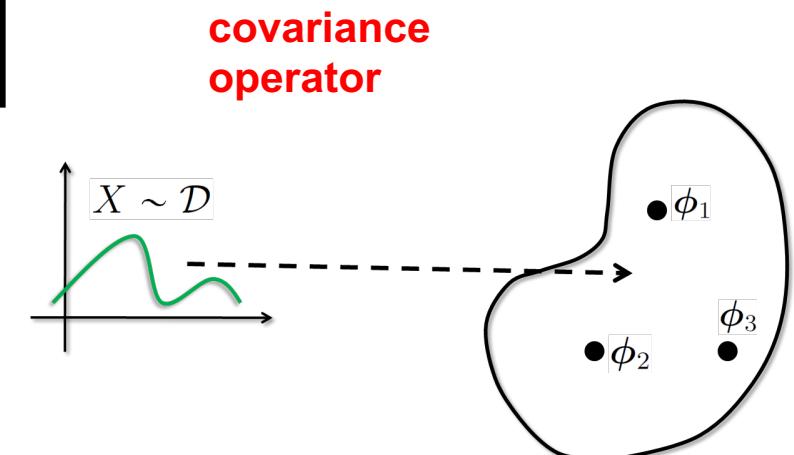
- ❑ Replace

$$\mathcal{P}[X_2, X_3] = \mathbb{E}[\delta_2 \otimes \delta_3] := \mathbb{E}[\delta_2 \delta_3^\top]$$

- ❑ with

$$\mathcal{C}[X_2, X_3] = \mathbb{E}[\phi_{X_2} \otimes \phi_{X_3}]$$

- ❑ (and similarly for other quantities)





Connection to Hilbert Space Embeddings

Discrete case:

$$\begin{aligned}\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}] &= \\ \mathcal{P}[X_{\{1,2\}}, X_3] \mathbf{V} (\mathbf{U}^\top \mathcal{P}[X_2, X_3] \mathbf{V})^{-1} \mathbf{U}^\top \mathcal{P}[X_2, X_{\{3,4\}}]\end{aligned}$$

Continuous case:

$$\begin{aligned}\mathcal{C}[X_{\{1,2\}}; X_{\{3,4\}}] &= \\ \mathcal{C}[X_{\{1,2\}}; X_3] \mathbf{V} (\mathbf{U}^\top \mathcal{C}[X_2, X_3] \mathbf{V})^{-1} \mathbf{U}^\top \mathcal{C}[X_2; X_{\{3,4\}}]\end{aligned}$$





Summary - EM & Spectral (Part I)

EM

- Aims to Find MLE so more “statistically” efficient
- Can get stuck in local-optima
- Lack of theoretical guarantees
- Slow
- Easy to derive for new models

Spectral

- Does not aim to find MLE so less statistically efficient.
- Local-optima-free
- Provably consistent
- Very fast
- Challenging to derive for new models (Unknown whether it can generalize to arbitrary loopy models)





Summary - EM & Spectral (Part II)

EM

- No issues with negative numbers
- Allows for easy modelling with conditional distributions
- Difficult to incorporate long-range features (since it increases treewidth).
- Generalizes poorly to non-Gaussian continuous variables.

Spectral

- Problems with negative numbers. Requires explicit normalization to compute likelihood.
- Allows for easy modelling with marginal distributions
- Easy to incorporate long-range features.
- Easy to generalize to non-Gaussian continuous variables via Hilbert Space Embeddings

