Use Chapter 3 of K&F as a reference for CSI
Reading for parameter learning: Chapter 12 of K&F

Context-specific independence
Parameter learning: MLE

Graphical Models – 10708

Carlos Guestrin
Carnegie Mellon University

October 5th, 2005

Announcements

- **Homework 2:**
  - **Out today/tomorrow**
  - **Programming part in groups of 2-3**
- **Class project**
  - **Teams of 2-3 students**
  - **Ideas on the class webpage, but you can do your own**
- **Timeline:**
  - **10/19: 1 page project proposal**
  - **11/14: 5 page progress report (20% of project grade)**
  - **12/2: poster session (20% of project grade)**
  - **12/5: 8 page paper (60% of project grade)**
  - **All write-ups in NIPS format (see class webpage)**

Clique trees versus VE

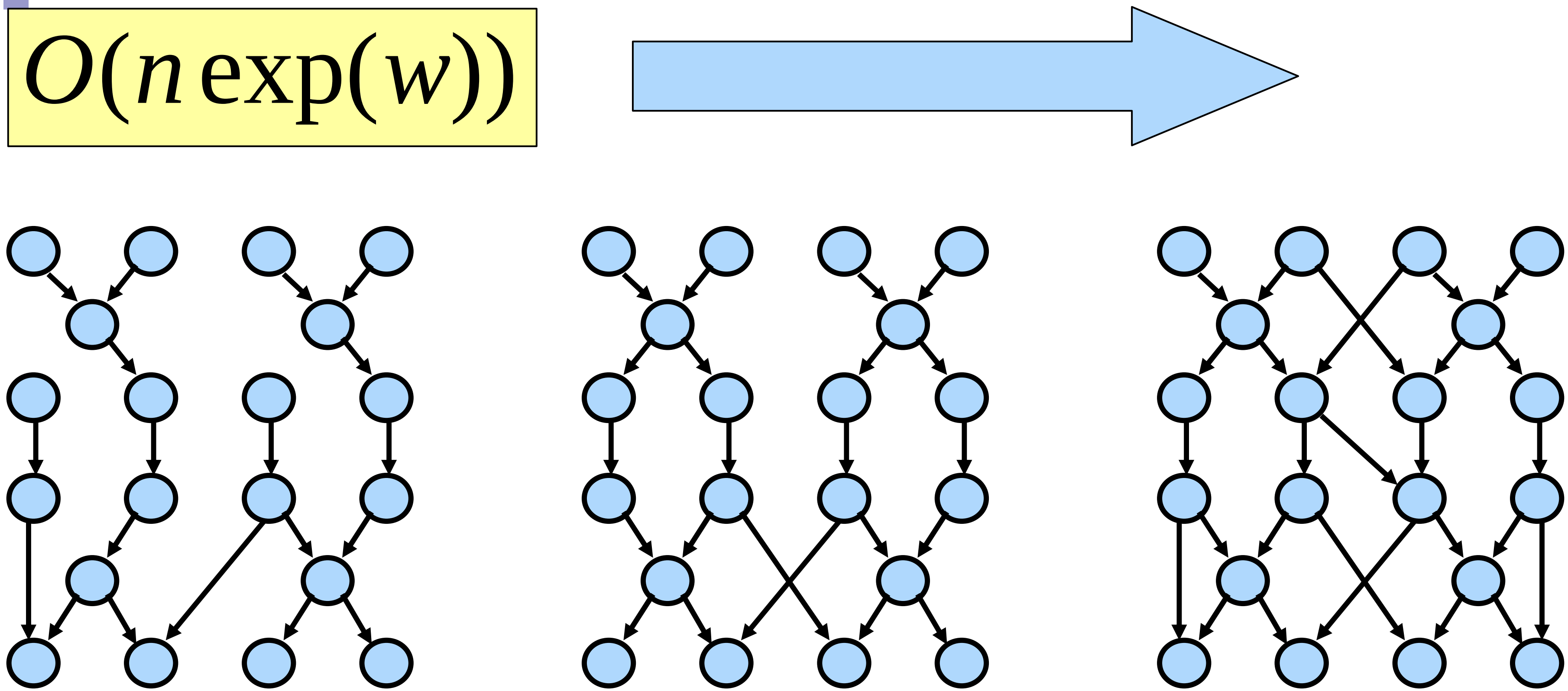
- Clique tree advantages
  - Multi-query settings
  - Incremental updates
  - Pre-computation makes complexity explicit
- Clique tree disadvantages
  - Space requirements – no factors are “deleted”
  - Slower for single query
  - Local structure in factors may be lost when they are multiplied together into initial clique potential

Clique tree summary

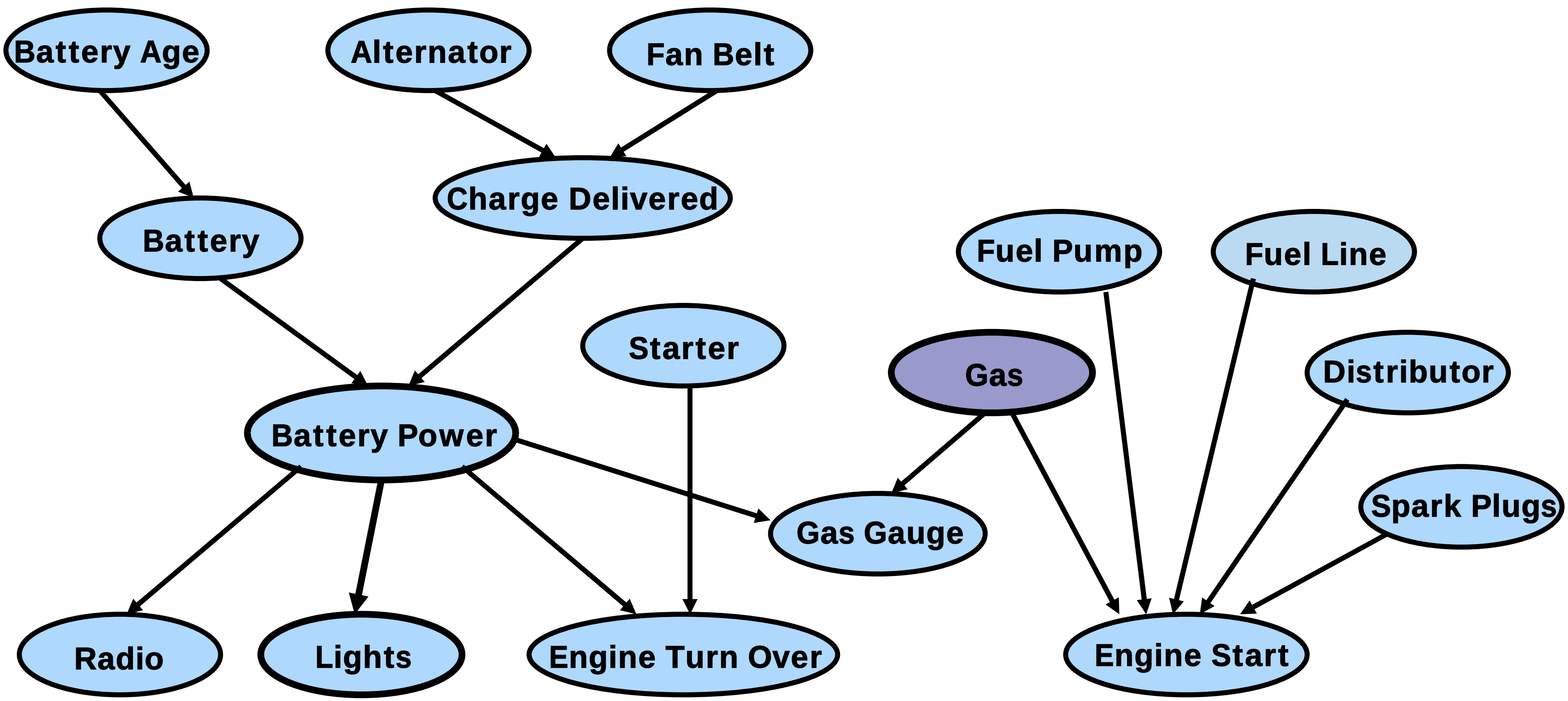
- Solve marginal queries for all variables in only twice the cost of query for one variable
- Cliques correspond to maximal cliques in induced graph
- Two message passing approaches
  - VE (the one that multiplies messages)
  - BP (the one that divides by old message)
- Clique tree invariant
  - Clique tree potential is always the same
  - We are only reparameterizing clique potentials
- Constructing clique tree for a BN
  - from elimination order
  - from triangulated (chordal) graph
- Running time (only) exponential in size of largest clique
  - Solve **exactly** problems with thousands (or millions, or more) of variables, and cliques with tens of nodes (or less)

Global Structure: Treewidth w

$O(n \exp(w))$

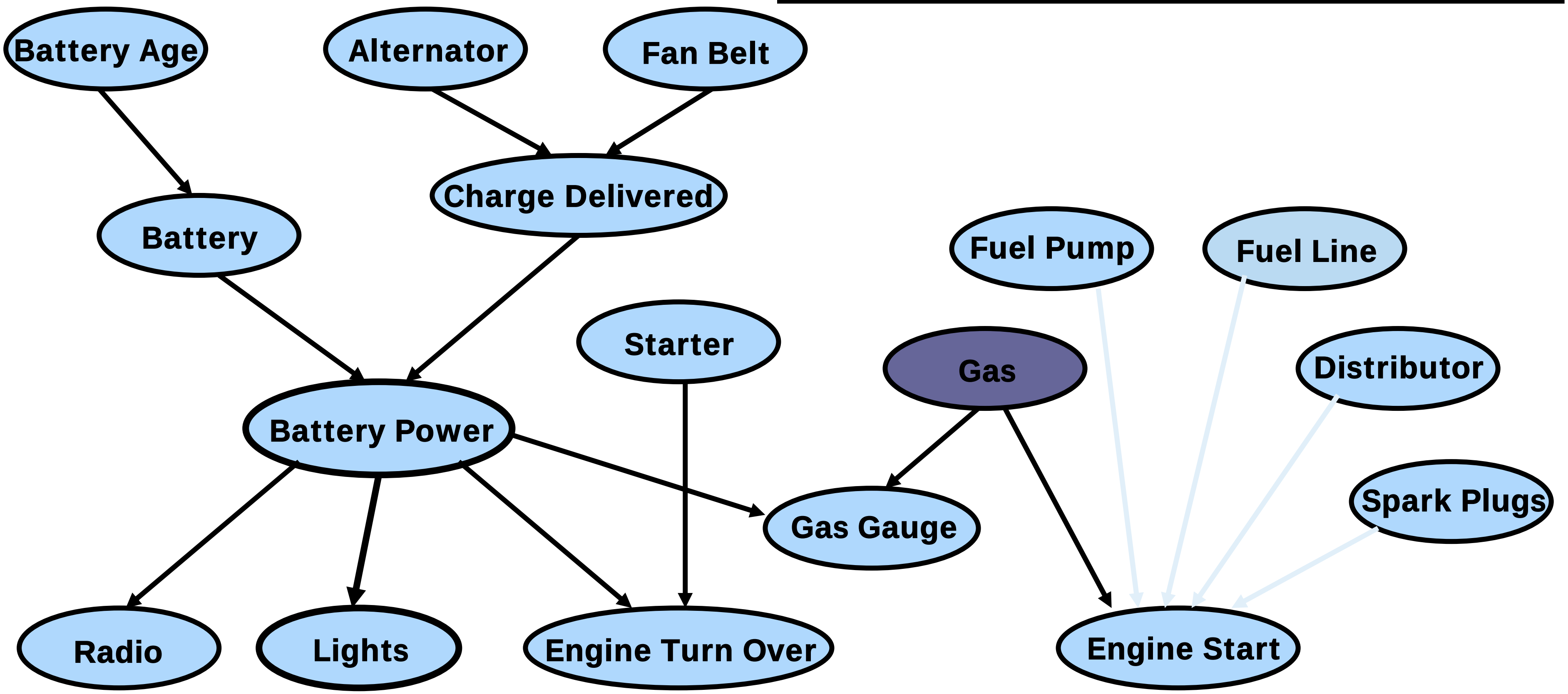


Local Structure 1: Context specific indepencence

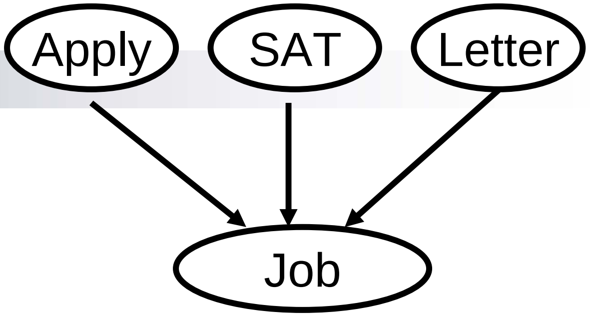


Local Structure 1: Context specific indepencence

Context Specific Independence (CSI)
After observing a variable, some vars become independent



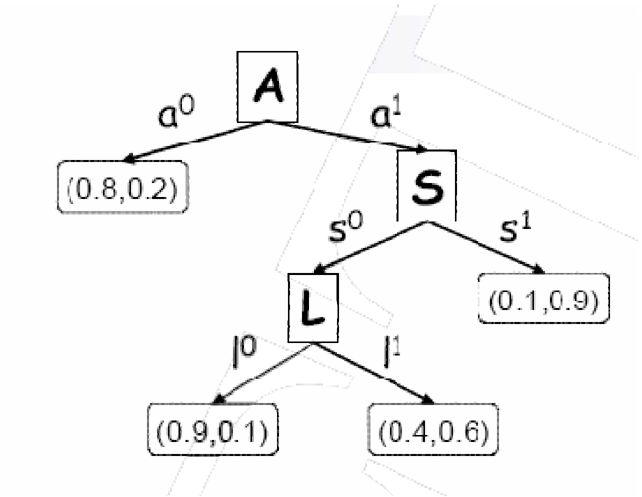
CSI example: Tree CPD



- Represent $P(X_i|\mathbf{Pa}_{X_i})$ using a decision tree
  - Path to leaf is an assignment to (a subset of) $\mathbf{Pa}_{X_i}$
  - Leaves are distributions over X_i given assignment of $\mathbf{Pa}_{X_i}$ on path to leaf
- **Interpretation of leaf**:
  - For specific assignment of $\mathbf{Pa}_{X_i}$ on path to this leaf – X_i is independent of other parents
- Representation can be exponentially smaller than equivalent table

Tabular VE with Tree CPDs

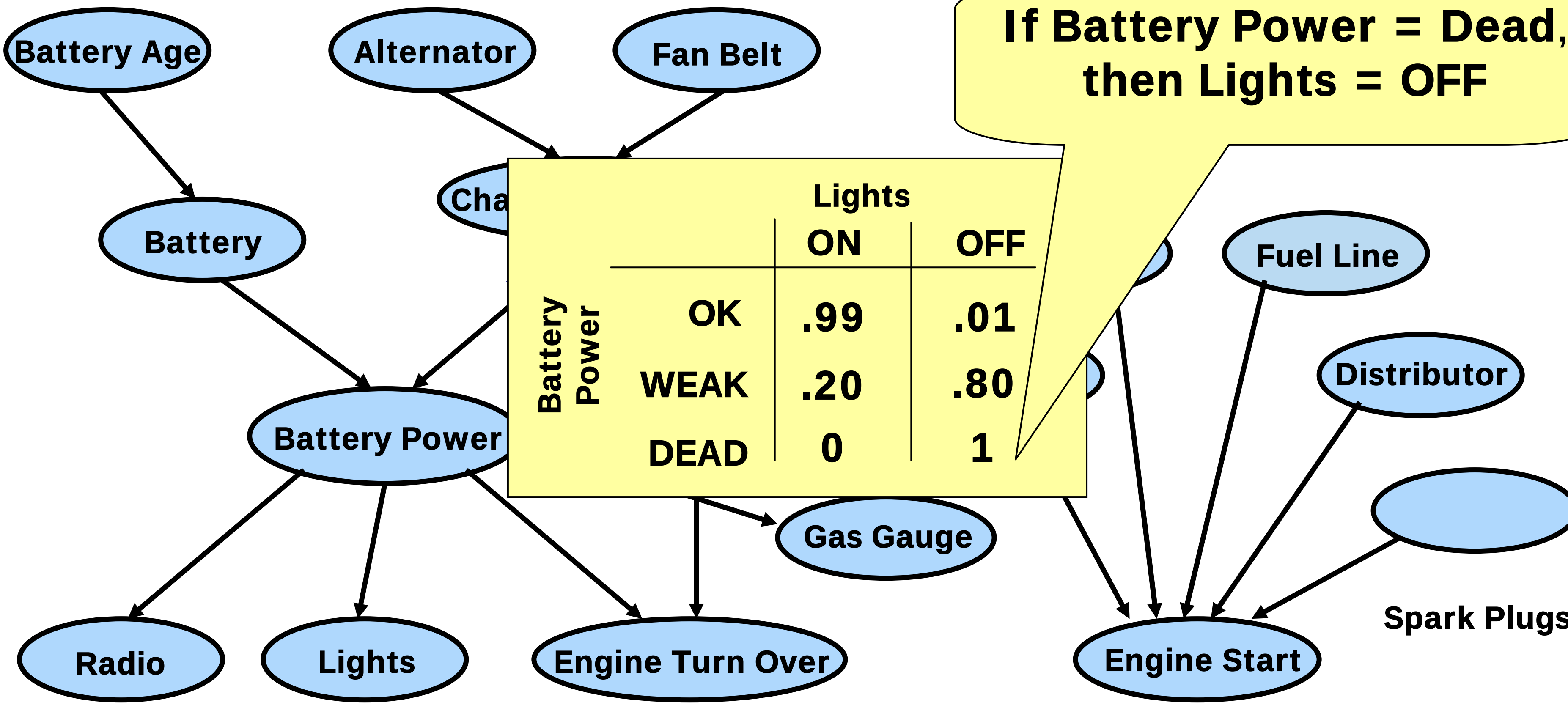
- If we turn a **tree CPD into table**
  - **“Sparsity” lost**!
- Need inference approach that **deals with tree CPD directly**!



Local Structure 2: Determinism

Determinism

If Battery Power = Dead, then Lights = OFF



Battery Power	Lights: ON	Lights: OFF
OK	.99	.01
WEAK	.20	.80
DEAD	0	1

Determinism and inference

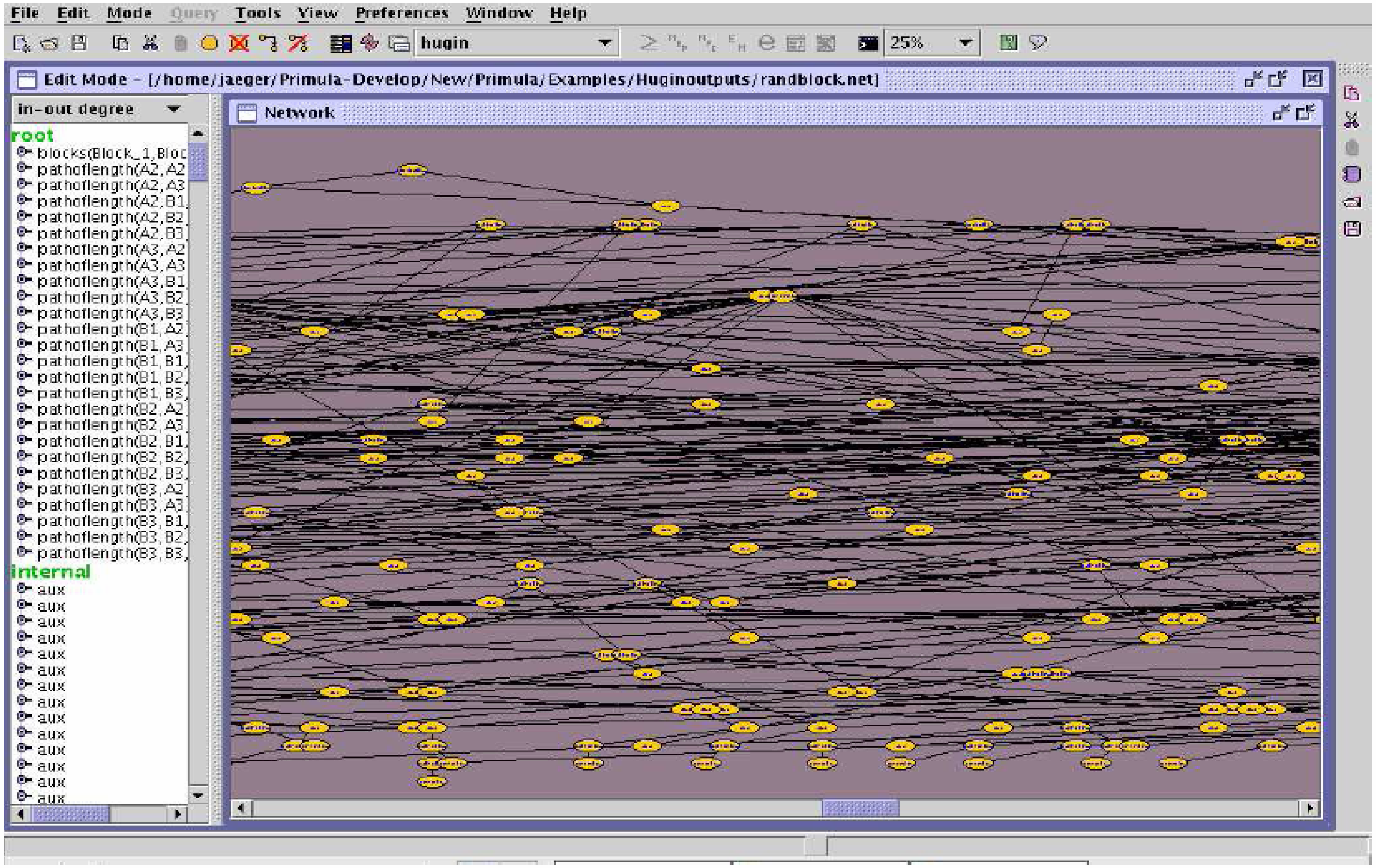
- Determinism gives a little sparsity in table, but much bigger impact on inference
- Multiplying deterministic factor with other factor introduces many new zeros
  - Operations related to theorem proving, e.g., unit resolution

Battery Power	Lights: ON	Lights: OFF
OK	.99	.01
WEAK	.20	.80
DEAD	0	1

Today’s Models …

- **Often characterized by:**
  - Richness in local structure (determinism, CSI)
  - Massiveness in size (10,000’s variables)
  - High connectivity (treewidth)
- **Enabled by:**
  - High level modeling tools: relational, first order
  - Advances in machine learning
  - New application areas (synthesis):
    - Bioinformatics (e.g. linkage analysis)
    - Sensor networks
- **Exploiting local structure a must!**

Exact inference in large models is possible…



- BN from a relational model

Recursive Conditioning

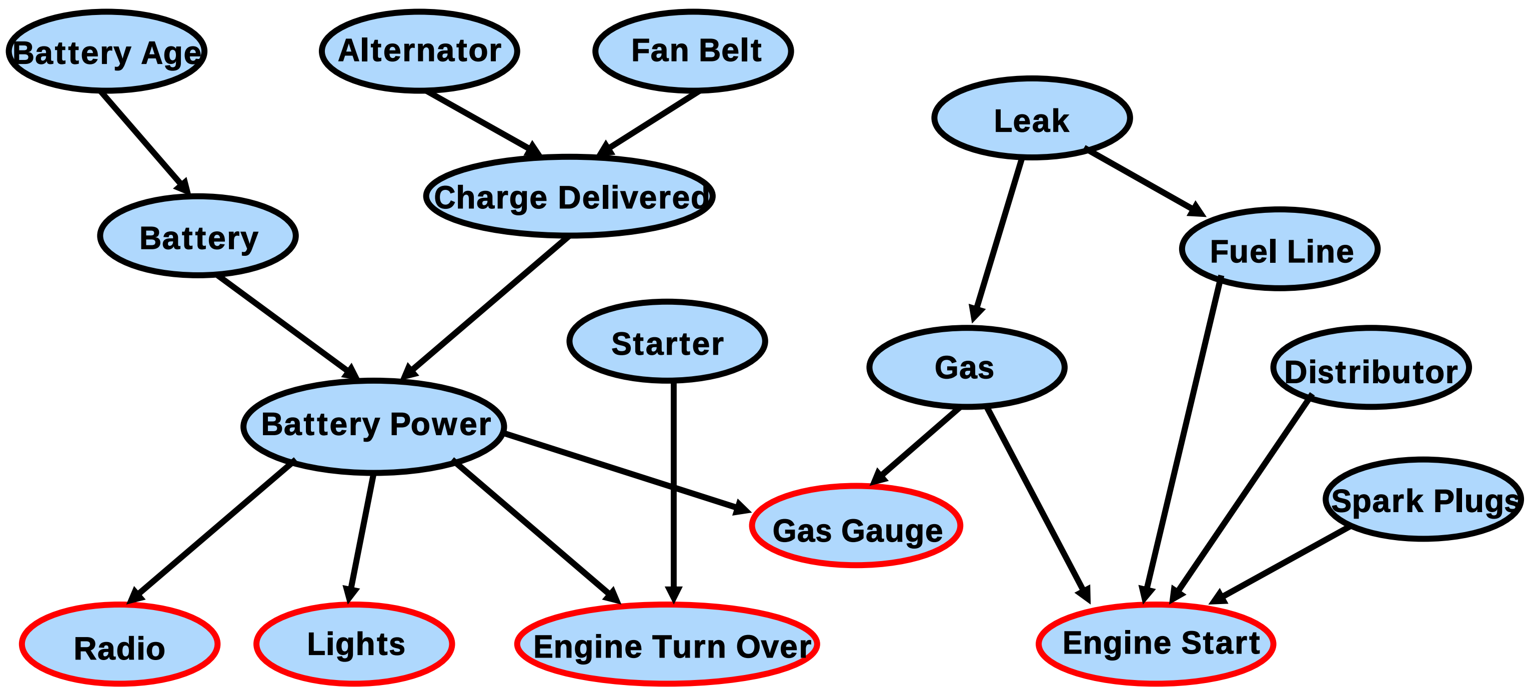
- Treewidth complexity (worst case)

- Better than treewidth complexity with local structure

- Provides a framework for time-space tradeoffs

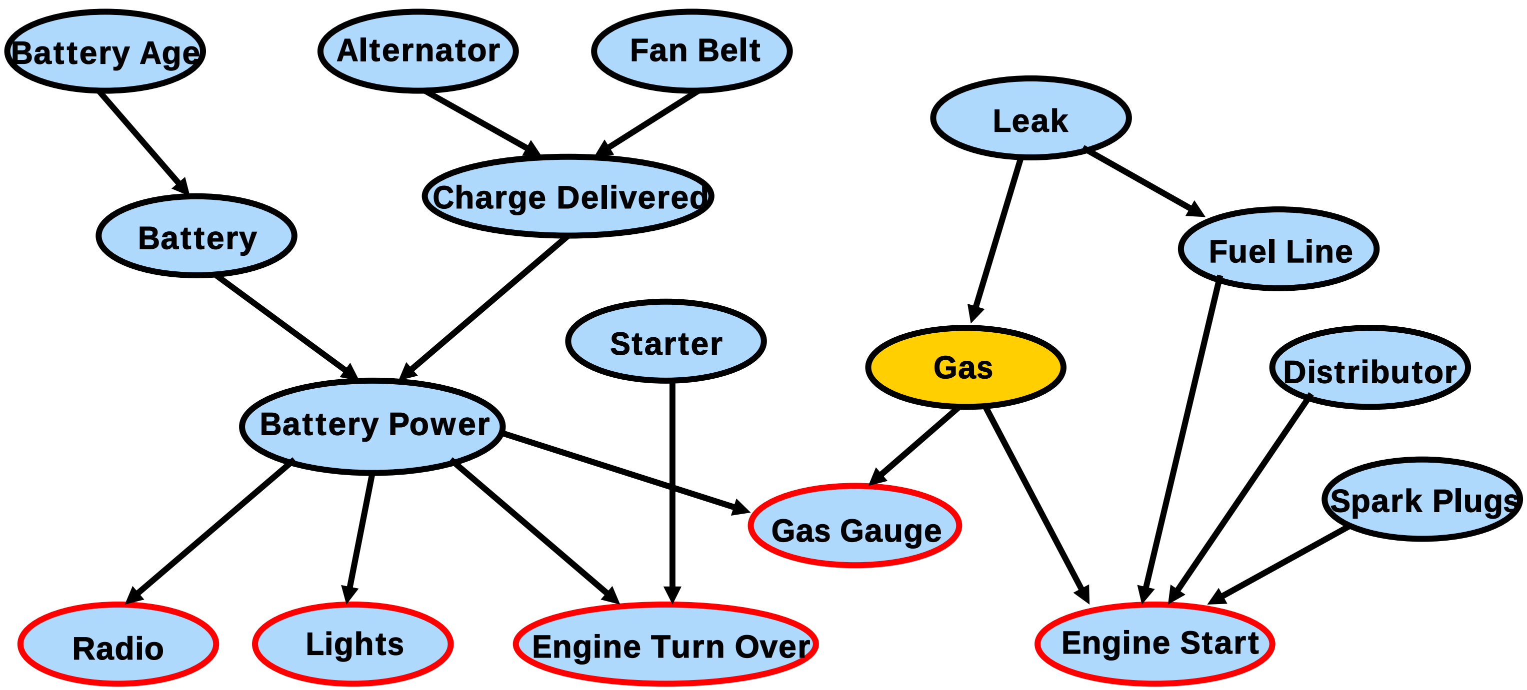
- Only quick intuition today, details:
  - Koller&Friedman: 3.1-3.4, 6.4-6.6
  - “Recursive Conditioning”, Adnan Darwiche. In *Artificial Intelligence Journal,* 125:1, pages 5-41

The Computational Power of Assumptions

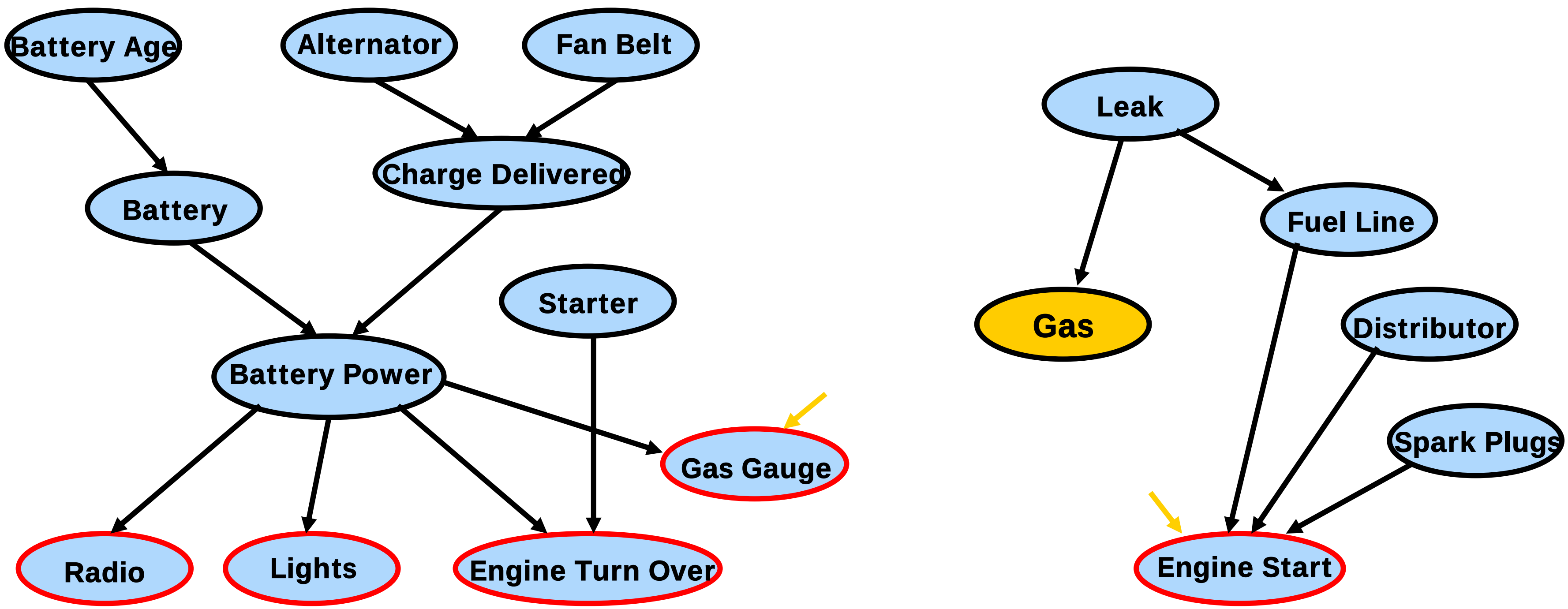


A. Darwiche

The Computational Power of Assumptions

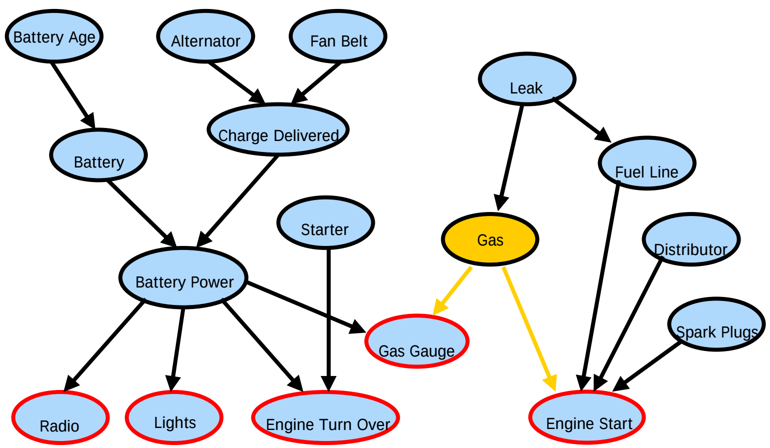


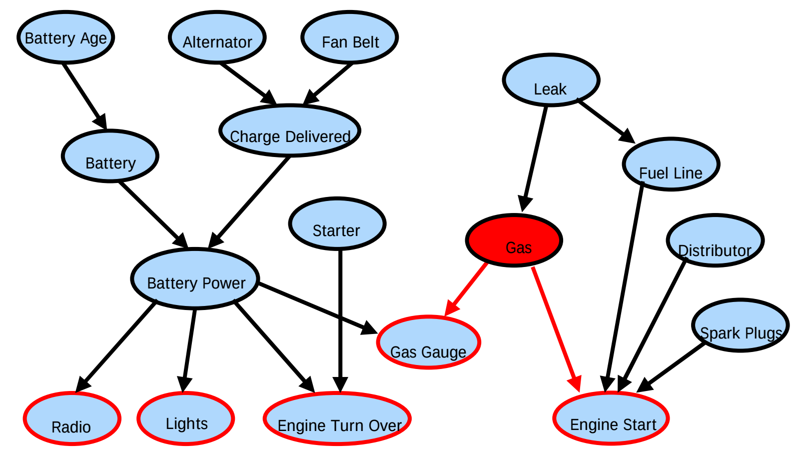
Decomposition



A. Darwiche

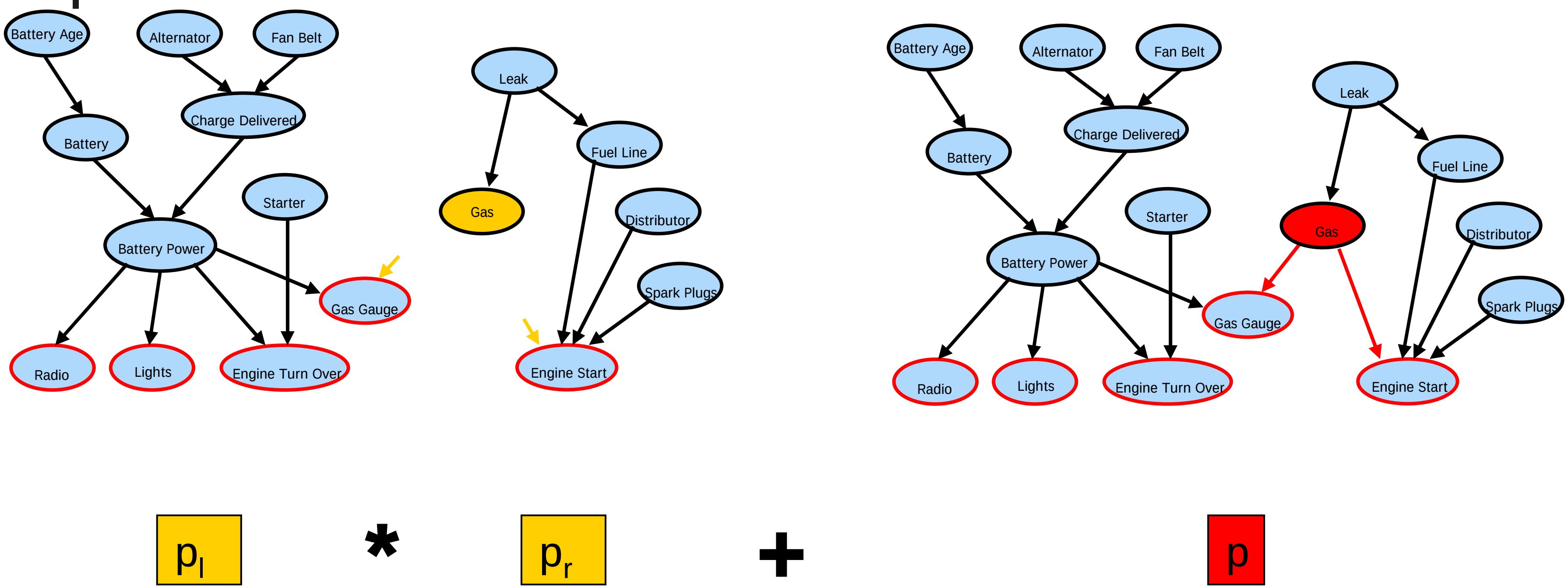
Case Analysis



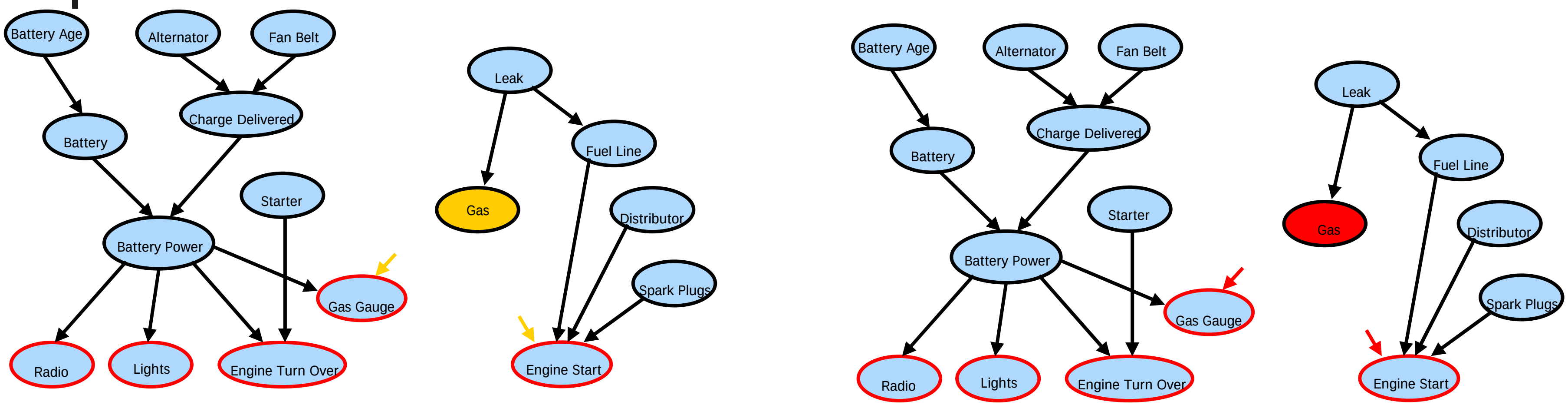


p + p

Case Analysis

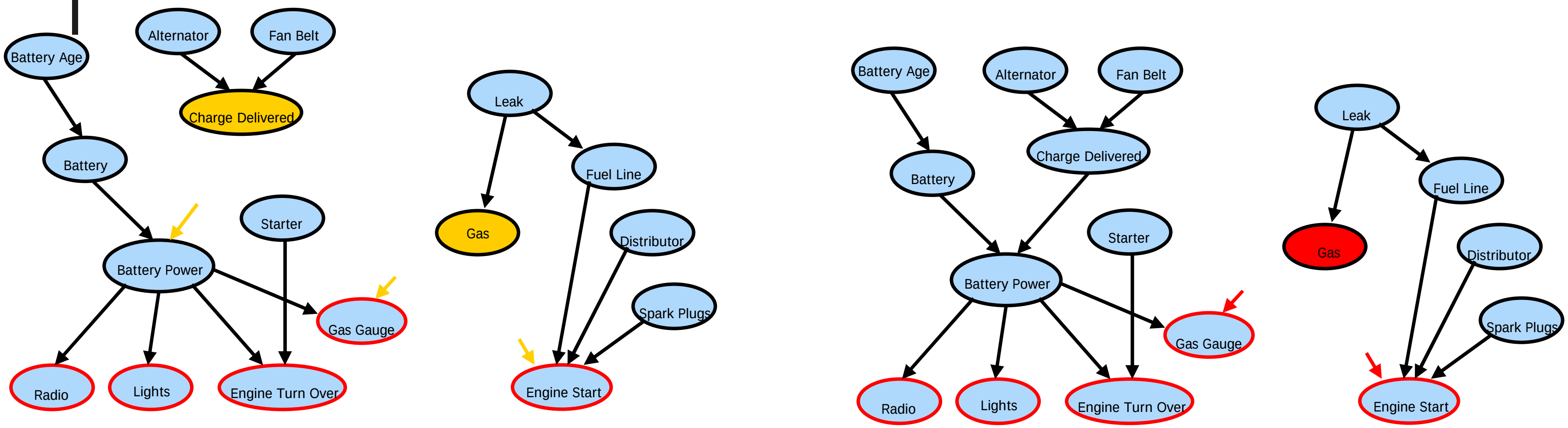


Case Analysis



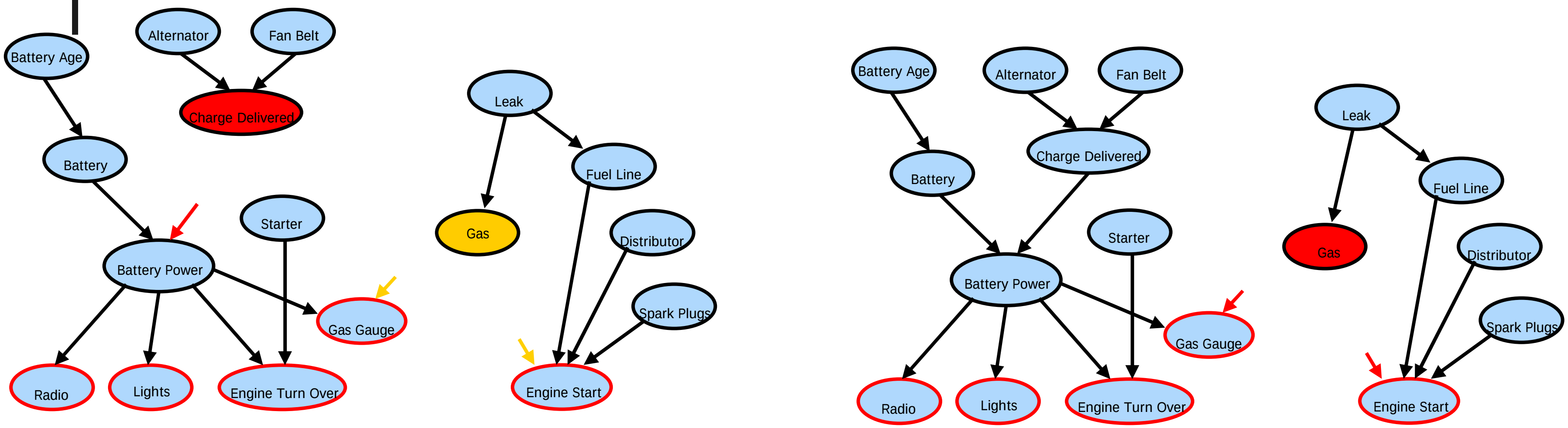
p_l * p_r + p_l * p_r

Case Analysis



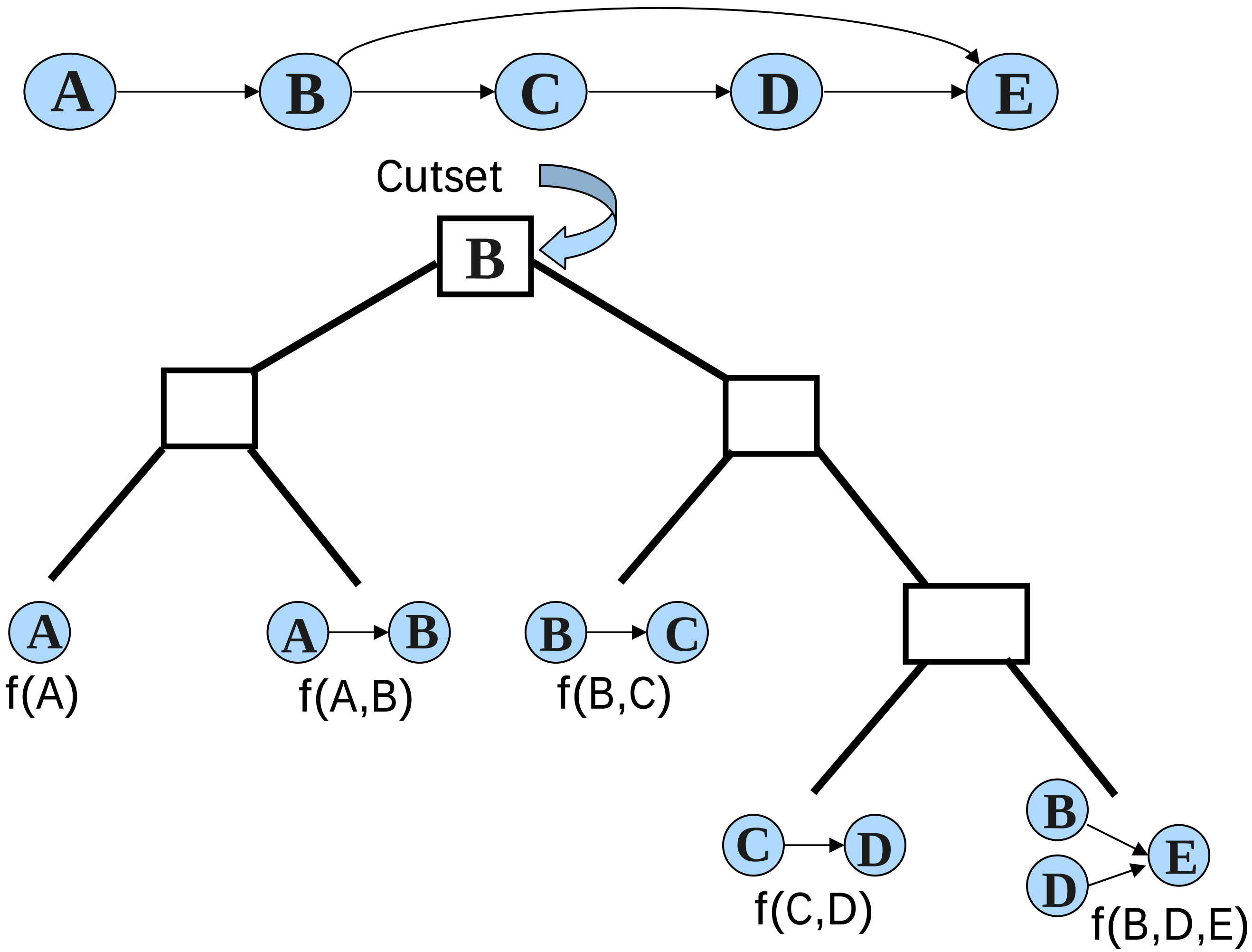
p_l * p_r + p_l * p_r

Case Analysis



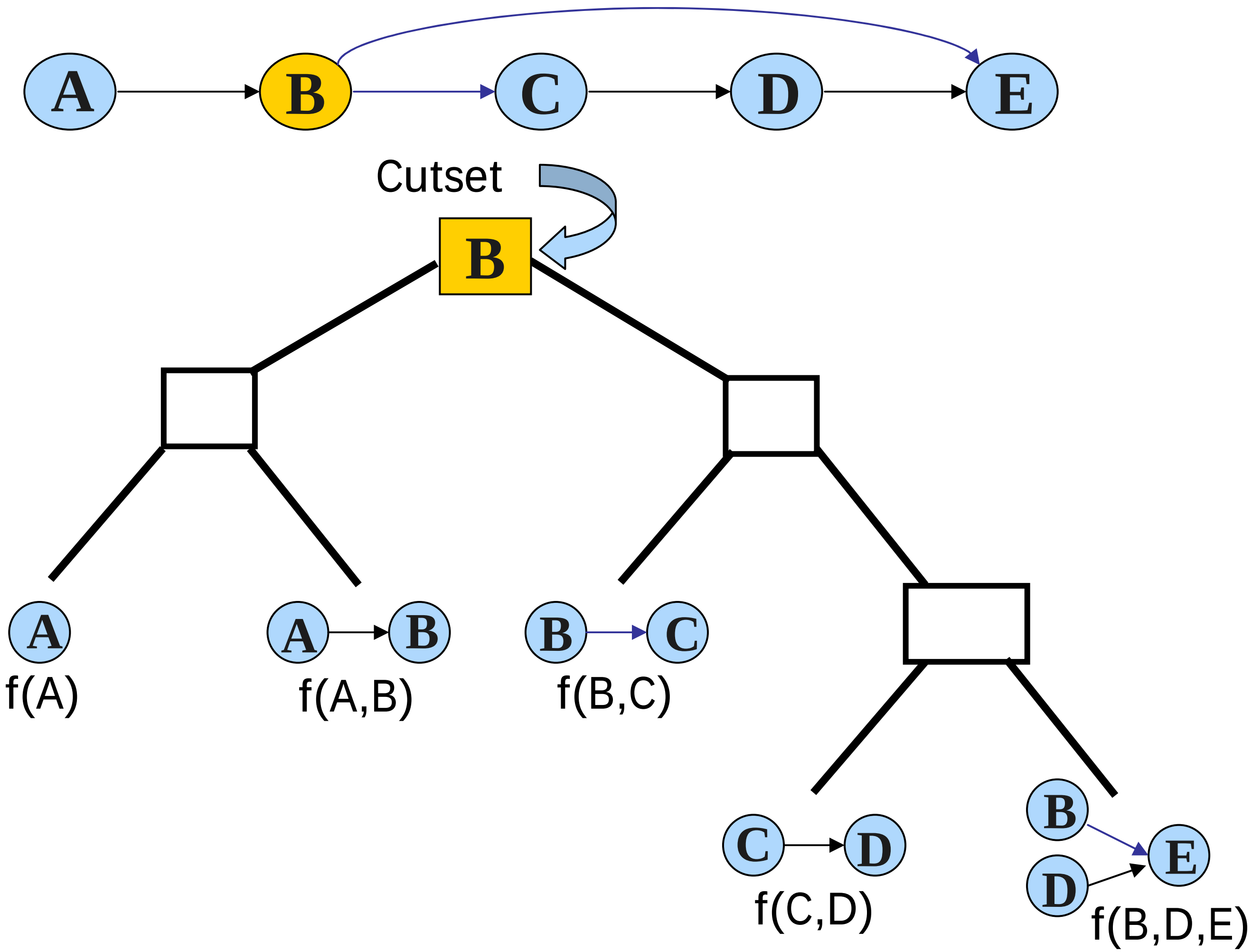
p_l * p_r + p_l * p_r

Decomposition Tree



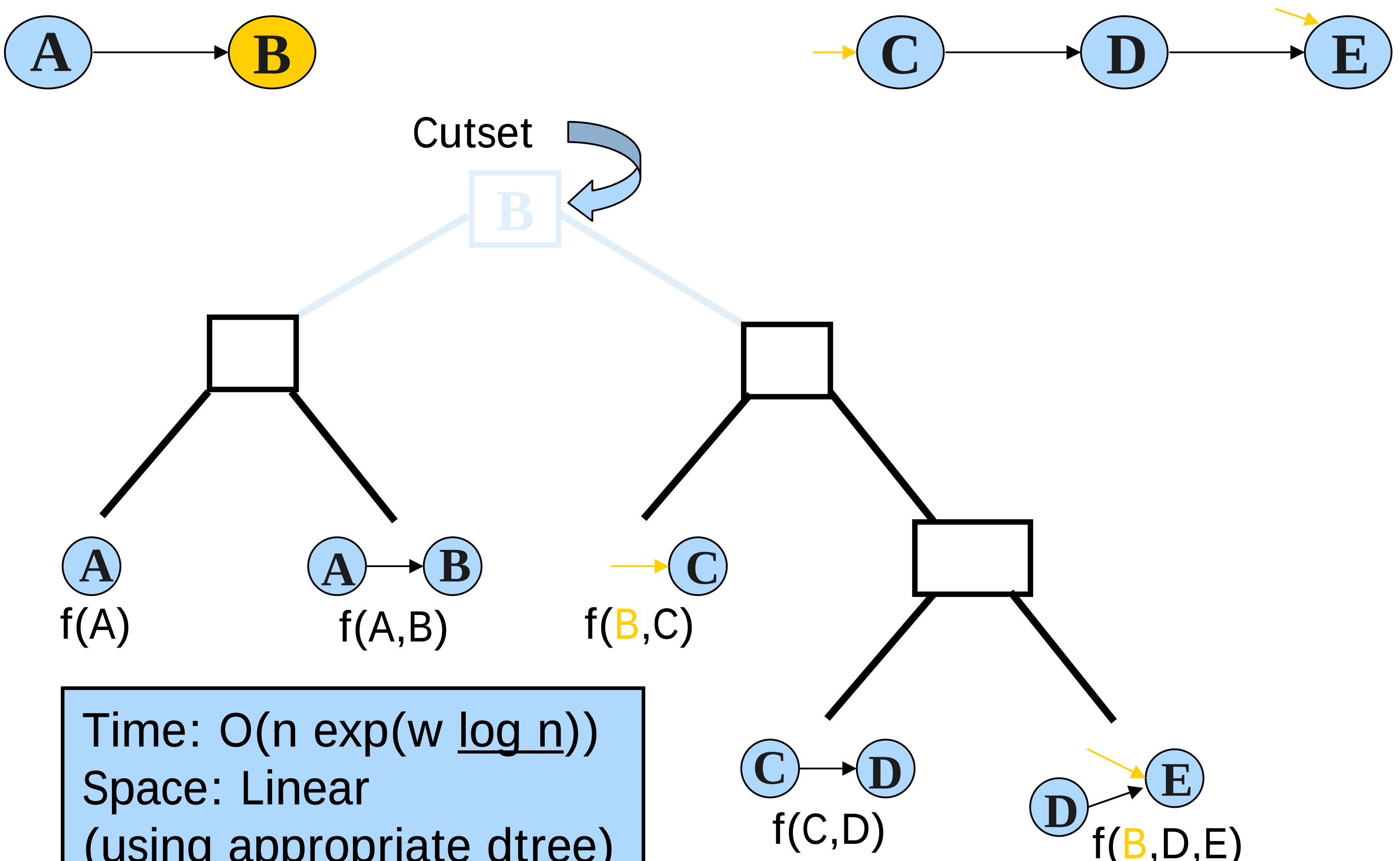
A. Darwiche

Decomposition Tree



A. Darwiche

Decomposition Tree



Time: O(n exp(w log n))
Space: Linear
(using appropriate dtree)

RC1

```
RC1(T,e)
    // compute probability of evidence e on dtree T

    If T is a leaf node
    Return Lookup(T,e)
    Else
        p := 0
        for each instantiation c of cutset(T)-E do
            p := p + RC1(Tl,ec) RC1(Tr,ec)
        return p
```

Lookup(T,**e**)

$\Theta_{X|\mathbf{U}}$: CPT associated with leaf T

If X is instantiated in **e**, then

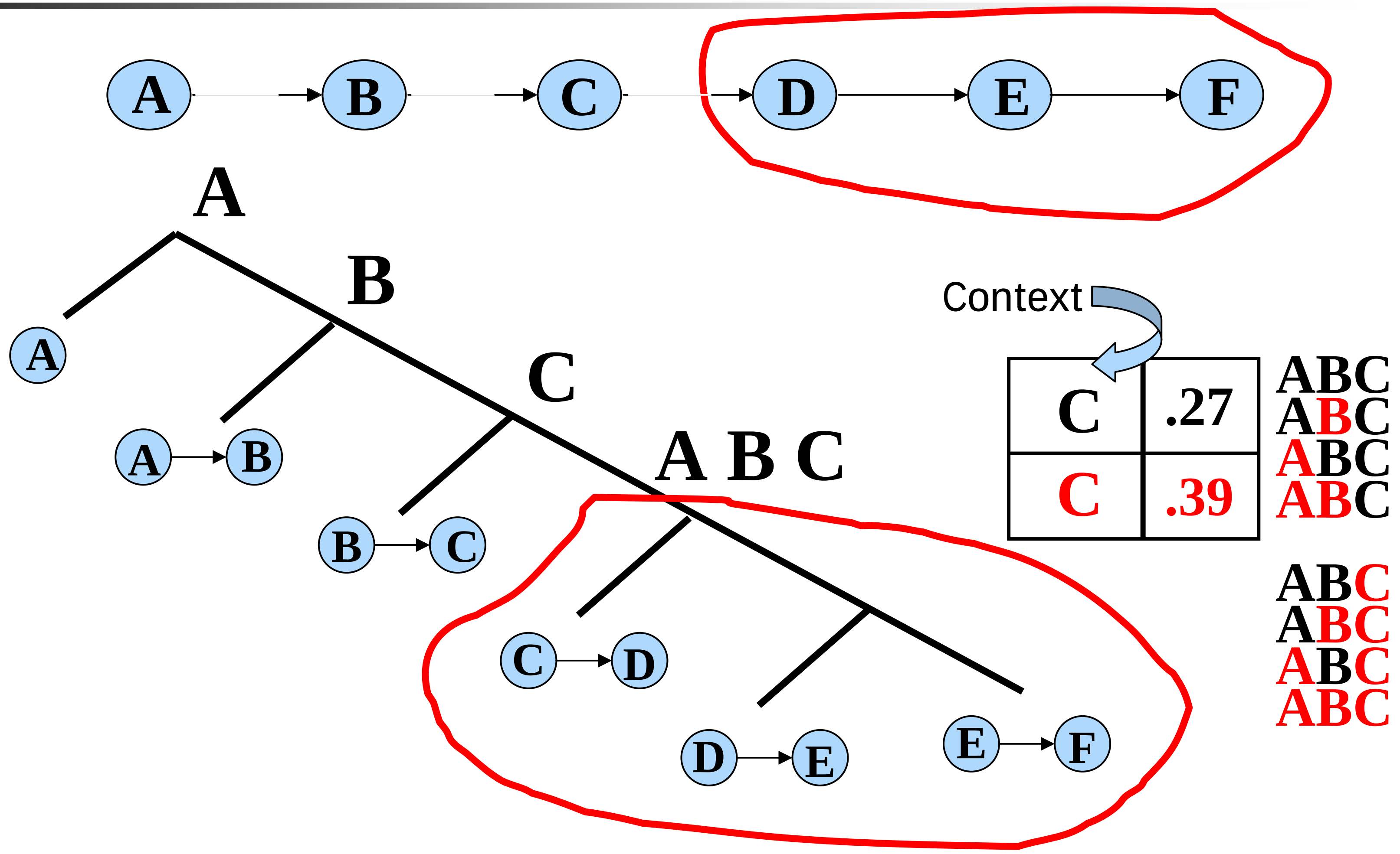
x: value of X in **e**

u: value of **U** in **e**

Return $\theta_{x|\mathbf{u}}$

Else return 1 = $\Sigma_x \ \theta_{x|\mathbf{u}}$

Caching


A B C D E F
A
B
C
A B C
A
A B
B C
C D
D E
E F
Context
C .27
C .39
ABC
ABC
ABC
ABC
ABC
ABC
ABC
ABC

Caching

Recursive Conditioning

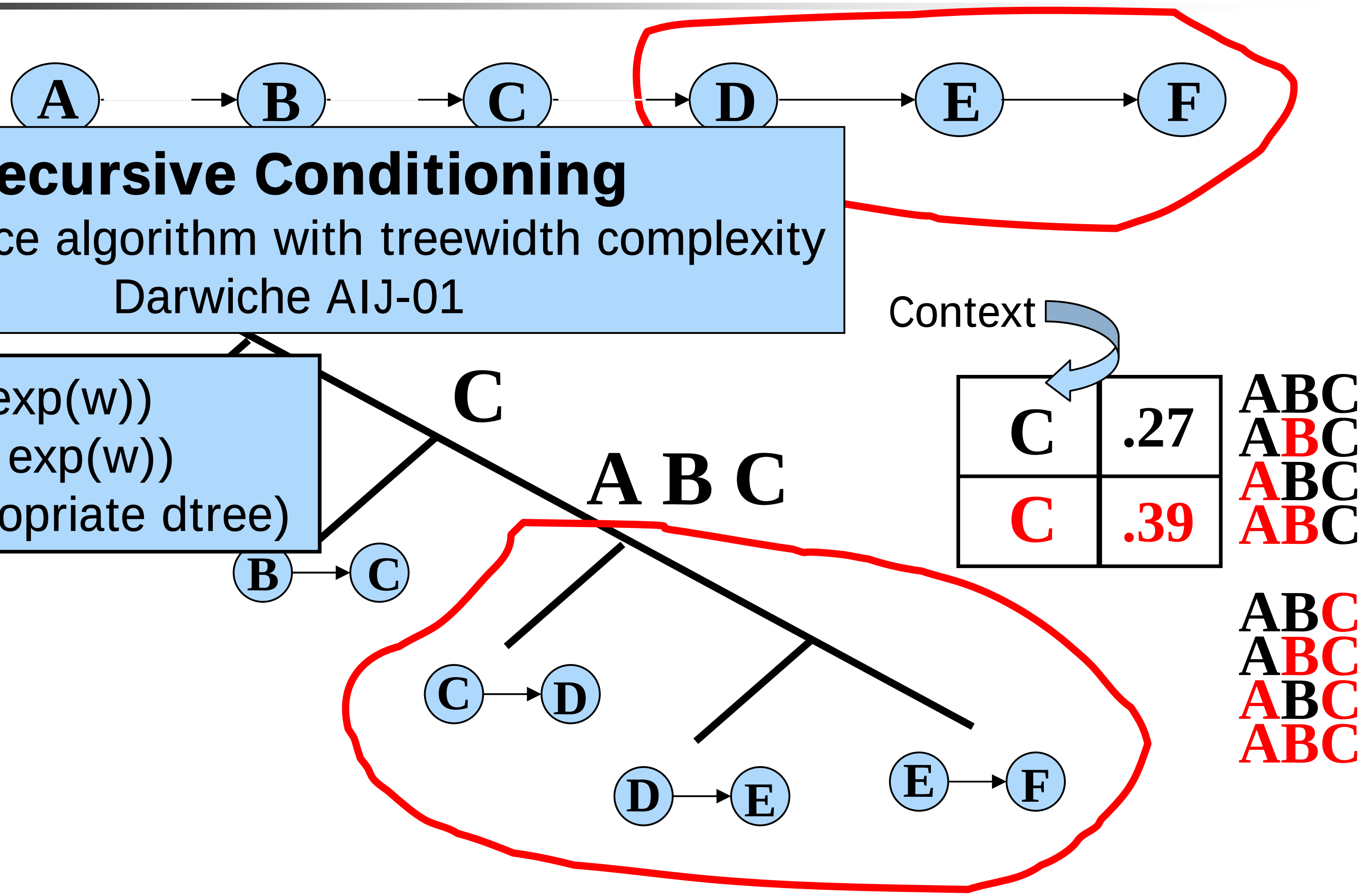
An any-space algorithm with treewidth complexity

Darwiche AIJ-01

Time: O(n exp(w))

Space: O(n exp(w))

(using appropriate dtree)



RC2

RC2(T,**e**)

If T is a leaf node, return Lookup(T,e)

y := instantiation of context(T)

If cache_T[**y**] <> nil, return cache_T[**y**]

p := 0

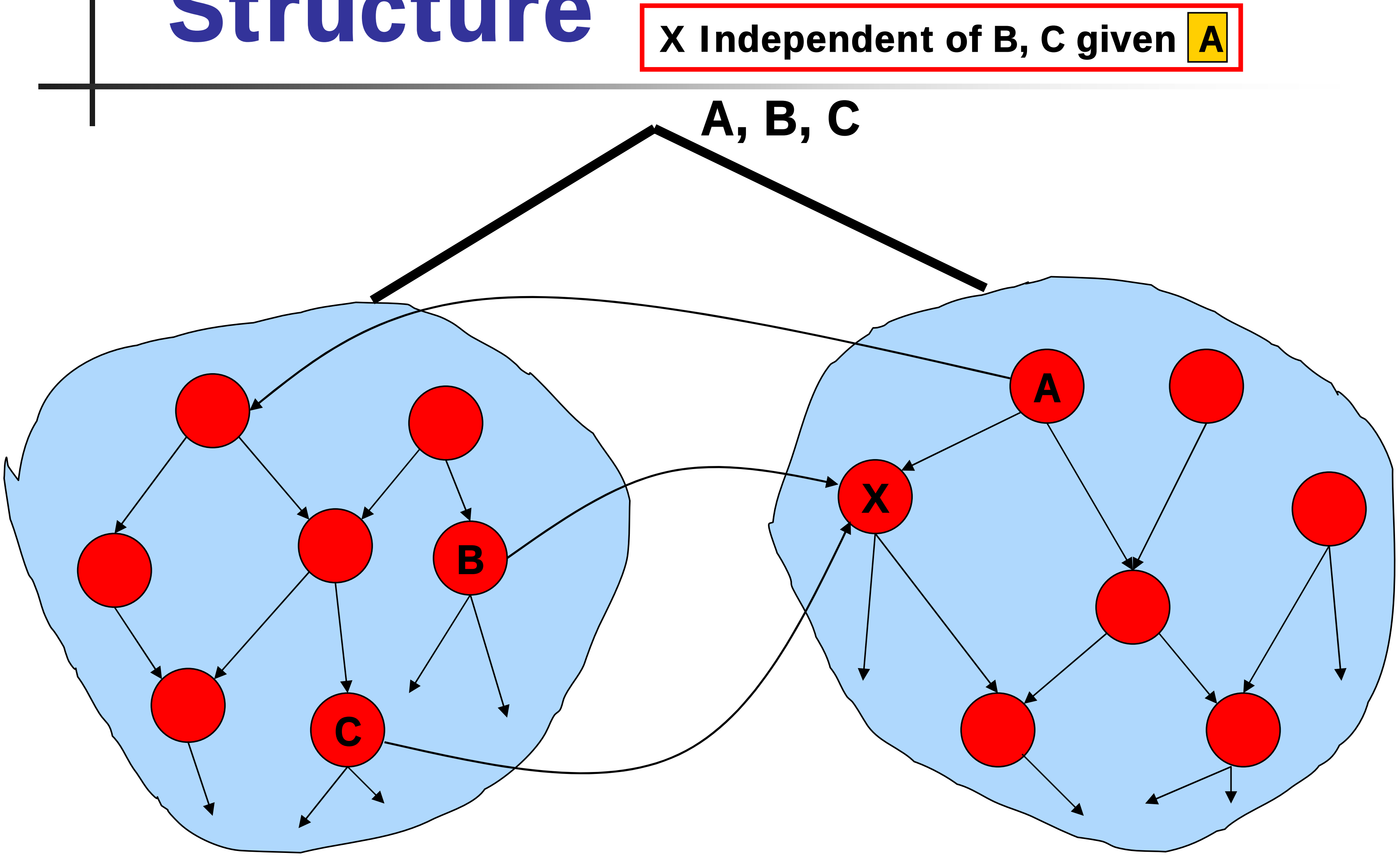
For each instantiation **c** of cutset(T)-**E** do

p := p + RC2(T^l,**ec**) RC2(T^r,**ec**)

cache_T[**y**] := p

Return p

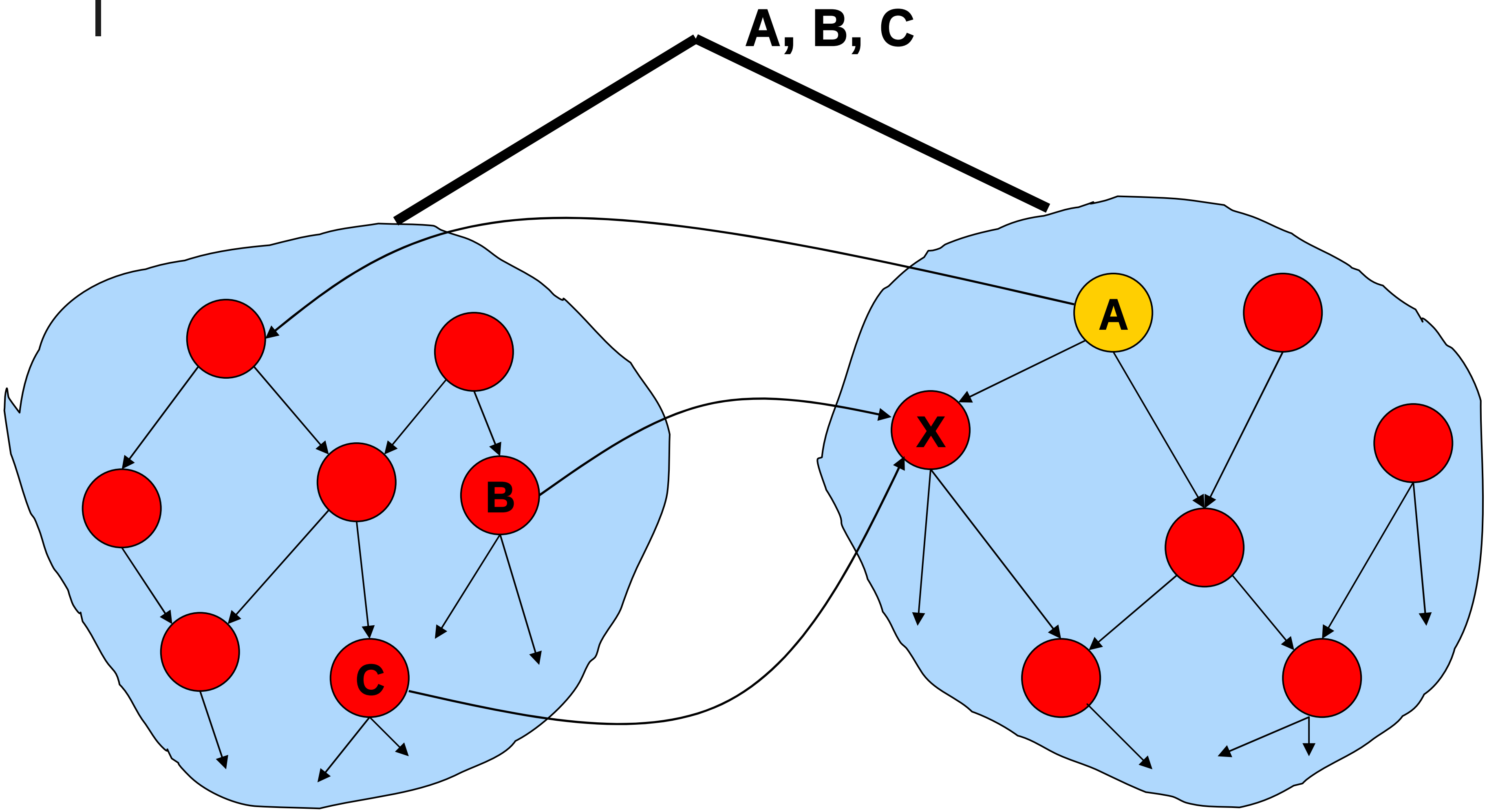
Decomposition with Local Structure



A. Darwiche

Decomposition with Local Structure

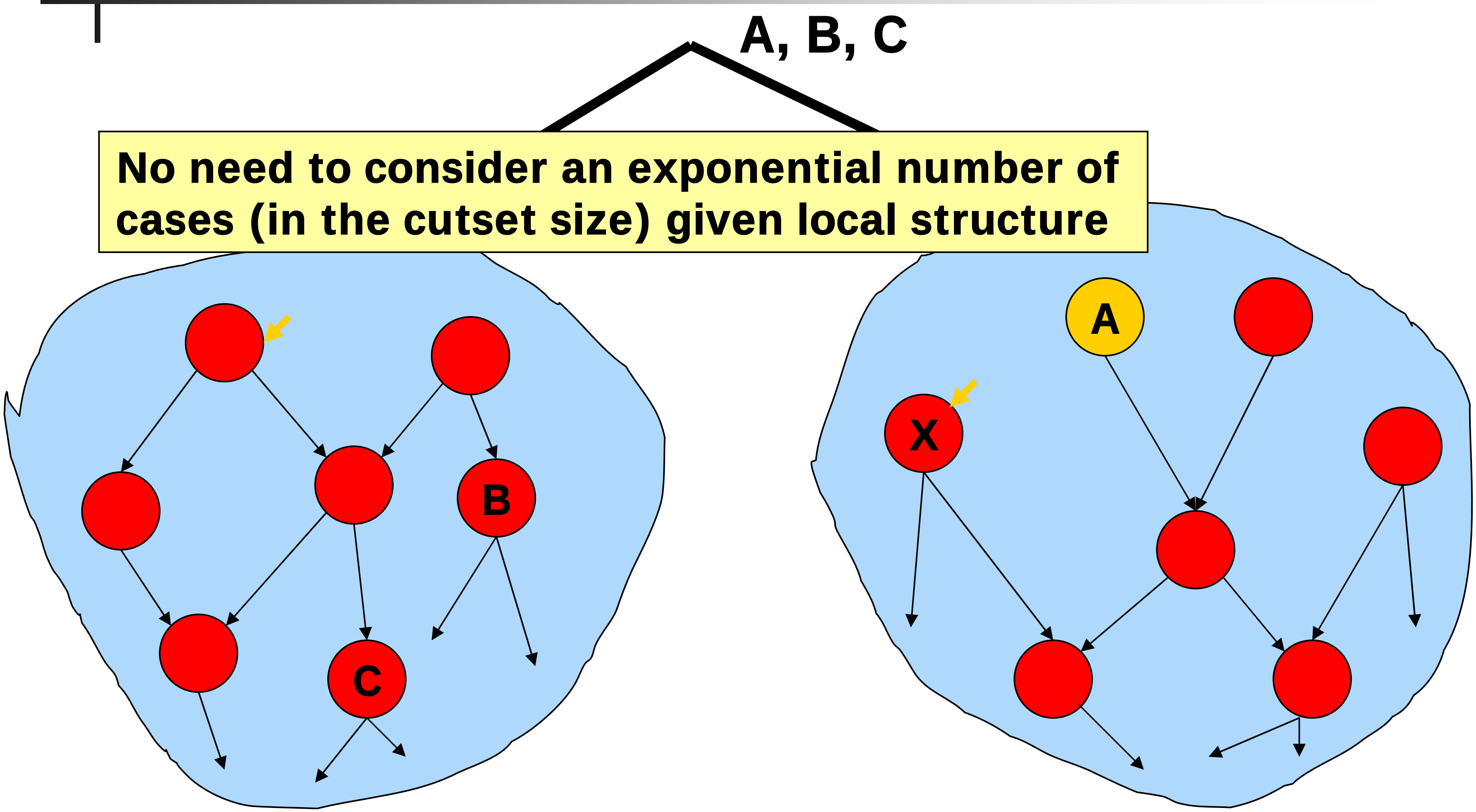
X Independent of B, C given A



A. Darwiche

Decomposition with Local Structure

X Independent of B, C given A



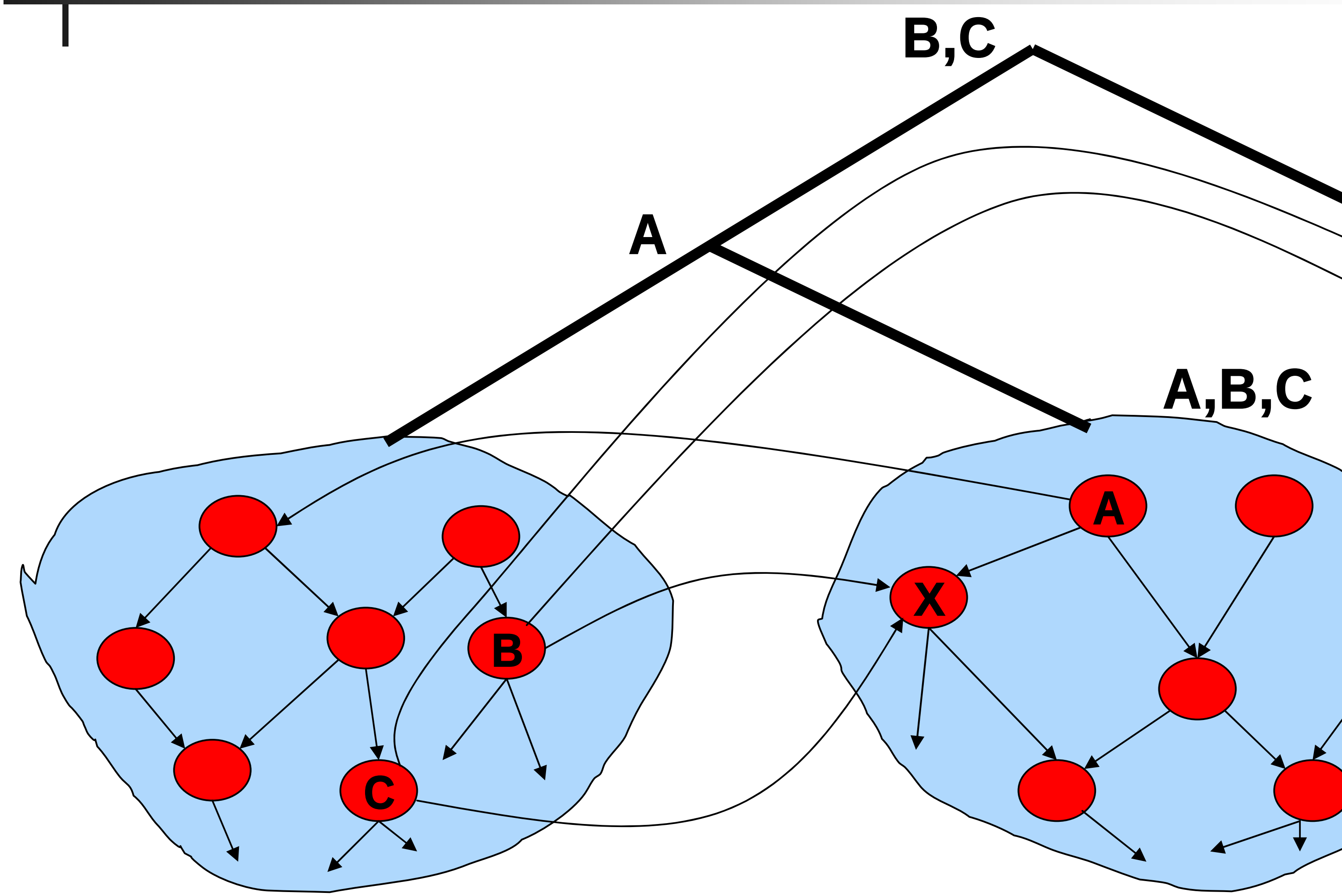
No need to consider an exponential number of cases (in the cutset size) given local structure

A. Darwiche

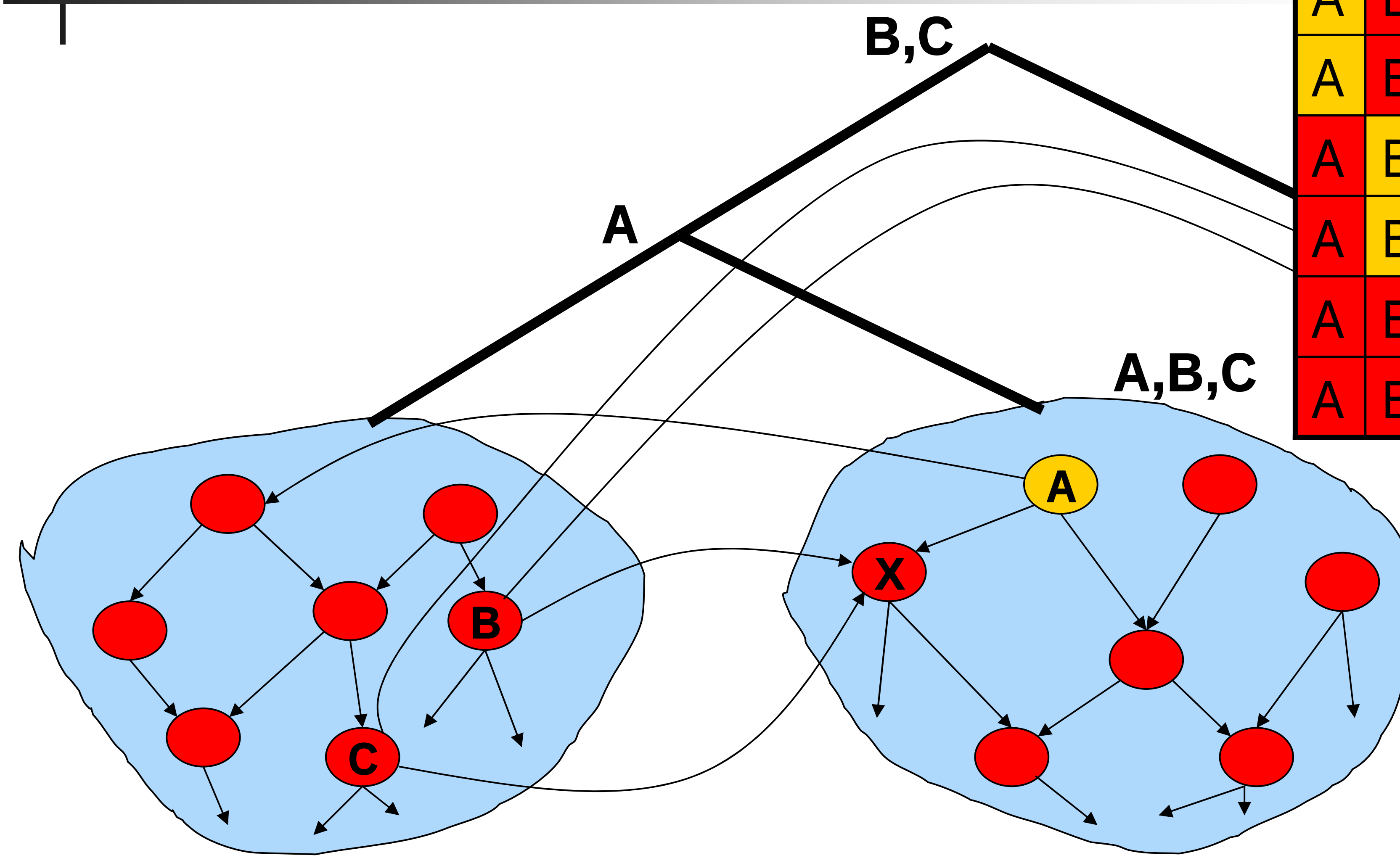
Caching with Local Structure

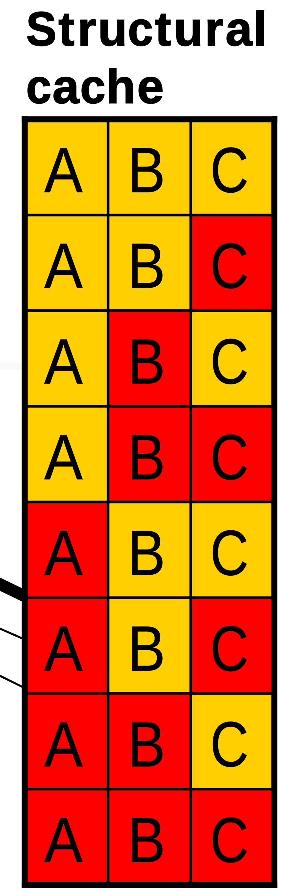
Structural cache

A	B	C
A	B	C
A	B	C
A	B	C
A	B	C
A	B	C
A	B	C
A	B	C



Caching with Local Structure

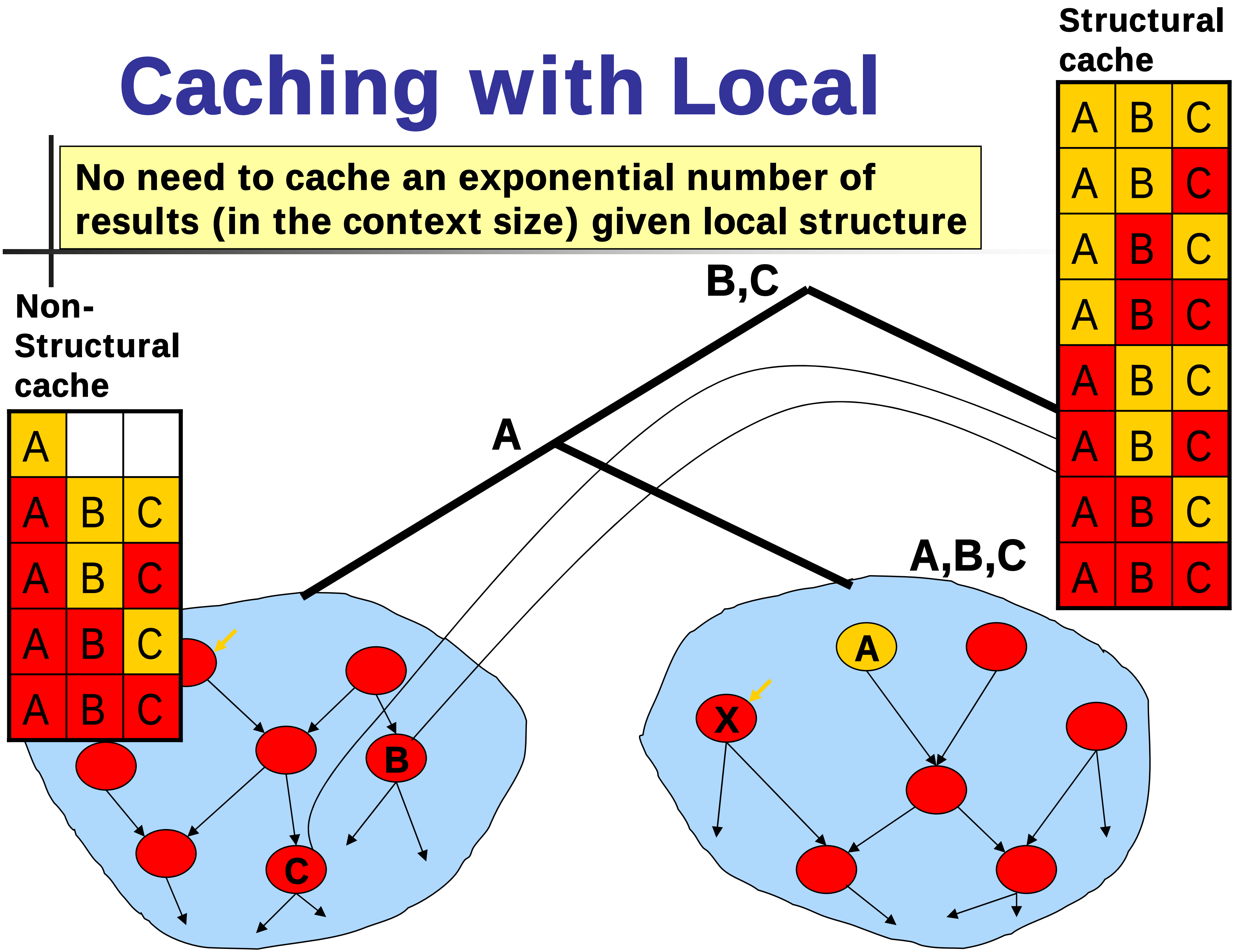




A. Darwiche

Caching with Local

No need to cache an exponential number of results (in the context size) given local structure



A. Darwiche

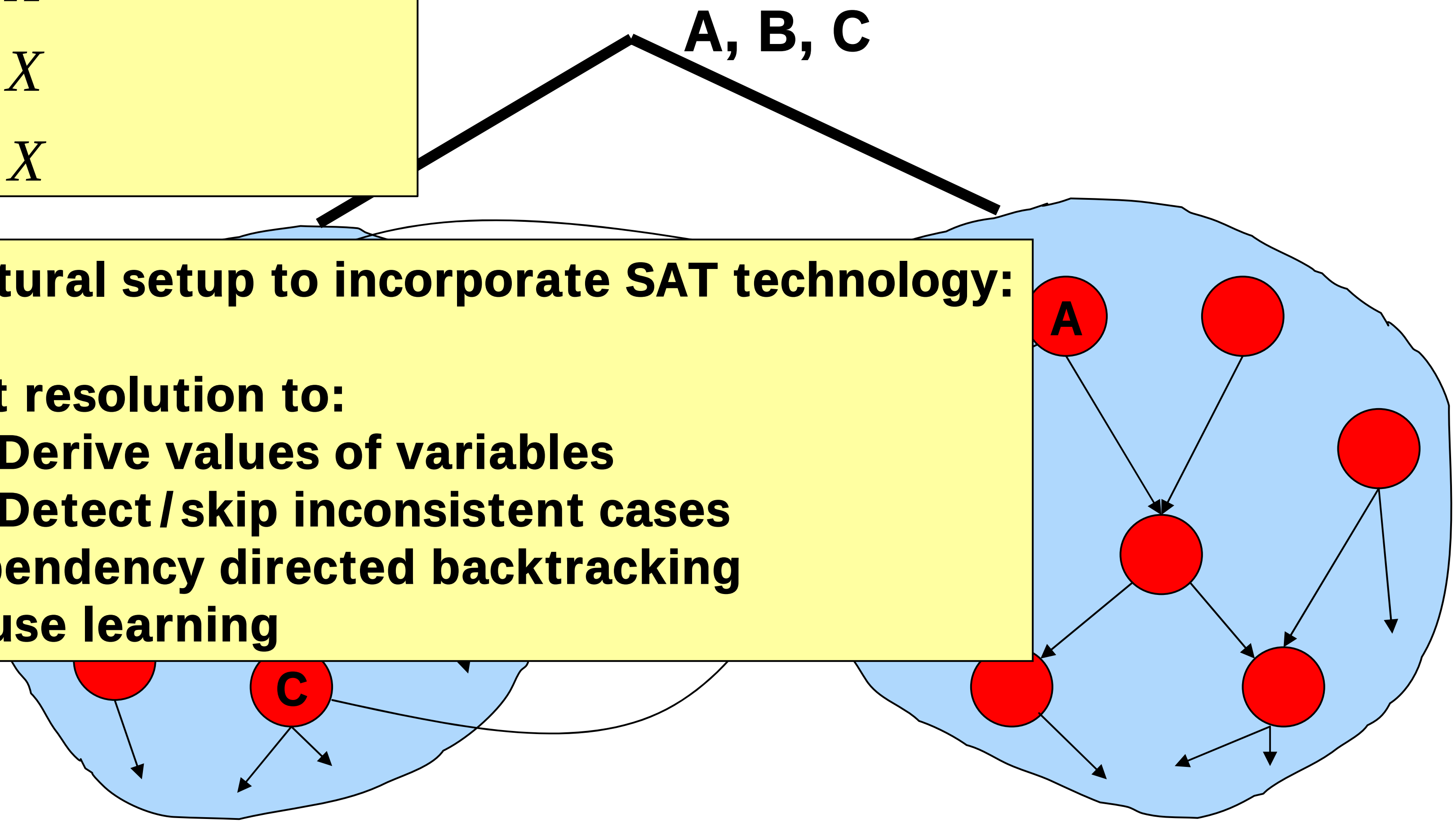
Determinism...

$$\neg A \wedge \neg B \wedge \neg C \Rightarrow \neg X$$

$$A \Rightarrow X$$

$$B \Rightarrow X$$

$$C \Rightarrow X$$



A natural setup to incorporate SAT technology:

- **Unit resolution to:**
  - **Derive values of variables**
  - **Detect/skip inconsistent cases**
- **Dependency directed backtracking**
- **Clause learning**

CSI Summary

- Exploit local structure
  - Context-specific independence
  - Determinism
- Significantly speed-up inference
  - Tackle problems with tree-width in the thousands

- Acknowledgements
  - Recursive conditioning slides courtesy of Adnan Darwiche
  - Implementation available:
    - http://reasoning.cs.ucla.edu/ace

Where are we?

- Bayesian networks
  - Represent exponentially-large probability distributions compactly
- Inference in BNs
  - Exact inference very fast for problems with low tree-width
  - Exploit local structure for fast inference
- **Now: Learning BNs**
  - Given structure, estimate parameters

Thumbtack – Binomial Distribution

- P(Heads) = θ, P(Tails) = 1-θ

- Flips are i.i.d.:
  - Independent events
  - Identically distributed according to Binomial distribution
- Sequence *D* of α_H Heads and α_T Tails

$$P(\mathcal{D} \mid \theta) = \theta^{\alpha_H}(1-\theta)^{\alpha_T}$$

Maximum Likelihood Estimation

- **Data:** Observed set *D* of α_H Heads and α_T Tails
- **Hypothesis:** Binomial distribution
- Learning θ is an optimization problem
  - What's the objective function?

- MLE: Choose θ that maximizes the probability of observed data:

$$\begin{aligned} \widehat{\theta} &= \arg\max_{\theta} \quad P(\mathcal{D} \mid \theta) \\ &= \arg\max_{\theta} \quad \ln P(\mathcal{D} \mid \theta) \end{aligned}$$

Your first learning algorithm

$$\begin{aligned}\widehat{\theta} &= \arg\max_{\theta} \quad \ln P(\mathcal{D} \mid \theta) \\ &= \arg\max_{\theta} \quad \ln \theta^{\alpha_H}(1-\theta)^{\alpha_T}\end{aligned}$$

- Set derivative to zero: $\boxed{\frac{d}{d\theta} \ln P(\mathcal{D} \mid \theta) = 0}$

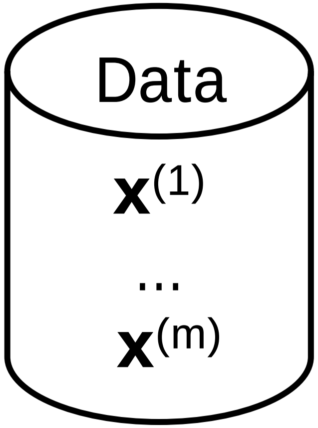
MLE for conditional probabilities

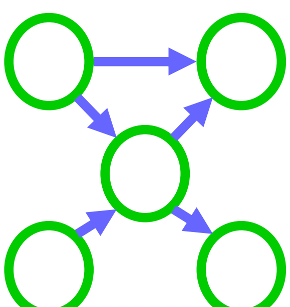
- MLE estimate of $P(X=x) =$

- MLE estimate of $P(X=x|Y=y)$
  - Only consider subset of data where $Y=y$

Learning the CPTs

MLE: $\hat{P}(X_i = x_i \mid X_{i-1} = x_{i-1}) = \dfrac{\text{Count}(X_i = x_i, X_{i-1} = x_{i-1})}{\text{Count}(X_{i-1} = x_{i-1})}$





MLE learning CPTs for general BN

- Vars $X_1,\ldots,X_n$ and BN structure given $P(X_1,\ldots,X_n) = \prod_{i=1}^{n} P\left(X_i \mid \mathbf{Pa}_{X_i}\right)$

- Each i.i.d. data point assigns a value all vars
- Likelihood of the data:

- MLE for CPT $P(X_i \mid \mathbf{Pa}_{X_i})$: $\widehat{P}(X_i = x_i \mid \mathbf{Pa}_{\mathbf{X_i}} = \mathbf{u}) = \frac{\text{Count}(X_i = x_i, \mathbf{Pa}_{\mathbf{X_i}} = \mathbf{u})}{\text{Count}(\mathbf{Pa}_{\mathbf{X_i}} = \mathbf{u})}$