Markov Decision Processes and Reinforcement Learning

Readings:

- Mitchell, chapter 13
- Kaelbling et al., [see class website]

Machine Learning 10-701
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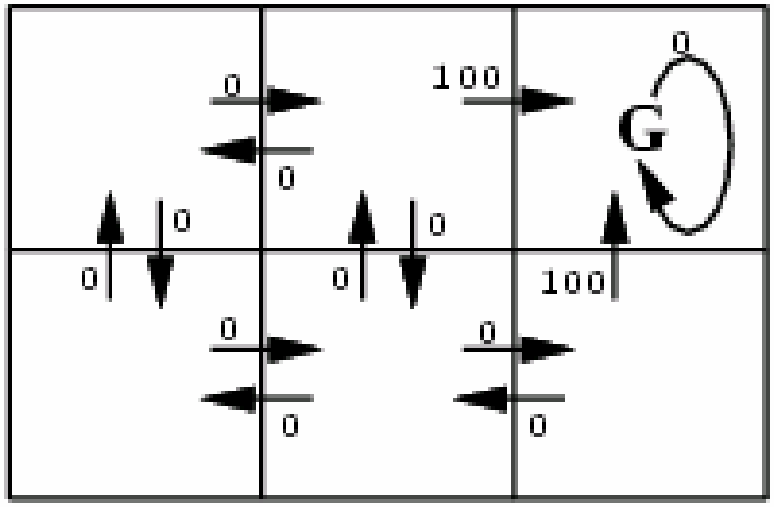
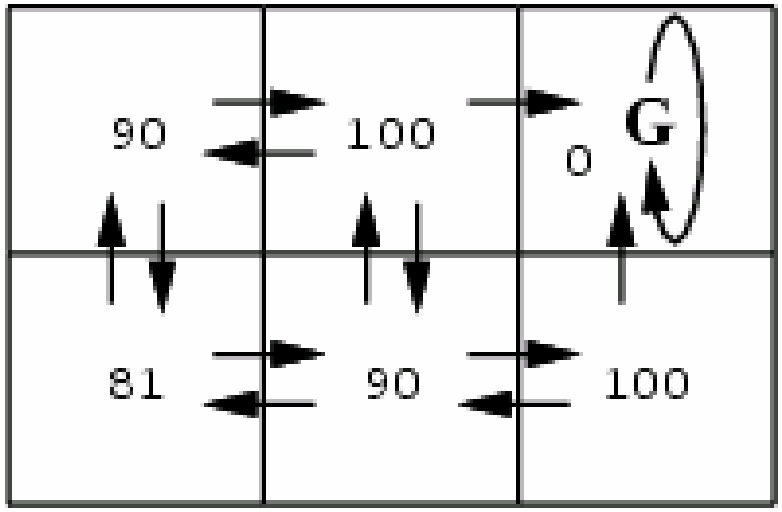
Reinforcement Learning

[Sutton and Barto 1981; Samuel 1957; ...]



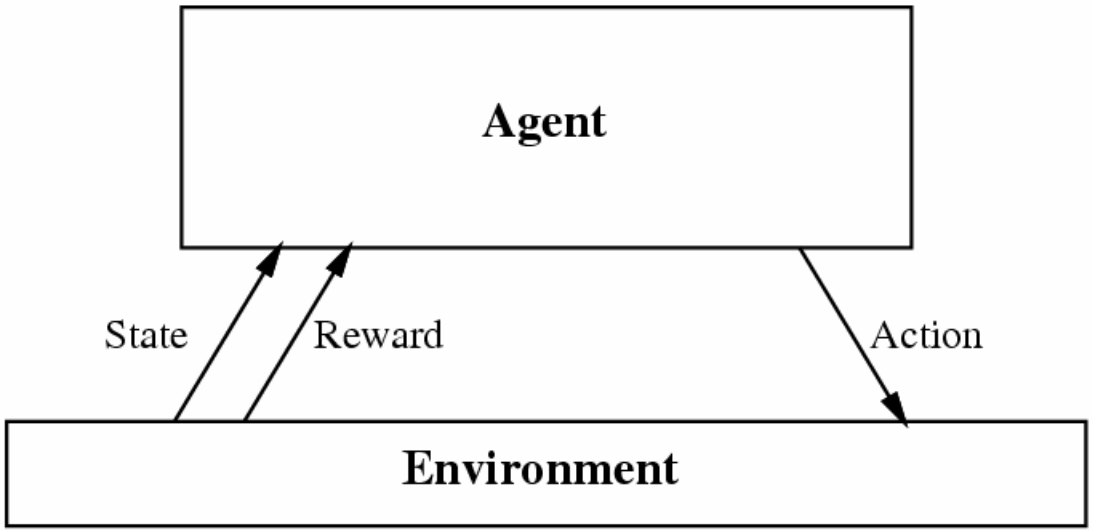


$$V^*(s) = E[r_t + \gamma\, r_{t+1} + \gamma^2 r_{t+2} + \ldots]$$

Outline

- Learning control strategies
  - Credit assignment and delayed reward
  - Discounted rewards
- Markov Decision Processes
  - Solving a known MDP
- Online learning of control strategies
  - When next-state function is known: value function $V^*(s)$
  - When next-state function unknown: learning $Q^*(s,a)$
- Role in modeling reward learning in animals

Reinforcement Learning Problem



$$s_0 \xrightarrow[r_0]{a_0} s_1 \xrightarrow[r_1]{a_1} s_2 \xrightarrow[r_2]{a_2} \ldots$$

Goal: Learn to choose actions that maximize

$$r_0 + \gamma r_1 + \gamma^2 r_2 + \ldots \text{ , where } 0 \leq \gamma < 1$$

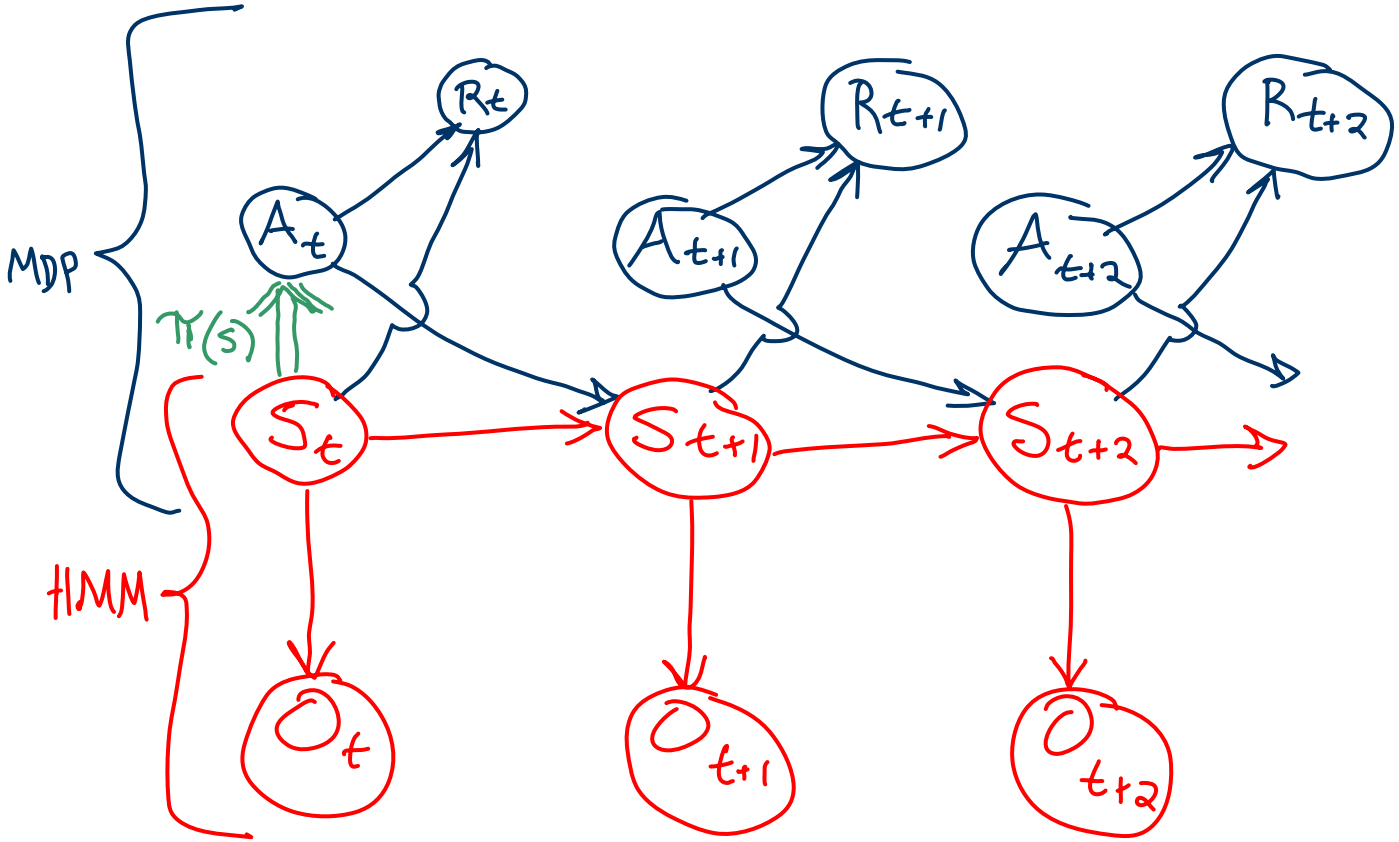
Markov Decision Process

- Set of states S
- Set of actions A
- At each time, agent observes state $s_t \in S$, then chooses action $a_t \in A$
- Then receives reward r_t , and state changes to s_{t+1}
- Markov assumption: $P(s_{t+1} \mid s_t, a_t, s_{t-1}, a_{t-1}, ...) = P(s_{t+1} \mid s_t, a_t)$
- Also assume $P(r_t \mid s_t, a_t, s_{t-1}, a_{t-1}, ...) = P(r_t \mid s_t, a_t)$

- The task: learn a policy π: S → A for choosing actions that maximizes $E[r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + ...]$

$$0 < \gamma \leq 1$$

HMM, Markov Process, Markov Decision Process



Reinforcement Learning Task for Autonomous Agent

Execute actions in environment, observe results, and

- Learn control policy π: S$\rightarrow$ A that maximizes $\sum_{t=0}^{\infty} \gamma^t E[r_t]$ from every state s $\in$ S
- Where 0 < $\lambda \leq$ 1 is the discount factor for future rewards

Note:

- Function to be learned is π: S$\rightarrow$ A
- But training examples of the form < <s,a>, r >
- $\rightarrow$ available training experience is not input-output pairs of the function to be learned !

Value Function for each Policy

- Given a policy π : S → A, define

$$V^{\pi}(s) = E[\sum_{t=0}^{\infty} \gamma^t r_t]$$

assuming actions are chosen according to π

- Then we want the policy π^* where

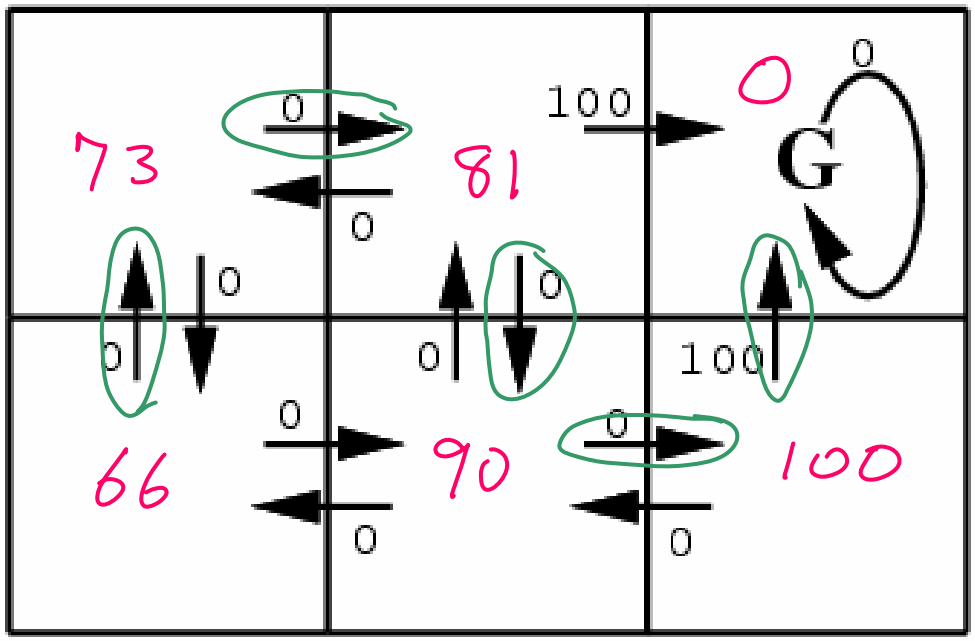
$$\pi^* = \arg\max_{\pi} V^{\pi}(s), \quad (\forall s)$$

- For any MDP, such a policy exists
- We'll write $V^*(s) = V^{\pi^*}(s)$
- Note if we have $V^*(s)$ and $P(s_{t+1}|s_t,a)$, we can compute $\pi^*(s)$

Value Function – what are the $V^{\pi}(s)$ values?

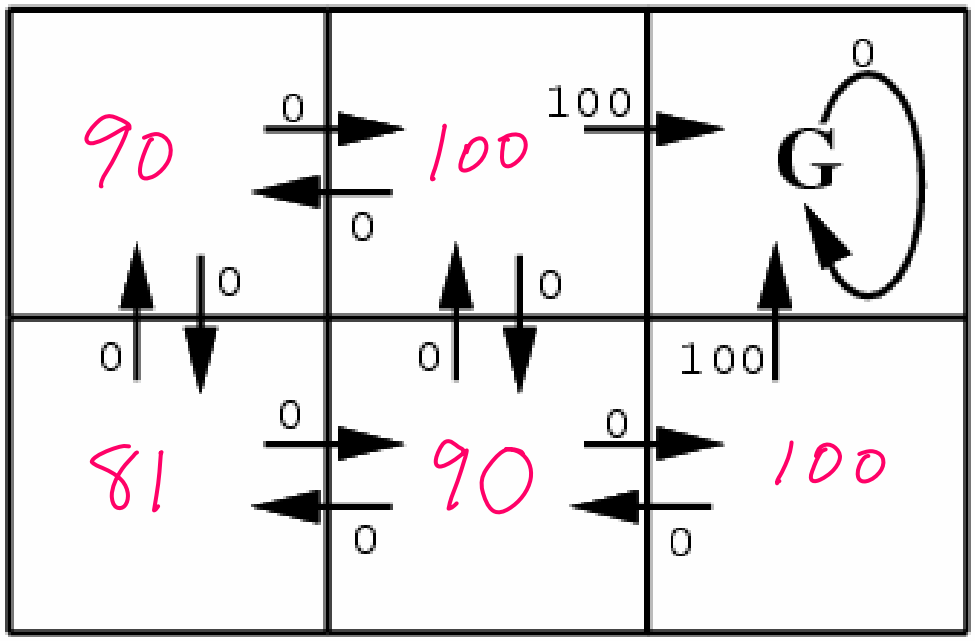
Suppose π is shown by circled action from each state

Suppose $\gamma = 0.9$



$r(s, a)$ (immediate reward)

Value Function – what are the $V^*(s)$ values?

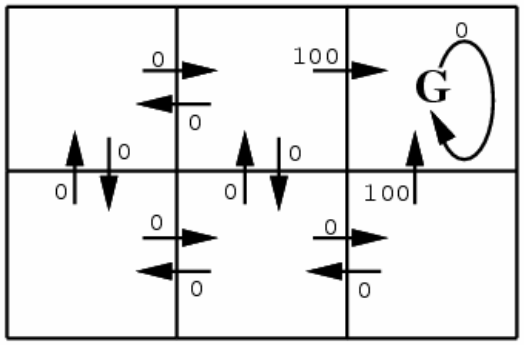


$r(s, a)$ (immediate reward)

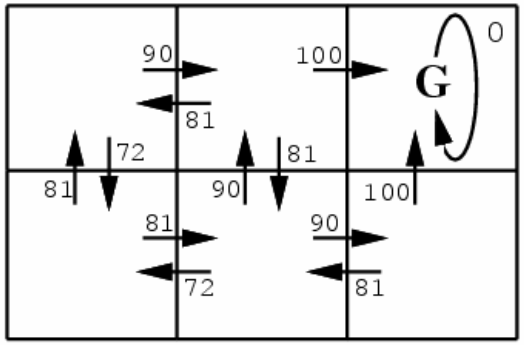
Immediate rewards r(s,a)

State values V*(s)

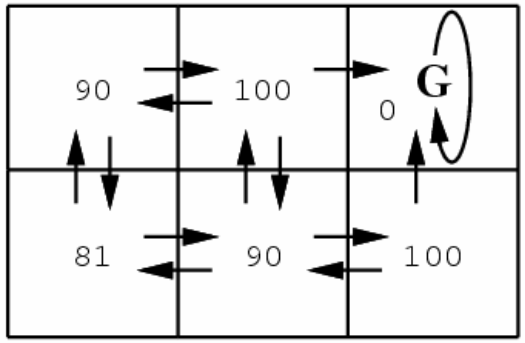
State-action values Q*(s,a)



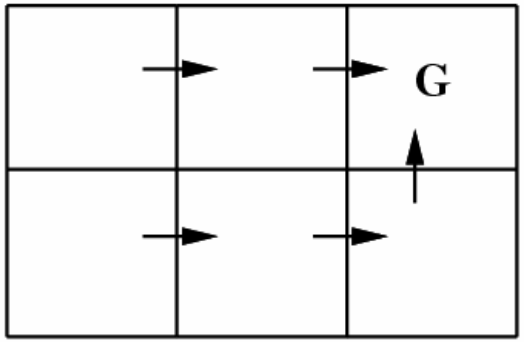
$r(s,a)$ (immediate reward) values



$Q(s,a)$ values



$V^*(s)$ values



One optimal policy

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Recursive definition for V*(S)

$$V^*(s) = E[\sum_{t=0}^{\infty} \gamma^t r_t]$$

assuming actions are chosen according to the optimal policy, π^*

$$V^*(s_1) = E[r(s_1, a_1)] + E[\gamma r(s_2, a_2)] + E[\gamma^2 r(s_3, a_3)] + \ldots]$$

$$V^*(s_1) = E[r(s_1, a_1)] + \gamma E_{s_2|s_1,a_1}[V^*(s_2)]$$

$$V^*(s) = E[r(s, \pi^*(s))] + \gamma E_{s'|s,\pi^*(s)}[V^*(s')]$$

Value Iteration for learning V* : assumes $P(S_{t+1}|S_t, A)$ known

Initialize V(s) arbitrarily

Loop until policy good enough

  Loop for s in S

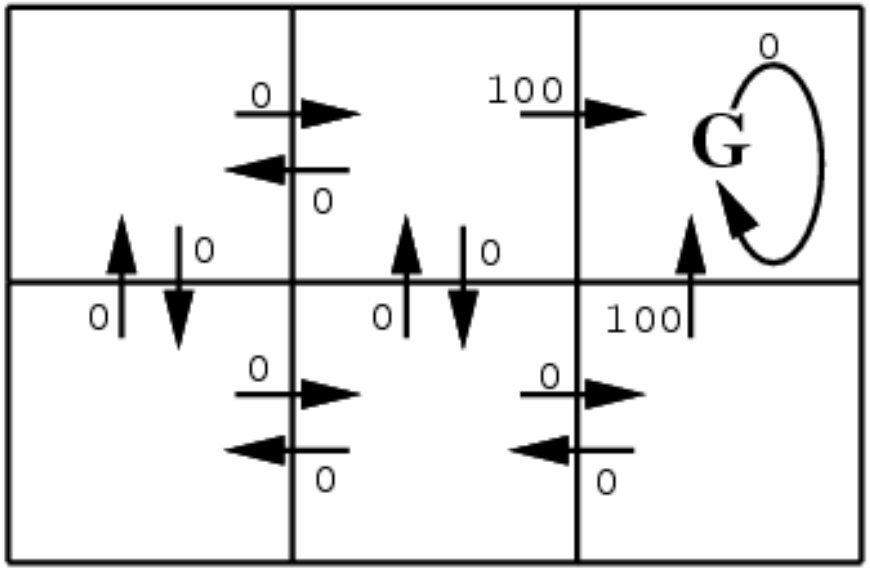
    Loop for a in A

- $Q(s,a) \leftarrow r(s,a) + \gamma \sum_{s' \in S} P(s'|s,a)V(s')$

    $V(s) \leftarrow \max_a Q(s,a)$

  End loop

End loop



V(s) converges to V*(s).

Same alg works if we randomly traverse the environment, as long as visit every transition repeatedly

Value Iteration

Interestingly, value iteration works even if we randomly traverse the environment instead of looping through each state and action methodically

- but we must still visit each state infinitely often on an infinite run
- For details: [Bertsekas 1989]
- Implications: online learning as agent randomly roams

If max (over states) difference between two successive value function estimates is less that ε, then the value of the greedy policy differs from the optimal policy by no more than $2\epsilon\gamma/(1-\gamma)$

So far: learning optimal policy when we know $P(s_t \mid s_{t-1}, a_{t-1})$

What if we don't?

Q learning

Define new function, closely related to V*

$$V^*(s) = E[r(s, \pi^*(s))] + \gamma E_{s'|s,\pi^*(s)}[V^*(s')]$$

$$Q(s,a) = E[r(s,a)] + \gamma E_{s'|s,a}[V^*(s')]$$

If agent knows Q(s,a), it can choose optimal action without knowing $P(s_{t+1}|s_t,a)$!

$$\pi^*(s) = \arg\max_a Q(s,a) \qquad V^*(s) = \max_a Q(s,a)$$

And, it can learn Q without knowing $P(s_{t+1}|s_t,a)$

Q Function

Consider first the deterministic case. P(s'| s,a) deterministic, denoted δ(s,a)

Define new function very similar to V^*

$$Q(s,a) \equiv r(s,a) + \gamma V^*(\delta(s,a))$$

If agent learns Q, it can choose optimal action even without knowing δ!

$$\pi^*(s) = \operatorname{argmax}_a [r(s,a) + \gamma V^*(\delta(s,a))]$$

$$\pi^*(s) = \operatorname{argmax}_a Q(s,a)$$

Q is the evaluation function the agent will learn

Training Rule to Learn Q

Note Q and V^* closely related:

$$V^*(s) = \max_{a'} Q(s, a')$$

Which allows us to write Q recursively as

$$\begin{aligned} Q(s_t, a_t) &= r(s_t, a_t) + \gamma V^*(\delta(s_t, a_t))) \\ &= r(s_t, a_t) + \gamma \max_{a'} Q(s_{t+1}, a') \end{aligned}$$

Nice! Let $\hat{Q}$ denote learner's current approximation to Q. Consider training rule

$$\hat{Q}(s, a) \leftarrow r + \gamma \max_{a'} \hat{Q}(s', a')$$

where s' is the state resulting from applying action a in state s

Q Learning for Deterministic Worlds

For each s, a initialize table entry $\hat{Q}(s, a) \leftarrow 0$

Observe current state s

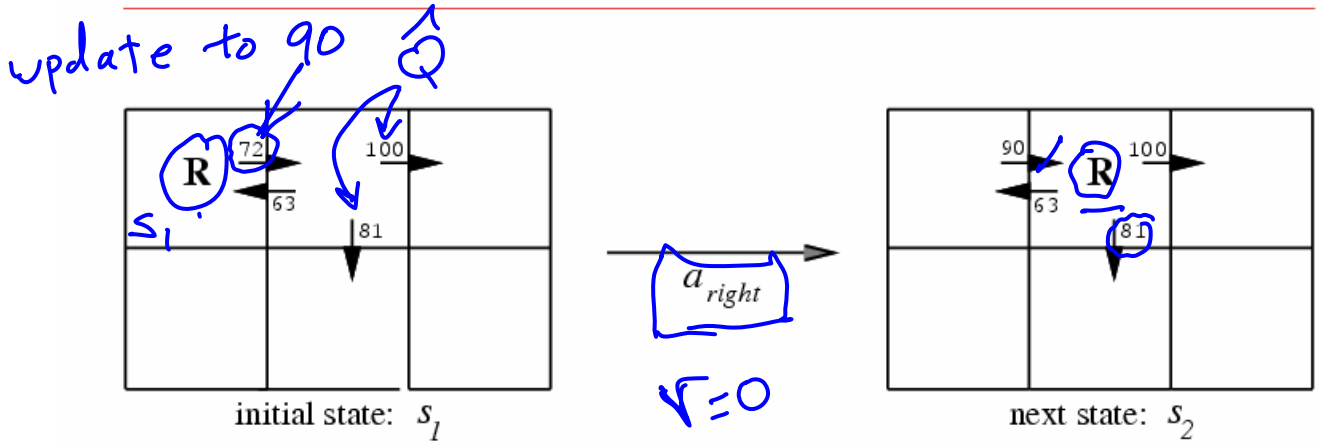
Do forever:

- Select an action a and execute it
- Receive immediate reward r
- Observe the new state s'
- Update the table entry for $\hat{Q}(s, a)$ as follows:

$$\hat{Q}(s, a) \leftarrow r + \gamma \max_{a'} \hat{Q}(s', a')$$

- $s \leftarrow s'$

Updating $\hat{Q}$



$$\hat{Q}(s_1, a_{right}) \leftarrow r + \gamma \max_{a'} \hat{Q}(s_2, a')$$
$$\leftarrow 0 + 0.9 \max\{63, 81, 100\}$$
$$\leftarrow 90$$

notice if rewards non-negative, then

$$(\forall s, a, n) \quad \hat{Q}_{n+1}(s, a) \geq \hat{Q}_n(s, a)$$

and

$$(\forall s, a, n) \quad 0 \leq \hat{Q}_n(s, a) \leq Q(s, a)$$

$\hat{Q}$ converges to Q. Consider case of deterministic world where see each $\langle s, a \rangle$ visited infinitely often.

Proof: Define a full interval to be an interval during which each $\langle s, a \rangle$ is visited. During each full interval the largest error in $\hat{Q}$ table is reduced by factor of γ

Let $\hat{Q}_n$ be table after n updates, and Δ_n be the maximum error in $\hat{Q}_n$; that is

$$\Delta_n = \max_{s,a} |\hat{Q}_n(s, a) - Q(s, a)|$$

For any table entry $\hat{Q}_n(s, a)$ updated on iteration $n + 1$, the error in the revised estimate $\hat{Q}_{n+1}(s, a)$ is

$$\begin{aligned}
|\hat{Q}_{n+1}(s, a) - Q(s, a)| &= |(r + \gamma \max_{a'} \hat{Q}_n(s', a')) \\
&\quad -(r + \gamma \max_{a'} Q(s', a'))| \\
&= \gamma |\max_{a'} \hat{Q}_n(s', a') - \max_{a'} Q(s', a')| \\
&\leq \gamma \max_{a'} |\hat{Q}_n(s', a') - Q(s', a')| \\
&\leq \gamma \max_{s'', a'} |\hat{Q}_n(s'', a') - Q(s'', a')| \\
|\hat{Q}_{n+1}(s, a) - Q(s, a)| &\leq \gamma \Delta_n
\end{aligned}$$

$|\max_a f_1(a) - \max_a f_2(a)| \leq \max_a |f_1(a) - f_2(a)|$

Nondeterministic Case

Q learning generalizes to nondeterministic worlds

Alter training rule to

$$\hat{Q}_n(s,a) \leftarrow (1-\alpha_n)\hat{Q}_{n-1}(s,a)+\alpha_n[r+\max_{a'}\hat{Q}_{n-1}(s',a')]$$

where

$$\alpha_n = \frac{1}{1+visits_n(s,a)}$$

Can still prove convergence of $\hat{Q}$ to Q [Watkins and Dayan, 1992]

Temporal Difference Learning

Q learning: reduce discrepancy between successive Q estimates

One step time difference:

$$Q^{(1)}(s_t, a_t) \equiv r_t + \gamma \max_a \hat{Q}(s_{t+1}, a)$$

Why not two steps?

$$Q^{(2)}(s_t, a_t) \equiv r_t + \gamma r_{t+1} + \gamma^2 \max_a \hat{Q}(s_{t+2}, a)$$

Or n?

$$Q^{(n)}(s_t, a_t) \equiv r_t + \gamma r_{t+1} + \cdots + \gamma^{(n-1)} r_{t+n-1} + \gamma^n \max_a \hat{Q}(s_{t+n}, a)$$

Blend all of these:

$$Q^{\lambda}(s_t, a_t) \equiv (1-\lambda) \left[Q^{(1)}(s_t, a_t) + \lambda Q^{(2)}(s_t, a_t) + \lambda^2 Q^{(3)}(s_t, a_t) \right.$$

Temporal Difference Learning

$$Q^{\lambda}(s_t, a_t) \equiv (1-\lambda)\left[Q^{(1)}(s_t, a_t) + \lambda Q^{(2)}(s_t, a_t) + \lambda^2 Q^{(3)}(s_t, a_t)\right.$$

Equivalent expression:

$$Q^{\lambda}(s_t, a_t) = r_t + \gamma[\ (1-\lambda)\max_a \hat{Q}(s_t, a_t) + \lambda\ Q^{\lambda}(s_{t+1}, a_{t+1})]$$

TD(λ) algorithm uses above training rule

- Sometimes converges faster than Q learning
- converges for learning V^* for any $0 \leq \lambda \leq 1$ (Dayan, 1992)
- Tesauro's TD-Gammon uses this algorithm

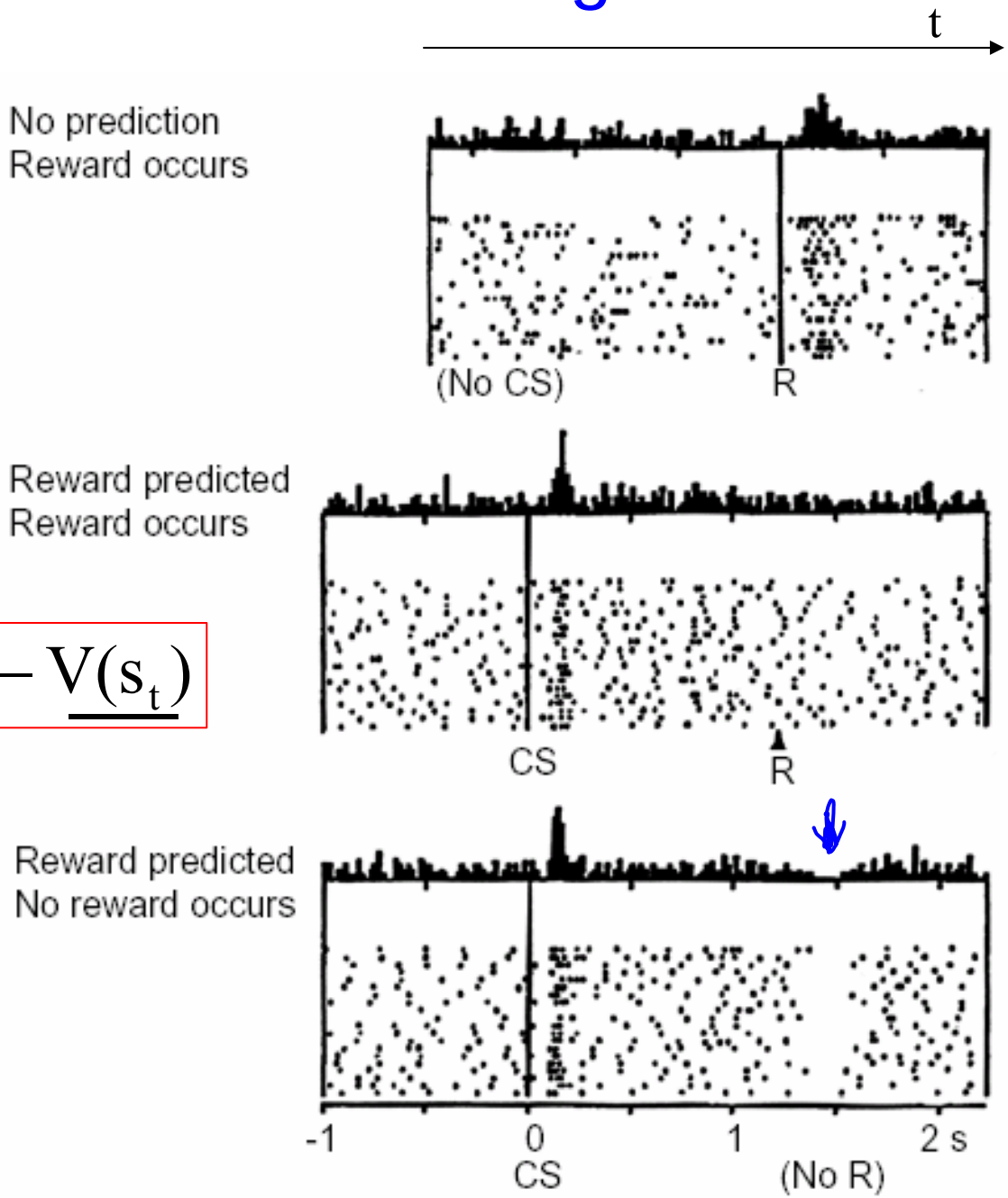
Subtleties and Ongoing Research

- Replace $\hat{Q}$ table with neural net or other generalizer
- Handle case where state only partially observable
- Design optimal exploration strategies
- Extend to continuous action, state
- Learn and use $\hat{\delta} : S \times A \rightarrow S$
- Relationship to dynamic programming

Reward-based learning in animals

Dopamine As Reward Signal

[Schultz et al., *Science*, 1997]



$$\text{error} = r_t + \gamma V(s_{t+1}) - V(s_t)$$

RL Models for Human Learning

[Seymore et al., Nature 2004]

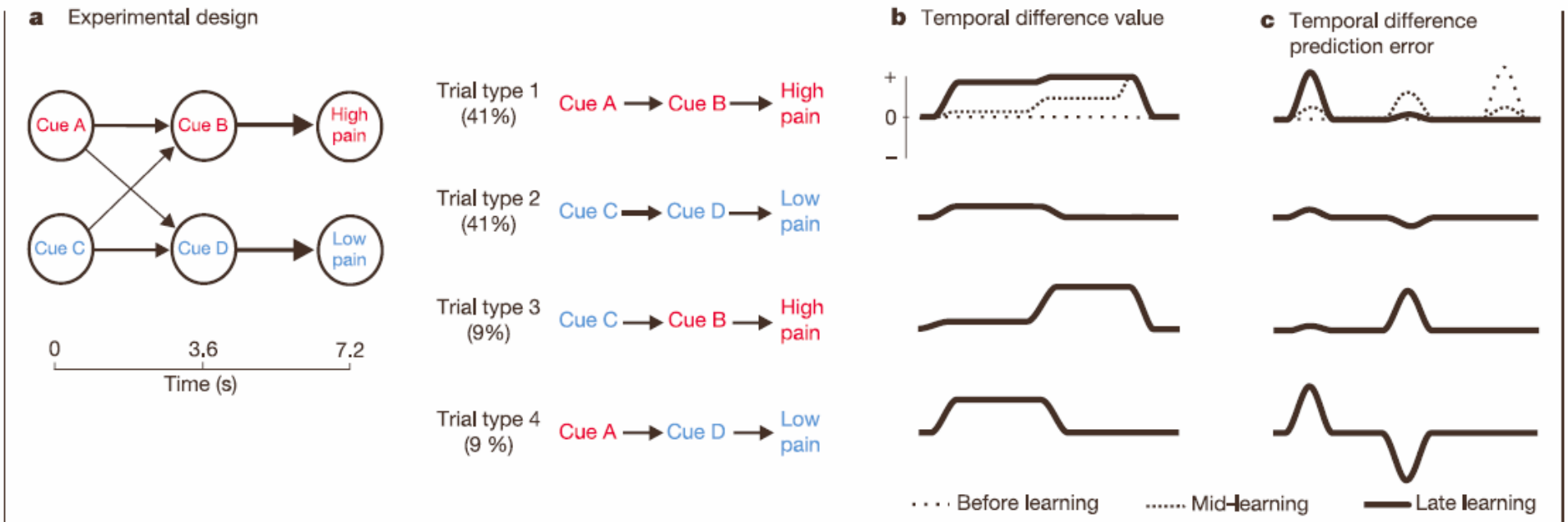
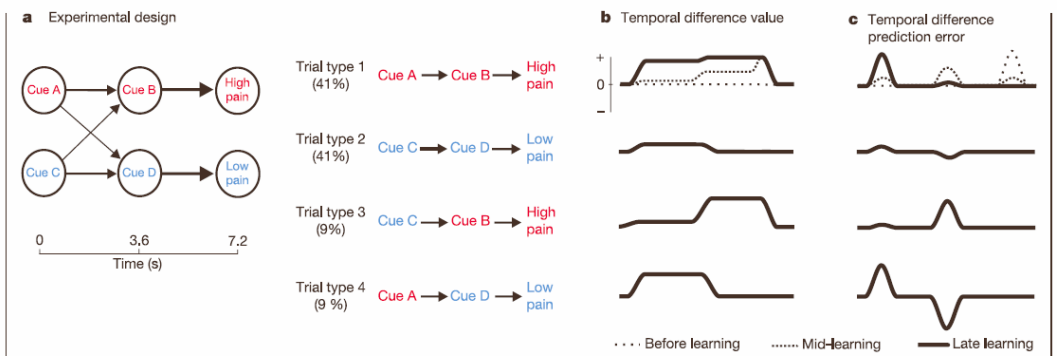


Figure 1 Experimental design and temporal difference model. **a**, The experimental design expressed as a Markov chain, giving four separate trial types. **b**, Temporal difference value. As learning proceeds, earlier cues learn to make accurate value predictions (that is, weighted averages of the final expected pain). **c**, Temporal difference prediction error; during learning the prediction error is transferred to earlier cues as they acquire the ability to make predictions. In trial types 3 and 4, the substantial change in prediction elicits a large positive or negative prediction error. (For clarity, before and mid-learning are shown only for trial type 1.)



[Seymore et al., Nature 2004]

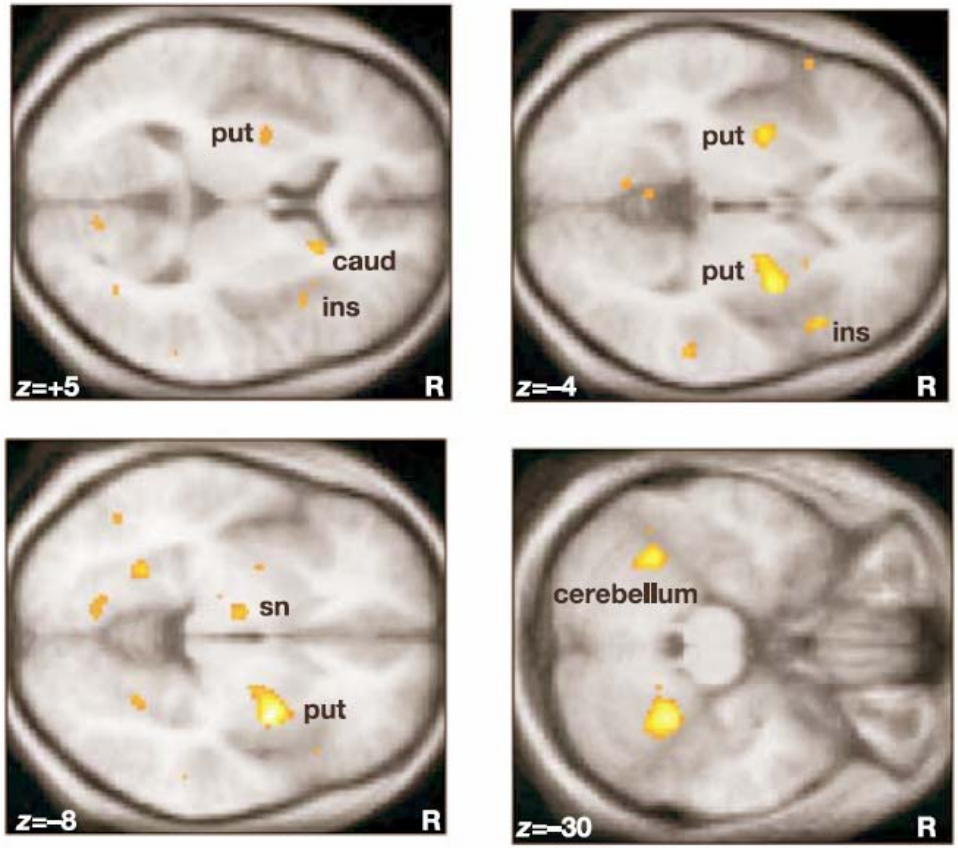


Figure 2 Temporal difference prediction error (statistical parametric maps). Areas coloured yellow/orange show significant correlation with the temporal difference

Human EEG responses to Pos/Neg Reward

from [Nieuwenhuis et al.]

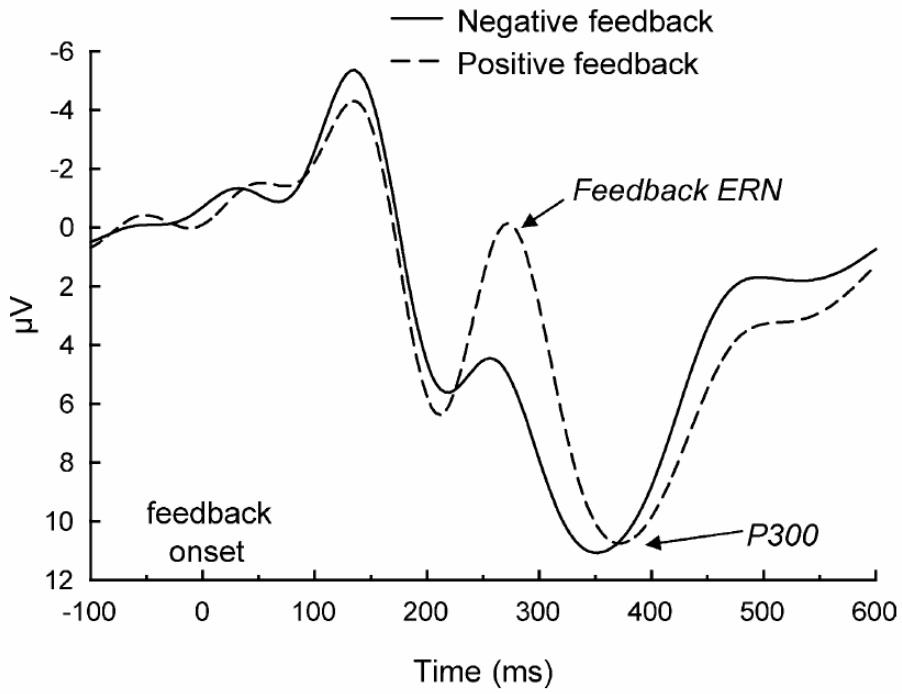


Fig. 1. Typical example of event-related brain potentials associated with negative and positive feedback (adapted from Ref. [25]). Negative is

Response due to feedback on timing task (press button exactly 1 sec after sound).

Neural source appears to be in anterior cingulate cortex (ACC)

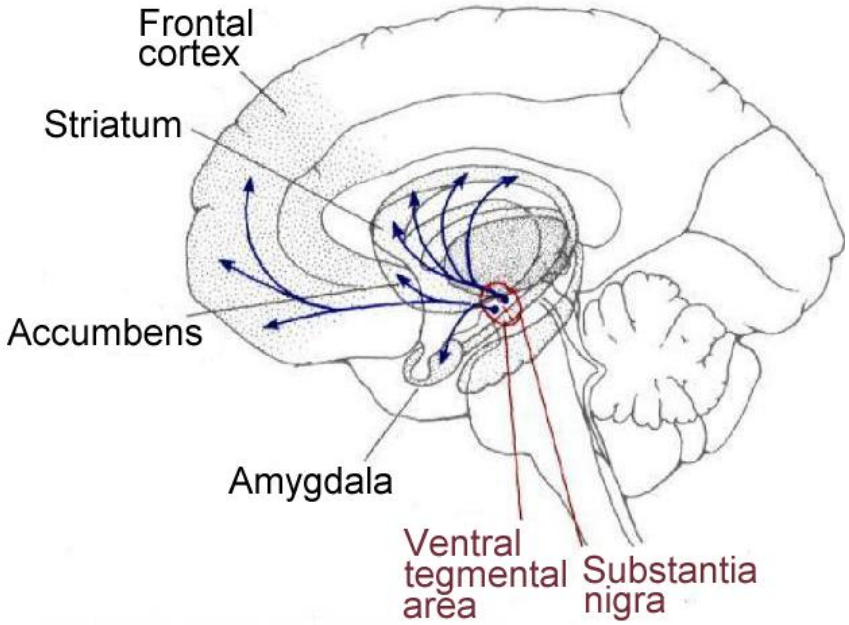
Response is abnormal in some subjects with OCD

One Theory of RL in the Brain

from [Nieuwenhuis et al.]

- Basal ganglia monitor events, predict future rewards
- When prediction revised upward (downward), causes increase (decrease) in activity of midbrain dopaminergic neurons, influencing ACC

- This dopamine-based activation somehow results in revising the reward prediction function. Possibly through direct influence on Basal ganglia, and via prefrontal cortex



Summary: Temporal Difference ML Model Predicts Dopaminergic Neuron Acitivity during Learning

- Evidence now of neural reward signals from
  - Direct neural recordings in monkeys
  - fMRI in humans (1 mm spatial resolution)
  - ERP in humans (1-10 msec temporal resolution)
- Dopaminergic responses track temporal difference error in RL
- Some differences, and efforts to refine HL model
  - Better information processing model
  - Better localization to different brain regions
  - Study timing (e.g., basal ganglia learns faster than PFC ?)

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What you should know

- Control learning
  - Credit assignment, learning from delayed rewards
- Markov Decision Processes
  - Discounted reward
  - Value iteration solution in case $P(S_{t+1}|S_t, A)$ is known
- When $P(S_{t+1}|S_t, A)$ is unknown
  - Learn Q(s,a) online
- Convergence, rote learning, generalizing function approximators
- Role in modeling reward learning in animals

Further Readings

- "Reinforcement Learning: A Survey" L. Kaelbling, M. Littman, A. Moore, JAIR (4), pp. 237-285, 1996.
- "R-max – A general polynomial time algorithm for near-optimal reinforcement learning," R. Brafman and M. Tennenholtz, JMLR (3), p.213, (2002).
- Tutorial slides by Andrew Moore, available at http://www.autonlab.org/tutorials/rl.html