Axioms and Theories

One important use of predicate logic is to pin down the properties mathematical objects. You fix a language, and a collection of formulae, so-called *axioms*, and then study all the models of these formulae.

In a while, we will generalize our deduction rules to predicate logic. The *theory* associated with a set of axioms Γ is

$$Th(\Gamma) = \text{ all } \varphi \text{ derivable from } \Gamma$$

The formulae in $Th(\Gamma)$ are called *theorems*.

▶ Since our deduction rules are sound, any formula in $Th(\Gamma)$ is valid in all models for Γ.

In other words, one single proof covers all models, we do not have to bother to prove the same fact over and over again in countless different models of the axioms.

▶ Usually try to keep axiom set small (ideally finite, at least very simple structure).

Peano Arithmetic (PA)

We use $\mathcal{L}(+, \cdot, S, 0; <)$ and omit the universal quantifiers. S stands for the *successor* function, $S(x) = x + 1$.

▶ Peano Axioms

$$S(x) \neq 0 \qquad S(x) = S(y) \rightarrow x = y$$

$$x + 0 = x \qquad x + S(y) = S(x + y)$$

$$x \cdot 0 = 0 \qquad x \cdot S(y) = (x \cdot y) + x$$

$$\neg(x < 0) \qquad x < S(y) \Leftrightarrow x = y \vee x < y$$

Induction Axiom:

$$\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(S(x))) \rightarrow \forall x \varphi(x)$$

The "Induction Axiom" is actually a so-called schema: there is one axiom for any formula φ.

All elementary number theory can be handled within this axiom system.

Computational Aspects

The Peano axioms are almost like programs.

In particular, the axioms provide recursive definitions of plus, times and less-than in terms of the successor function S.

```
add( x, y )
{
   if( y == 0 )  return x;
   return   S( x, y–1 );
}
```

```
less( x, y )
{
   if( y == 0 ) return false;
   return   ( x == y–1 ) || less( x, y–1 );
}
```

A Theorem of (PA)

Claim: $\forall x\,(0 + x = x)$

Proof.

Consider the formula $\varphi(x) \equiv (0 + x = x)$.

Then $\varphi(0)$ is the first addition axiom (more precisely, replace x by 0 there).

Now assume $\varphi(x)$. Then by the second addition axiom

$$0 + S(x) = S(0 + x) = S(x)$$

Hence we have shown $\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(S(x)))$.

By the Induction Axiom and modus ponens we get $\forall x \varphi(x)$.

Likewise, one can prove

$$\forall x, y, z\,(x + (y + z) = (x + y) + z)$$

$$\forall x, y\,(x + y = y + x)$$

Hence, it follows from the Peano axioms that $\langle\, \mathbb{N}; +, 0\,\rangle$ is associative, commutative, and has an identity.

Primes in Peano Arithmetic

Remember our formula that expresses primality?

$$\varphi(x) \equiv S(0) < x \wedge \forall y, z(x = y \cdot z \to x = y \vee x = z)$$

With some more effort one could derive from (PA)

$$\forall x \exists y \, (x < y \wedge \varphi(y))$$

In other words, there are infinitely many primes.

Likewise, one can show in (PA) that every number can be decomposed uniquely into a product of primes, and so on.

All results of basic arithmetic can be deduced from just (PA).

▶ Hence we have a very succinct representation of the essential features of arithmetic: just 8 axioms and one axiom schema (induction).

Recall: Natural Deduction Rules

And

$$\frac{\phi \quad \psi}{\phi \wedge \psi} \wedge i \qquad \frac{\phi \wedge \psi}{\phi} \wedge e_1 \qquad \frac{\phi \wedge \psi}{\psi} \wedge e_2$$

Or

$$\frac{\phi}{\phi \vee \psi} \vee i_1 \qquad \frac{\psi}{\phi \vee \psi} \vee i_2 \qquad \frac{\phi \vee \psi \quad \begin{array}{c}[\phi]\\ \vdots \\ \chi\end{array} \quad \begin{array}{c}[\psi]\\ \vdots \\ \chi\end{array}}{\chi} (\vee\, e)$$

Implication

$$\frac{\phi \quad \phi \to \psi}{\psi} (\to\ e) \qquad \frac{\neg\psi \quad \phi \to \psi}{\neg\phi} (\to\ mt) \qquad \frac{\begin{array}{c}[\phi]\\ \vdots \\ \psi\end{array}}{\phi \to \psi} (\to\ i)$$

Double negation

$$\frac{\neg\neg\phi}{\phi} (\neg\neg\, e) \qquad \frac{\phi}{\neg\neg\phi} (\neg\neg\, i)$$

Falsum

$$\frac{\bot}{\phi} (\bot\, e) \qquad \frac{\neg\phi \quad \phi}{\bot} (\bot\, i)$$

Derivations in Predicate Logic

We need to augment our deduction rules. We keep all the rules from propositional logic, and add rules for the quantifiers.

Intuitively, we would like to use

$$\frac{\phi(t)}{\exists x \phi(x)}\,(\exists\, i) \qquad \frac{\exists x \phi(x)}{\phi(c)}\,(\exists\, e)$$

$$\frac{\phi(x)}{\forall x \phi(x)}\,(\forall\, i) \qquad \frac{\forall x \phi(x)}{\phi(t)}\,(\forall\, e)$$

where x is a variable, c a constant, and t a term.

▶ Correct in spirit.

Alas, as stated these rules are not sound.

Counterexample

Suppose we adopt the quantifier rules from above. Then we can perform the following derivation.

$\forall x \exists y (x < y)$	premise
$\exists y (x < y)$	$\forall$ e
$(x < c)$	$\exists$ e
$\forall x (x < c)$	$\forall$ i
$\exists y \forall x (x < y)$	$\exists$ i

- Disaster!

The premise is valid over $\mathcal{N}$, but the conclusion is not. This is exactly the wrong direction of the valid implication $\exists x \forall y \varphi(x, y) \to \forall y \exists x \varphi(x, y)$

The problem is that the c really depends on x.

To address this and similar problems, one has to add certain technical conditions to the quantifier rules.

Amended Quantifier Rules

A term t is *substitutable* for x in $\varphi(x)$ iff no variable in t becomes bound in $\varphi(t)$.

$$\frac{\phi(t)}{\exists x \phi(x)} \ (\exists\, i)$$

where t is substitutable for x in φ.

$$\frac{\exists x \phi(x) \qquad \begin{array}{c}[\varphi(x)] \\ \vdots \\ \psi\end{array}}{\psi} \ (\exists\, e)$$

where x is not free in ψ, and does not occur in active assumptions.

$$\frac{\phi(x)}{\forall x \phi(x)} \ (\forall\, i)$$

where x is not free in any assumptions for φ.

$$\frac{\forall x \phi(x)}{\phi(t)} \ (\forall\, e)$$

where t is substitutable for x in φ.

Example

An example for a correct derivation, according to our rules:

$$\exists x(\varphi(x) \to \psi(x)) \vdash \forall x \varphi(x) \to \exists x \psi(x)$$

This is valid (our structures are never empty).

Here is the proof tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x \varphi(x)]}{\varphi(x)}\,(\forall\, e) \qquad [\varphi(x) \to \psi(x)]}{\psi(x)}\,(\to\ e)}{\exists x \psi(x)}\,(\exists\, i)}{\forall x \varphi(x) \to \exists x \psi(x)}\,(\to\ i) \qquad \exists x(\varphi(x) \to \psi(x))}{\forall x \varphi(x) \to \exists\, x \psi(x)}\,(\exists\, e)$$

Enough?