The Mayer-Vietoris Sequence in HoTT

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- Axioms for Cohomology
- A Model in HoTT
- Mayer-Vietoris?
- Cubes
- Mayer-Vietoris

(Axioms for Cohomology, A Model in HoTT: Shulman and IAS)

(Cubes: Licata)

All results are formalized in Agda!

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Cohomology Theory

A cohomology theory is:

- family of contravariant functors $C^n : \text{Type}_* \to \text{AbGrp}$ for $n : \mathbb{Z}$
- satisfying certain axioms (Eilenberg-Steenrod Axioms)

Think homotopy groups: associate a group $C^n(X)$ to each dimension n of a space X.

Note! Types will always be pointed, and functions basepoint-preserving.

$$\text{Type}_* \equiv \sum_{A:\text{Type}} A \qquad (A, a_0) \to (B, b_0) \equiv \sum_{f:A\to B} f\ a_0 = b_0$$

Cohomology Axioms in HoTT

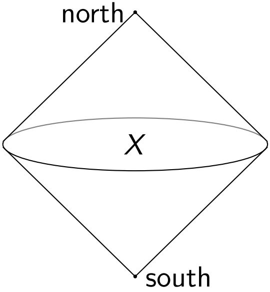
Eilenberg-Streenrod Axioms

1. Suspension Axiom
2. Exactness Axiom
3. Additivity Axiom (?)

Eilenberg-Steenrod Axioms (1/3)

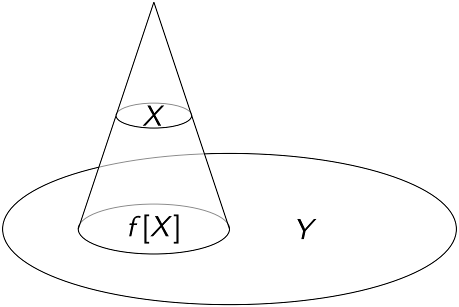
1. Suspension Axiom: $C^n(X) = C^{n+1}(\Sigma X)$

data ΣX where
north : ΣX
south : ΣX
merid : $X \to$ north $=$ south



Eilenberg-Steenrod Axioms (2/3)

For $f : X \to Y$, there is the cofiber space:



data $\text{Cof}(f)$ where

cfbase : $\text{Cof}(f)$

cfcod : $Y \to \text{Cof}(f)$

cfglue : $(x : X) \to \text{cfbase} = \text{cfcod}(f(x))$

Eilenberg-Steenrod Axioms (2/3)

Thus for each $f : X \to Y$ a sequence

$$X \xrightarrow{f} Y \xrightarrow{\text{cfcod}} \text{Cof}(f)$$

2. Exactness Axiom:

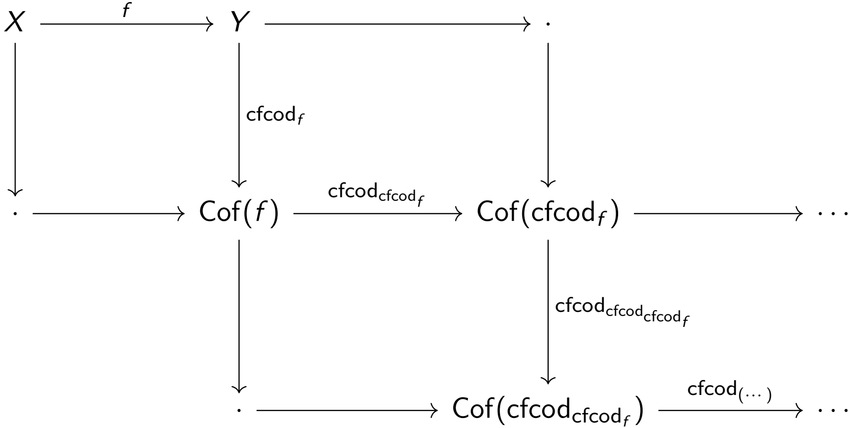
   For $f : X \to Y$, an exact sequence:

   $$C^n(\text{Cof}(f)) \xrightarrow{\text{cfcod}^*} C^n(Y) \xrightarrow{f^*} C^n(X)$$

   "The image of cfcod* is the kernel of f^*."
   That is, for $v : C^n(Y)$, $f^* v = e$ if and only if there *merely exists* $u : C^n(\text{Cof}(f))$ such that $\text{cfcod}^* u = v$.

Exactness Axiom

Extending the short exact sequence:



Exactness Axiom

Extending the short exact sequence:

$$
\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \longrightarrow & \cdot & & \\
\downarrow & & \downarrow{\scriptstyle \mathsf{cfcod}_f} & & \downarrow & & \\
\cdot & \longrightarrow & \mathsf{Cof}(f) & \xrightarrow{\mathsf{extglue}} & \Sigma X & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle \Sigma f} & & \\
 & & \cdot & \longrightarrow & \Sigma Y & \xrightarrow{\Sigma \mathsf{cfcod}_f} & \cdots
\end{array}
$$

Eilenberg-Steenrod Axioms (3/3)

3. Additivity Axiom (?):

   For suitable (?) I and $Z : I \to \text{Type}_*$,

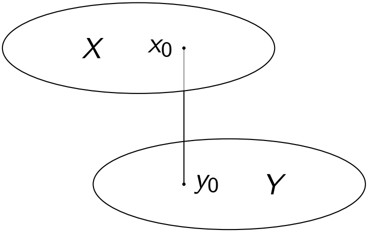
$$C^n(\bigvee_{i:I} Z_i) = \prod_{i:I} C^n(Z_i)$$

data $X \vee Y$ where

winl : $X \to X \vee Y$

winr : $Y \to X \vee Y$

wglue : winl x_0 = winr y_0



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A Model in HoTT

$K(G, n)$, for G : AbGrp and $n : \mathbb{N}$, is the nth *Eilenberg-MacLane space*, which satisfies

- $\pi_k(K(G, n)) = \begin{cases} G, & k = n \\ 0, & k \neq n \end{cases}$
- $\Omega K(G, n+1) = K(G, n)$

Formalized in Agda by Dan Licata:
$K(G, 1)$ is a HIT, $K(G, n+1) \equiv ||\Sigma^n K(G, 1)||_{n+1}$.

These are classically known to be representing spaces for cohomology theories.

A Model in HoTT

Fix G : AbGrp. Define

$$C^n(X) \equiv \|X \to K(G,n)\|_0 \qquad f^*(|g|_0) \equiv |g \circ f|_0$$

(with $C^n(X) \equiv 1$ for $n < 0$.)

Group structure on $C^n(X)$ inherited from $K(G,n) = \Omega K(G,n{+}1)$.

Analogous to $\pi_n(X) \equiv \|S^n \to X\|_0$; the property
$K(G,n) = \Omega K(G,n{+}1)$ is dual to $\Sigma S^n = S^{n+1}$.

Axioms in the Model (1/3)

1. Suspension Axiom: $C^n(X) = C^{n+1}(\Sigma X)$

$$\begin{aligned} C^{n+1}(\Sigma X) &= \|\Sigma X \to K(G, n+1)\|_0 \\ &= \|X \to \Omega K(G, n+1)\|_0 \\ &= \|X \to K(G, n)\|_0 \\ &= C^n(X) \end{aligned}$$

Axioms in the Model (2/3)

2. Exactness Axiom:

   For $f : X \to Y$, an exact sequence:

$$C^n(\mathrm{Cof}(f)) \xrightarrow{\mathrm{cfcod}^*} C^n(Y) \xrightarrow{f^*} C^n(X)$$

For $|g|_0 : C^n(Y)$, have $|g \circ f|_0 = e$ iff there is $|h|_0 : C^n(\mathrm{Cof}(f))$ such that $|g|_0 = |h \circ \mathrm{cfcod}|_0$.

Recall the definition of the cofiber space...
A function $\mathrm{Cof}(f) \to K(G, n)$ is (approximately) a function $Y \to K(G, n)$ which maps the "subset" $f[X]$ to the basepoint.

Axioms in the Model (3/3)

3. Additivity Axiom (?):

   For $X : I \to \mathrm{Type}_*$, $C^n(\bigvee_{i:I} X_i) = \prod_{i:I} C^n(X_i)$.

$$C^n\left(\bigvee_{i:I} X_i\right) = \left\|\bigvee_{i:I} X_i \to K(G,n)\right\|_0 = \left\|\prod_{i:I}(X_i \to K(G,n))\right\|_0$$

$$\prod_{i:I} C^n(X_i) = \prod_{i:I} \|X_i \to K(G,n)\|_0$$

Does Π commute with truncation? (Not often)

Mayer-Vietoris Sequence

Cohomology of spheres is easy in our model:

$$C^n(S^k) = \left\|S^k \to K(G,n)\right\|_0 = \pi_k(K(G,n)) = \begin{cases} G, & n = k \\ 0, & n \neq k \end{cases}$$

Many spaces can be built from spheres using pushouts. What is the cohomology of a homotopy pushout?

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ \downarrow f & & \downarrow \\ X & \longrightarrow & X \sqcup_Z Y \end{array}$$

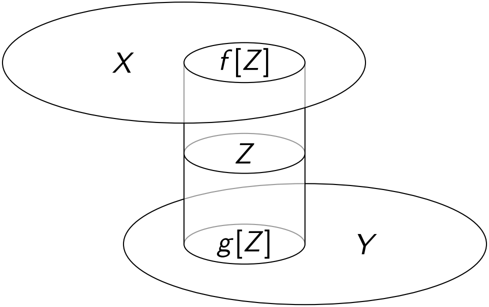
Mayer-Vietoris Sequence

data $X \sqcup_Z Y$ where

left : $X \to X \sqcup_Z Y$

right : $Y \to Y \sqcup_Z Y$

glue : $(z : Z) \to \text{left}\,(f\ z) = \text{right}\,(g\ z)$



Mayer-Vietoris Sequence

Classically, for $X \xleftarrow{f} Z \xrightarrow{g} Y$, a long exact sequence

$$\cdots \to C^{n-1}(Z) \to C^n(X \sqcup_Z Y) \to C^n(X) \times C^n(Y) \to C^n(Z) \to \cdots$$

Try for a short exact sequence

$$C^n(\Sigma Z) \to C^n(X \sqcup_Z Y) \to C^n(X \vee Y)$$

working from

$$X \vee Y \to X \sqcup_Z Y \to \Sigma Z$$

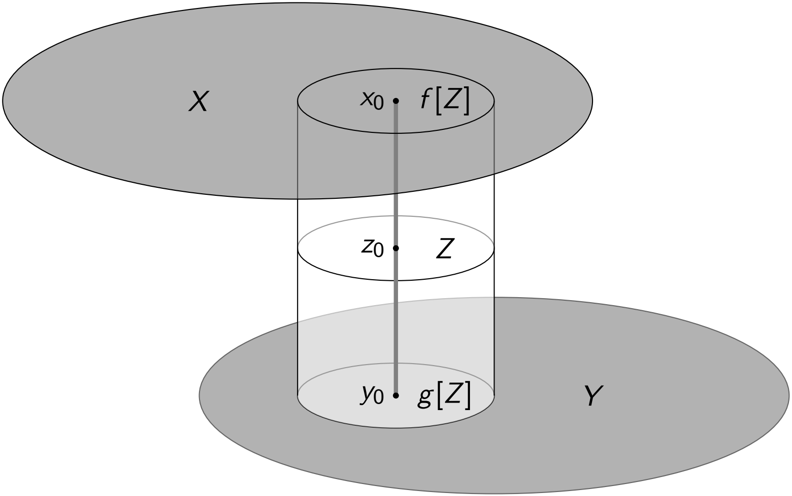
Mayer-Vietoris Sequence

Start with a map $X \vee Y \to X \sqcup_Z Y$:

$$\begin{aligned}
&\mathsf{reglue} : X \vee Y \to X \sqcup_Z Y \\
&\mathsf{reglue}\ (\mathsf{winl}\ x) = \mathsf{left}\ x \\
&\mathsf{reglue}\ (\mathsf{winr}\ y) = \mathsf{right}\ y \\
&\mathsf{ap}_{\mathsf{reglue}}\ \mathsf{wglue} = \mathsf{glue}\ z_0
\end{aligned}$$

What is the cofiber space of this map?

Mayer-Vietoris Sequence



Mayer-Vietoris Sequence

An equivalence $\mathrm{Cof}(\mathrm{reglue}) \simeq \Sigma Z$ gives us:

$$
\begin{array}{ccccccc}
C^n(\Sigma(X \vee Y)) & \to & C^n(\mathrm{Cof}(\mathrm{reglue})) & \to & C^n(X \sqcup_Z Y) & \to & C^n(X \vee Y) \\
& \searrow & \updownarrow \simeq & \nearrow & & & \\
& & C^n(\Sigma Z) & & & &
\end{array}
$$

Mayer-Vietoris Sequence

To prove $\text{Cof}(\text{reglue}) \simeq \Sigma Z$, need maps

$$\text{into} : \text{Cof}(\text{reglue}) \to \Sigma Z \qquad \text{out} : \Sigma Z \to \text{Cof}(\text{reglue})$$

and need to prove

$$\text{out-into} : (\kappa : \text{Cof}(\text{reglue})) \to \text{out}\ (\text{into}\ \kappa) = \kappa$$

(and more).

How in general to construct

$$(\kappa : \text{Cof}(\text{reglue})) \to h\,\kappa = k\,\kappa$$

for $h, k : \text{Cof}(\text{reglue}) \to C$?

Mayer-Vietoris Sequence

To prove $p : (\kappa : \mathsf{Cof}(\mathsf{reglue})) \to h\,\kappa = k\,\kappa$ by induction on the cofiber space, we need to give

1. $p_{\mathsf{cfbase}} : h\,\mathsf{cfbase} = k\,\mathsf{cfbase}$
2. $p_{\mathsf{cfcod}} : (\gamma : X \sqcup_Z Y) \to h\,(\mathsf{cfcod}\gamma) = k\,(\mathsf{cfcod}\gamma)$
3. A proof that, for $w : X \vee Y$, $p_{\mathsf{cfbase}} =^{\kappa.h\kappa=k\kappa}_{\mathsf{cfglue}\,w} p_{\mathsf{cfcod}}\,(\mathsf{reglue}\,w)$

Proving the third by induction on w would mean constructing a dependent path in the fibration
$w.\, p_{\mathsf{cfbase}} =^{\kappa.h\kappa=k\kappa}_{\mathsf{cfglue}\,w} p_{\mathsf{cfcod}}\,(\mathsf{reglue}\,w)$...

How do we build such a path?
Idea: Represent these paths as cubes.

Cubes

For $p : x = y$, $u : f\,x = g\,x$, and $v : f\,y = g\,y$, the dependent path type $u =_{p}^{z.fz=gz} v$ is equivalent to the type of commutative squares

$$\begin{array}{ccc} f\,x & \xrightarrow{\mathsf{ap}_f\ \ p} & f\,y \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ g\,x & \xrightarrow{\mathsf{ap}_g\ \ p} & g\,y \end{array}$$

Cubes

We can express the type of commutative squares of paths

$$\begin{array}{ccc} a & \xrightarrow{p} & c \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle s} \\ b & \xrightarrow{r} & d \end{array}$$

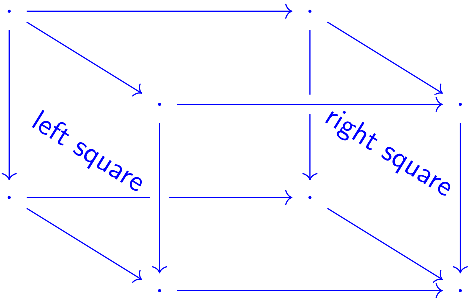
as Square p q r s where Square is inductively defined as

data Square : $(a = b) \rightarrow (a = c) \rightarrow (b = d) \rightarrow (c = d) \rightarrow$ Type where

  srefl : Square refl refl refl refl

Cubes

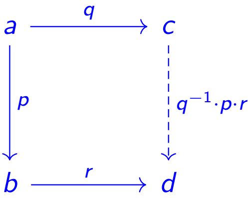
- Dependent type in a family of paths is a square,
- Dependent type in a family of squares is a cube.



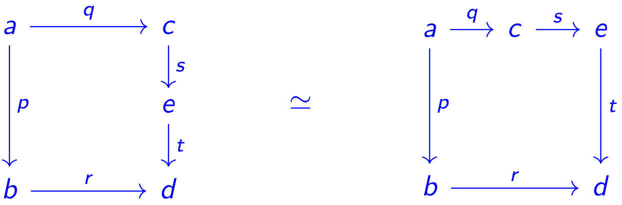
$$\text{data Cube} : (\cdots \text{six faces} \cdots) \to \text{Type where}$$
$$\text{crefl} : \text{Cube srefl srefl srefl srefl srefl srefl}$$

Cubes

Propositionally unique fillers exist:



Shifting faces around gives equivalent types:



The case of Cof(reglue)

Were trying to prove $p : (\kappa : \mathsf{Cof}(\mathsf{reglue})) \to h\,\kappa = k\,\kappa$.
Needed, for $w : X \vee Y$, a dependent path
$p_{\mathsf{cfbase}} =^{\kappa . h\kappa = k\kappa}_{\mathsf{cfglue}\,w} p_{\mathsf{cfcod}}\,(\mathsf{reglue}\,w)$. Equivalently, a square

$$\begin{array}{ccc}
h\,\mathsf{cfbase} & \xrightarrow{\mathsf{ap}_h\ (\mathsf{cfglue}\,w)} & h\,(\mathsf{cfcod}\,(\mathsf{reglue}\ w)) \\
\downarrow{\scriptstyle p_{\mathsf{cfbase}}} & & \downarrow{\scriptstyle p_{\mathsf{cfcod}}\,(\mathsf{reglue}\ w)} \\
k\,\mathsf{cfbase} & \xrightarrow{\mathsf{ap}_k\ (\mathsf{cfglue}\,w)} & k\,(\mathsf{cfcod}\,(\mathsf{reglue}\ w))
\end{array}$$

To give this by induction on $w : X \vee Y$...

To give this by induction on $w : X \vee Y$...

$$
\text{l-square} : (x : X) \quad \to \quad
\begin{array}{ccc}
h\,\text{cfbase} & \xrightarrow{\cdots} & h\,(\text{cfcod}\,(\text{reglue}\;(\text{winl}\;x))) \\
\Big\downarrow{\scriptstyle p_{\text{cfbase}}} & & \Big\downarrow{\scriptstyle \vdots} \\
k\,\text{cfbase} & \xrightarrow{\cdots} & k\,(\text{cfcod}\,(\text{reglue}\;(\text{winl}\;x)))
\end{array}
$$

$$
\text{r-square} : (y : Y) \quad \to \quad
\begin{array}{ccc}
h\,\text{cfbase} & \xrightarrow{\cdots} & h\,(\text{cfcod}\,(\text{reglue}\;(\text{winr}\;y))) \\
\Big\downarrow{\scriptstyle p_{\text{cfbase}}} & & \Big\downarrow{\scriptstyle \vdots} \\
k\,\text{cfbase} & \xrightarrow{\cdots} & k\,(\text{cfcod}\,(\text{reglue}\;(\text{winr}\;y)))
\end{array}
$$

and a cube with (among other faces) left face l-square x_0 and right face r-square y_0.

Given l-square and r-square, we can define a replacement r-square′ which automatically satisfies the cube requirement.

$\text{r-square}'\, y =$

$$\begin{array}{ccccc}
h\,\text{cfbase} & \xrightarrow{\text{refl}} & h\,\text{cfbase} & \xrightarrow{\cdots} & h\,(\text{cfcod}\,(\text{reglue}\ (\text{winr}\ y))) \\
\Big\downarrow{\scriptstyle p_{\text{cfbase}}} & \text{base-filler} & \Big\downarrow{\scriptstyle p_{\text{cfbase}}} & \text{rsquare y} & \Big\downarrow{\scriptstyle \vdots} \\
k\,\text{cfbase} & \xrightarrow{\text{refl}} & k\,\text{cfbase} & \xrightarrow{\cdots} & k\,(\text{cfcod}\,(\text{reglue}\ (\text{winr}\ y)))
\end{array}$$

where base-filler is the filling face giving the correct cube between l-square x_0 and base-filler $\cdot^h$ r-square y_0.

done!

Remainder...

Showed that when proving $p : (\kappa : \mathsf{Cof}(\mathsf{reglue})) \to h\,\kappa = k\,\kappa$ we can get the highest coherence condition automatically.

Rest of Mayer-Vietoris:

- Formalization: github.com/HoTT/HoTT-Agda, at cohomology.MayerVietoris
- Paper proof: www.contrib.andrew.cmu.edu/~ecavallo