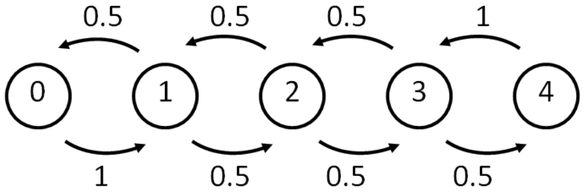
Periodic Markov chains

A random walk with reflecting boundaries



$$\begin{array}{c} \\ E_0 \\ E_1 \\ E_2 \\ E_3 \\ E_4 \end{array}\begin{bmatrix} & E_0 & E_1 & E_2 & E_3 & E_4 \\ E_0 & 0 & 1 & 0 & 0 & 0 \\ E_1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ E_2 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ E_3 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ E_4 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{1}$$

$$\varphi(0) = (0, 0, 1, 0, 0)$$

$$\varphi(1) = (0, \frac{1}{2}, 0, \frac{1}{2}, 0)$$
$$\varphi(2) = (\frac{1}{4}, 0, \frac{1}{2}, 0, \frac{1}{4})$$
$$\varphi(3) = (0, \frac{1}{2}, 0, \frac{1}{2}, 0)$$
$$\varphi(4) = (\frac{1}{4}, 0, \frac{1}{2}, 0, \frac{1}{4})$$

$$\varphi(0) = (0, 1, 0, 0, 0)$$

$$\varphi(0) = (\mathbf{0.0000}, 1.0000, \mathbf{0.0000}, \mathbf{0.0000}, \mathbf{0.0000})$$
$$\varphi(1) = (0.5000, \mathbf{0.0000}, 0.5000, \mathbf{0.0000}, \mathbf{0.0000})$$
$$\varphi(2) = (\mathbf{0.0000}, 0.7500, \mathbf{0.0000}, 0.2500, \mathbf{0.0000})$$
$$\varphi(3) = (0.3750, \mathbf{0.0000}, 0.5000, \mathbf{0.0000}, 0.1250)$$
$$\varphi(4) = (\mathbf{0.0000}, 0.6250, \mathbf{0.0000}, 0.3750, \mathbf{0.0000})$$
$$\varphi(5) = (0.3125, \mathbf{0.0000}, 0.5000, \mathbf{0.0000}, 0.1875)$$
$$\varphi(6) = (\mathbf{0.0000}, 0.5625, \mathbf{0.0000}, 0.4375, \mathbf{0.0000})$$

A finite, irreducible Markov chain is aperiodic, if

1. The chain contains a state with a self-loop; i.e., $P_{xx} > 0$ for some state E_x.
2. All elements of the n-step transition matrix P^n are greater than 0 for some positive integer n.
3. The Markov chain contains a state E_x such that $P_{xx}^{n_1} > 0$ and $P_{xx}^{n_2} > 0$, and the $\gcd(n_1, n_2) = 1$.
4. The Markov chain contains an aperiodic state.

Even number of time steps

$$P^{(n)} \to \begin{array}{c|ccccc} & E_0 & E_1 & E_2 & E_3 & E_4 \\ \hline E_0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} \\ E_1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ E_2 & \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} \\ E_3 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ E_4 & \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} \end{array} \tag{2}$$

for sufficiently large n.

Odd number of time steps

$$P^{(2n+1)} \to \begin{array}{c|ccccc} & E_0 & E_1 & E_2 & E_3 & E_4 \\ \hline E_0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ E_1 & \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} \\ E_2 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ E_3 & \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} \\ E_4 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{array} \tag{3}$$

for sufficiently large n.

$P_{xx}^{(2n)} > 0$ and $P_{xx}^{(2n+2)} > 0, \forall x$. The greatest common divisor is 2.