A Denotational Semantics for Low-Level Probabilistic Programs with Nondeterminism

Di Wang\textsuperscript{a} Jan Hoffmann\textsuperscript{a} Thomas Reps\textsuperscript{b,c}

\textsuperscript{a} Carnegie Mellon University
\textsuperscript{b} University of Wisconsin
\textsuperscript{c} GrammaTech, Inc.

Abstract

Probabilistic programming is an increasingly popular formalism for modeling randomness and uncertainty. Designing semantic models for probabilistic programs has been extensively studied, but is technically challenging. Particular complications arise when trying to account for (i) unstructured control-flow, a natural feature in low-level imperative programs; (ii) general recursion, an extensively used programming paradigm; and (iii) nondeterminism, which is often used to represent adversarial actions in probabilistic models, and to support refinement-based development. This paper presents a denotational-semantics framework that supports the three features mentioned above, while allowing nondeterminism to be handled in different ways. To support both probabilistic choice and nondeterministic choice, the semantics is given over control-flow hyper-graphs. The semantics follows an algebraic approach: it can be instantiated in different ways as long as certain algebraic properties hold. In particular, the semantics can be instantiated to support nondeterminism among either program states or state transformers. We develop a new formalization of nondeterminism based on powerdomains over sub-probability kernels. Semantic objects in the powerdomain enjoy a notion we call generalized convexity, which is a generalization of convexity. As an application, the paper sketches an algebraic framework for static analysis of probabilistic programs, which has been proposed in a companion paper.

Keywords: Probabilistic programming, denotational semantics, control-flow hyper-graphs, nondeterminism, powerdomains

1 Introduction

Probabilistic programming provides a powerful framework for implementing randomized algorithms \cite{2}, cryptographic protocols \cite{3}, cognitive models \cite{31}, and machine-learning algorithms \cite{29}. One important focus of recent studies on probabilistic programming is to reason rigorously about probabilistic programs and systems. The first step in such works is to provide a suitable formal semantics for probabilistic programs.

Despite the fact that lots of existing work focuses on high-level probabilistic programs, e.g., lambda calculus \cite{8}, higher-order functions \cite{32,20}, and recursive types \cite{63}, we observe that low-level features could arise naturally. For example, when developing a compiler for a probabilistic programming language \cite{26,56}, we need a semantics for the imperative target language to prove compiler correctness. There have been studies on denotational semantics for well-structured imperative programs \cite{43,44,47,48,62,34,38,55,7}, as well as operational semantics for control-flow graphs (CFGs) based on Markov chains (MCs) and Markov decision processes (MDPs) \cite{25,14,15}. On the one hand, we prefer CFGs as program representations because they enable rich low-level features such as unstructured flows, e.g., those introduced by break and continue. On the other hand, from the perspective of rigorous reasoning, a denotational semantics (i) abstracts from details about program executions and focuses on program effects, and (ii) is compositional in the sense that the semantics of a program fragment is established from the semantics of the fragment’s proper constituents.

Therefore, in this paper, we devise a denotational semantics for low-level probabilistic programs. Our work makes three main contributions:

- We use hyper-graphs as the representation for low-level probabilistic programs with unstructured control-flow, general recursion, and nondeterminism.

This paper is electronically published in
Electronic Notes in Theoretical Computer Science
URL: www.elsevier.nl/locate/entcs
• We develop a domain-theoretic characterization of a new model of nondeterminism for probabilistic programming, which involves nondeterminacy among state transformers, opposed to a common model that involves nondeterminacy among program states.

• We devise an algebraic framework for denotational semantics. The advantage of having a framework is that it can be instantiated with different models of nondeterminism. We show how to instantiate the framework using two different approaches to formalizing nondeterminism in Ex. 5.2. We also show that for programs without procedure calls and nondeterminism, the resulting denotational semantics is equivalent to a distribution-based operational semantics (§5.2).

We define the denotational semantics directly as an interpretation of the control-flow hyper-graphs (CFHG) of low-level probabilistic programs, introduced in §2. Hyper-graphs consist of hyper-edges, each of which connects one source node and possibly several destination nodes. For example, probabilistic choices are represented by weighted hyper-edges with two destinations. Nondeterminism is then represented by multiple hyper-edges starting in the same node. The interpretation of hyper-edges is also different from standard edges. If the CFHG were treated as a standard graph, the subpaths from each successor of a branching node would be analyzed independently. In contrast, our hyper-graph approach interprets a probabilistic-choice hyper-edge with probability \( p \) as a function \( \lambda a . \lambda b . a \not\in b \), where \( \not\in \) is an operation that weights the subpaths through the two successors by \( p \) and \( 1 - p \). In other words, we do not reason about subpaths starting from a node individually, instead we analyze these subpaths jointly as a probability distribution. If a node has two outgoing probabilistic-choice hyper-edges, it represents two “worlds” of subpaths, each of which carries a probability distribution with respect to the probabilistic choice made in this “world.”

Some high-level decision choices about nondeterminism arise when we are developing the low-level semantics. Nondeterminism itself is an important feature from two perspectives: (i) it arises naturally from probabilistic models, such as the agent for an MDP [6], or the unknown input distribution for modeling fault tolerance [40], and (ii) it is required by the common paradigm of abstraction and refinement on programs [19,48]. While nondeterminism has been well studied for standard programming languages, the combination of probabilities and nondeterminism turns out to be tricky. One substantial question is when the nondeterminism is resolved. A well-studied model for nondeterminism in probabilistic programming is to resolve program inputs prior to nondeterminism [18,50,47,51,48,62]. This model follows a commonplace principle of semantics research that represents a nondeterministic function as a set-valued function that maps an input to a collection of possible outputs, i.e., an element in \( X \rightarrow \wp(X) \), where \( X \) is a program state space and \( \wp(\cdot) \) is the powerset operator. However, it is sometimes desirable to resolve nondeterminism prior to program inputs, i.e., a nondeterministic program should represent a collection of elements in \( \wp(X \rightarrow X) \). For example, one may want to show for every refined version of a nondeterministic program with each nondeterministic choice replaced by a conditional, its behavior on all inputs is indistinguishable. We call the common model nondeterminism-first and the other nondeterminism-last. In §4, we present a domain-theoretic study of nondeterminism-first. Technically, we propose a notion of generalized convexity (\( g \)-convexity, for short), which expresses that a set of state transformers is stable under refinements (while standard convexity describes that a set of states is stable under refinements), as well as devise a \( g \)-convex powerdomain that characterizes expressible semantic objects.

To achieve our ultimate goal of developing a denotational semantics, instead of restricting ourselves to one specific model for nondeterminism, we propose a general algebraic denotational semantics in §5, which can be instantiated with different treatments of nondeterminism. The semantics is algebraic in the sense that it performs reasoning in some space of program states and state transformers, while the transformers should obey some algebraic laws. For instance, the program command \texttt{skip} should be interpreted as the \texttt{identity} element for sequencing in an algebra of program-state transformers. In addition, the algebraic approach is a good fit for static analysis of probabilistic programs. In §6, we sketch a static-analysis framework proposed in a companion paper [64], as an application of the denotational semantics.

The algebraic approach we take in this paper is challenging in the setting of probabilistic programming. In contrast, for standard, non-probabilistic programming languages, it is almost trivial to derive a low-level denotational semantics \texttt{once} one has a semantics for well-structured programs at hand. The trick is to first define the semantic operations as a Kleene algebra [41,16,42,45], which admits an extend operation, used for sequencing, a combine operation, used for branching, and a closure operation, used for looping; then extract from the CFG a regular expression that captures all execution paths by Tarjan’s path-expression algorithm [61]; and finally use the Kleene algebra to reinterpret the regular expression to obtain the semantics for the CFG. However, this approach fails when both probabilities and nondeterminism come into the picture. Consider the probabilistic program with a nondeterministic choice \( \ast \) in Fig. 1. The program is intended to draw a random value \( t \) from either a fair coin flip or a biased one. If one adopts the path-expression approach, one ends up with a regular expression that describes a single collection of four program executions: (i) \( t := 0 \) with probability \( 1/2 \),
The term “deterministic” is used in the sense “not nondeterministic.”

2 An Operational Semantics for Low-Level Probabilistic Programs

In this section, we sketch an operational semantics for an imperative, single-procedure, deterministic, probabilistic programming language, following the approach of Borgström et al.’s distribution-based semantics [8]. We use the operational semantics to (i) illustrate how to model executions of probabilistic programs operationally, and (ii) justify the development of a denotational semantics in later sections.

2.1 A Hyper-Graph Program Model

We define the operational semantics on CFHGs of programs. We adopt a common approach for standard CFGs in which the nodes represent program locations, and edges labeled with instructions describe transitions among program locations (e.g., [24,54,46]). Instead of standard directed graphs, we make use of hyper-graphs [27].

**Definition 2.1** A hyper-graph $H$ is a quadruple $(V,E,v^\text{entry},v^\text{exit})$, where $V$ is a finite set of nodes, $E$ is a set of hyper-edges, $v^\text{entry} \in V$ is a distinguished entry node, and $v^\text{exit} \in V$ is a distinguished exit node. A hyper-edge $e = (x,Y)$ in $E$, we use src$(e)$ to denote $x$ and dst$(e)$ to denote $Y$. Following the terminology from graphs, we say that $e$ is an outgoing edge of $x$ and an incoming edge of each of the nodes $y \in Y$. We assume $v^\text{entry}$ does not have incoming edges, and $v^\text{exit}$ has no outgoing edges.

**Definition 2.2** A probabilistic program contains a finite set of procedures $\{H_i\}_{1 \leq i \leq n}$, where each procedure $H_i = (V_i,E_i,v^\text{entry}_i,v^\text{exit}_i)$ is a control-flow hyper-graph (CFHG) in which each node except $v^\text{exit}_i$ has at least one outgoing hyper-edge, and $v^\text{exit}_i$ has no outgoing hyper-edge. Define $V = \bigcup_{1 \leq i \leq n} V_i$. To assign meanings to probabilistic programs modulo data actions Act and deterministic conditions Cond that can be probabilistic, we associate with each hyper-edge $e \in E = \bigcup_{1 \leq i \leq n} E_i$ a control-flow action Ctrl$(e)$ that has one of the following three forms:

\[
\text{Ctrl} \;::=\; \text{seq}[\text{act}] \mid \text{cond}[\varphi] \mid \text{call}[i \mapsto j], \text{where } 1 \leq i,j \leq n
\]

where the number of destination nodes $|\text{dst}(e)|$ of a hyper-edge $e$ is 1 if $\text{Ctrl}(e)$ is seq[act] or call[i → j], and 2 otherwise.

**Example 2.3** Fig. 2(b) shows the CFHG of the program in Fig. 2(a), where $v_0$ is the entry and $v_3$ is the exit. The hyper-edge $(v_2, \{v_3\})$ is associated with a sequencing action $\text{seq}[n := n + 1]$, while $(v_1, \{v_2, v_4\})$ is assigned a deterministic-choice action $\text{cond}[\text{prob}(0.5) \land \text{prob}(0.5)]$, i.e., an event where two coin flips both show heads.

Note that break, continue (and also goto) are not data actions, and are encoded directly as edges in CFHGs in a standard way. The grammar below defines data actions Act and deterministic conditions Cond that could be used for an arithmetic program, where $p \in [0,1]$, $c \in \mathbb{Q}$, $a,b \in \mathbb{Z}$, and $n \in \mathbb{N}$.

\[
\begin{align*}
\text{Act} &::= x := e \mid x \sim D \mid \text{observe}(\varphi) \mid \text{skip} \\
\text{Exp} &::= c \mid e_1 + e_2 \mid e_1 \times e_2 \\
\text{Dist} &::= \text{Binomial}(n,p) \mid \text{Uniform}(a,b) \mid \text{Geometric}(p) \mid \cdots
\end{align*}
\]

Dist stands for a collection of discrete probability distributions. For example, Binomial$(n,p)$ with $n \in \mathbb{N}$ and $p \in [0,1]$ describes the distribution of the number of successes in $n$ independent experiments, each of which succeeds with probability $p$; Uniform$(a,b)$ represents a discrete uniform distribution on $[a,b] \cap \mathbb{Z}$.

\[2\] The term “deterministic” is used in the sense “not nondeterministic.”
\[ n := 0; \]
\[ \textbf{while } \text{prob}(0.5) \land \text{prob}(0.5) \text{ do} \]
\[ n := n + 1; \]
\[ \textbf{if } n \geq 10 \text{ then break} \]
\[ \textbf{else continue} \]
\[ \textbf{od} \]

Fig. 2. (a) An example of probabilistic programs; (b) The corresponding CFHG

2.2 A Distribution-Based Small-Step Operational Semantics

The next step is to define a semantics based on CFHGs. We adopt Borgström et al.’s distribution-based small-step operational semantics for lambda calculus [8] to our hyper-graph setting, while we suppress the features of multiple procedures and nondeterminism for now.

Three components are used to define the semantics:

- A function \( \langle \varphi \rangle \) from program states to \( \{0, 1\} \)-valued functions \( \varphi \). Intuitively, \( \langle \varphi \rangle (\omega) \) is the probability that the condition \( \varphi \) holds in state \( \omega \in \Omega \).

The point distribution \( \delta(\omega) \) is defined as \( \lambda \omega'. [\omega = \omega'] \) where \( [\psi] \) is an Iverson bracket that evaluates to 1 if \( \psi \) is true and 0 otherwise. If \( \Delta \) is a distribution, then \( \Delta \cdot r \) for the distribution \( \lambda \omega . r \cdot \Delta(\omega) \).

Fig. 3 shows interpretation of the data actions and deterministic conditions given in §2.1, where \( \omega(e) \) evaluates expression \( e \) in state \( \omega \), \( \omega(v) \) updates \( x \) in \( \omega \) with \( v \), and \( \Delta_D : Q \rightarrow [0, 1] \) is the probability mass function of the distribution \( D \). If \( \varphi \) does not contain any probabilistic choices \( \text{prob}(p) \), then \( \langle \varphi \rangle (\omega) \) is either 0 or 1. Intuitively, \( \langle \varphi \rangle (\omega) \) is the probability that \( \varphi \) is true in the state \( \omega \), w.r.t. a probability space specified by all the \( \text{prob}(p) \)'s in \( \varphi \). Then the probability of \( \varphi_1 \land \varphi_2 \) is defined as the product of the individual probabilities of \( \varphi_1 \) and \( \varphi_2 \), because \( \varphi_1 \) and \( \varphi_2 \) are interpreted w.r.t. probabilistic choices in \( \varphi_1 \) and \( \varphi_2 \), respectively, and these two sets of choices are disjoint, thus independent.

Suppose that \( P = \langle V, E, v_{\text{entry}}, v_{\text{exit}} \rangle \) is a single-procedure deterministic program. Therefore, each node in \( P \) except \( v_{\text{exit}} \) is associated with exactly one hyper-edge. The program configurations \( T = V \times \Omega \) are pairs of the form \( (v, \omega) \), where \( v \in V \) is a node in the CFHG, and \( \omega \in \Omega \) is a program state.

We define one-step evaluation as a relation \( (v, \omega) \rightarrow \Delta \) between configurations \( (v, \omega) \) and distributions \( \Delta \) on configurations, as shown in Fig. 4.

Fig. 4. One-step evaluation relation
Example 2.4 For the program in Fig. 2, some one-step evaluations are $\langle v_0, \{n \mapsto 233\} \rangle \rightarrow \delta(\{n \mapsto 0\})$, $\langle v_1, \{n \mapsto 0\} \rangle \rightarrow 2.5 \cdot \delta(\{n \mapsto 1\}) + 0.75 \cdot \delta(\{n \mapsto 1\})$, and $\langle v_3, n \mapsto 9 \rangle \rightarrow \delta(\{v_1, \{n \mapsto 9\}\})$.

We now define step-indexed evaluation as the family of $n$-indexed relations $\langle v, \omega \rangle \rightarrow_n \Delta$ between configurations $\langle v, \omega \rangle$ and distributions $\Delta$ on program states inductively, as shown in Fig. 5.

\[
\begin{align*}
\langle v, \omega \rangle & \rightarrow_0 \lambda \omega'.0 \\
\langle v^{\text{exit}}, \omega \rangle & \rightarrow_n \delta(\omega) \\
\langle v, \omega \rangle & \rightarrow_{n+1} \sum_{\tau \in \text{supp}(\Delta)} \Delta(\tau) \cdot \Delta'_{\tau} \quad \text{if } n > 0
\end{align*}
\]

where $\langle v, \omega \rangle \rightarrow \Delta$ and $\tau \rightarrow_n \Delta'_{\tau}$ for any $\tau \in \text{supp}(\Delta)$

Fig. 5. Step-indexed evaluation relation.

Example 2.5 For the program in Fig. 2, some one-step evaluations are $\langle v_4, \{n \mapsto 10\} \rangle \rightarrow_1 \delta(\{n \mapsto 10\})$, $\langle v_1, \{n \mapsto 0\} \rangle \rightarrow_2 0.75 \cdot \delta(\{n \mapsto 0\})$, and $\langle v_1, \{n \mapsto 0\} \rangle \rightarrow_5 0.75 \cdot \delta(\{n \mapsto 0\}) + 0.1875 \cdot \delta(\{n \mapsto 1\})$.

For the program $P = \langle V, E, v^{\text{entry}}, v^{\text{exit}} \rangle$, we define its semantics $\llbracket P \rrbracket_{\text{ds}}(\omega) \overset{\text{def}}{=} \sup_{n \in N} \{ \Delta \mid \langle v^{\text{entry}}, \omega \rangle \rightarrow_n \Delta \}$.

Example 2.6 For the program $P$ in Fig. 2, $\llbracket P \rrbracket_{\text{ds}}(\omega)$ for any initial state $\omega$ with $n \in \text{dom}(\omega)$ is given by $\sum_{k=0}^{9}(0.75 \times 0.25^k) \cdot \delta([n \mapsto k]|\omega) + 0.00000095367431640625 \cdot \delta([n \mapsto 10]|\omega)$.

2.3 Why is a Denotational Semantics Desirable?

We have already shown how probabilistic programs execute operationally. As mentioned in §1, we are instead interested in developing a denotational semantics, which concentrates on the effects of programs and abstracts from how the program executes. This characterization of denotational semantics is indeed beneficial for rigorous reasoning about programs, such as static analysis and model checking, because one usually only cares whether programs satisfy certain properties, e.g., if they terminate on all possible inputs. Even better, a denotational semantics is often compositional—that is, the property of a whole program can be established from properties of its proper constituents. In other words, one could develop local—and thus scalable—reasoning techniques based on a denotational semantics. In contrast, the operational semantics in §2.2 is not compositional—it takes into account the whole program $P$ to define $\llbracket P \rrbracket_{\text{ds}}$.

Another benefit of a denotational semantics is that it is often easier to extend than an operational one. In the rest of this section, we briefly compare the complexity of adding procedure calls and nondeterminism to an operational semantics versus a denotational semantics. To support multiple procedures and procedure calls in the semantics proposed in §2.2, one needs to introduce a notion of stacks to keep track of procedure calls, as in [22,23,55]. Then the program configurations become triples of call stacks, control-flow-graph nodes, and program states. As a consequence, the one-step and step-indexed evaluation relations in Figs. 4 and 5 would become more complex. However, such an extension is almost trivial for a denotational semantics. Suppose we are able to compose semantic objects, e.g., $[C_1; C_2]_{\text{ds}} = [C_2]_{\text{ds}} \circ [C_1]_{\text{ds}}$, where $C_1, C_2$ are program fragments, $\circ$ denotes a composition operation, and $[C]_{\text{ds}}$ gives the denotation of $C$. If $C_1$ is indeed a procedure call call $Q$ where $Q$ is a procedure, because we can obtain the denotation $[Q]_{\text{ds}}$ of $Q$, we can interpret $\llbracket \text{call } Q; C_2 \rrbracket_{\text{ds}}$ merely as $[C_2]_{\text{ds}} \circ [Q]_{\text{ds}}$. By this means, we do not need to reason about stacks explicitly.

Another important programming feature is nondeterminism. For operational semantics of probabilistic programs, nondeterminism is often formalized using the notion of a scheduler, which resolves a nondeterministic choice from the computation that leads up to it (e.g., [25,14,15]). When the scheduler is fixed, a program can be executed deterministically (as shown in §2.2). To reason about nondeterministic programs with respect to an operational semantics, one needs to take all possible schedulers into consideration. However, if one only cares about the effects of a program, it is possible to sidestep these schedulers by switching to a denotational semantics. For example, let $C_1, C_2$ be two program fragments and $[C_1]_{\text{ds}}, [C_2]_{\text{ds}}$ be their denotations, which should be maps from initial states to a collection of possible final states. Then the denotation $\llbracket \text{if } * \text{ then } C_1 \text{ else } C_2 \rrbracket_{\text{ds}}$ of a nondeterministic-choice between $C_1$ and $C_2$ could be something like $\lambda \omega.[C_1]_{\text{ds}}(\omega) \cup [C_2]_{\text{ds}}(\omega)$. Note that this approach does not need to consider schedulers explicitly.

3 A Summary of Existing Domain-Theoretic Developments

Our development of models for nondeterminism makes great use of existing domain-theoretic studies of powerdomains, thus in this section, we present a brief summary of them. We review some standard notions from
domain theory [33,49], as well as some results on probabilistic powerdomains [36,35] and nondeterministic powerdomains [18,50,47,51,48,62].

3.1 Background from Domain Theory

Let $P$ be a nonempty set with a partial order $\subseteq$, i.e., a poset. The lower closure of a subset $A$ is defined as $\downarrow A \triangleq \{ x \in P \mid \exists a \in A : x \subseteq a \}$. The upper closure of a subset $A$ is defined as $\uparrow A \triangleq \{ x \in P \mid \exists a \in A : a \subseteq x \}$. A subset $A$ satisfying $\downarrow A = A$ is called a lower set. A subset $A$ satisfying $\uparrow A = A$ is called an upper set. If all elements of $P$ are above a single element $x \in P$, then $x$ is called the least element, denoted commonly by $\bot$. A function $f : P \to Q$ between two posets $P$ and $Q$ is monotone if for all $x, y \in P$ such that $x \subseteq y$, we have $f(x) \subseteq f(y)$. A subset $A$ of $P$ is directed if it is nonempty and each pair of elements in $A$ has an upper bound in $A$. If $A$ is totally ordered and isomorphic to natural numbers, then $A$ is called an $\omega$-chain. If a directed set $A$ has a supremum, then it is denoted by $\bigcup A$.

A poset $D$ is called directed complete or a dcpo if each directed subset $A$ of $D$ has a supremum $\bigcup A$ in $D$. A function $f : D \to E$ between two dcpos $D$ and $E$ is Scott-continuous if it is monotone and preserves directed suprema, i.e., $f(\bigcup A) = \bigcup f(A)$ for all directed subsets $A$ of $D$.

Let $D$ be a dcpo. For two elements $x, y$ of $D$, we say that $x$ approximates $y$, denoted by $x \ll y$, if for all directed subsets $A$ of $D$, we have $y \in \bigcup A$ implies $x \subseteq a$ for some $a \in A$. We define $\downarrow A \triangleq \{ x \in D \mid \exists a \in A : x \subseteq a \}$ and $\uparrow A \triangleq \{ x \in D \mid \exists a \in A : a \subseteq x \}$. The dcpo $D$ is called continuous if there exists a subset $B$ of $D$ such that for every element $x$ of $D$, the set $\downarrow x \cap B$ is directed and $x = \bigcup (\downarrow x \cap B)$. The set $B$ is called a basis of $D$.

Let $D$ be a dcpo. A subset $A$ is Scott-closed if $A$ is a lower set and is closed under directed suprema. The complement $D \setminus A$ of a Scott-closed subset $A$ is called Scott-open. These Scott-open subsets form the Scott-topology on $D$. The closure of a subset $A$ is the smallest Scott-closed set containing $A$ as a subset, denoted by $\bar{A}$.

Let $X$ be a topological space whose open sets are denoted by $\mathcal{O}(X)$. A cover $\mathcal{C}$ of a subset $A$ of $X$ is a collection of subsets whose union contains $A$ as a subset. A sub-cover of $\mathcal{C}$ is a subset of $\mathcal{C}$ that still covers $A$. The cover $\mathcal{C}$ is called an open-cover if each of its members is an open set. A subset $A$ is compact if every open-cover of $A$ contains a finite sub-cover. A subset $A$ is saturated if $A$ is an intersection of its neighborhoods. The saturation of a subset $A$ is the intersection of its neighborhoods. In dcpo’s equipped with the Scott-topology, saturated sets are precisely the upper sets, and the saturation of a subset $A$ is given by $\uparrow A$. The Lawson-topology on a dcpo $D$ is generated by Scott-open sets and sets of the form $D \setminus \downarrow x$. A lens is a nonempty subset that is the intersection of a Scott-closed subset and a Scott-compact saturated subset. Lenses are always Scott-closed.

A continuous dcpo $D$ is called coherent if the intersection of any two Scott-compact saturated subsets is also Scott-compact. The Lawson-topology on a coherent dcpo is compact.

We are going to use the following theorems in our technical development.

Proposition 3.1 (Kleene fixed-point theorem) Suppose $(D, \subseteq)$ is a dcpo with a least element $\bot$, and let $f : D \to D$ be a Scott-continuous function. Then $f$ has a least fixed point which is the supremum of the ascending Kleene chain of $f$ (i.e., the $\omega$-chain $\bot \subseteq f(\bot) \subseteq f(f(\bot)) \subseteq \cdots \subseteq f^n(\bot) \subseteq \cdots$), denoted by $\text{fix}_D f$.

Proposition 3.2 (Cor. of [33, Hofmann-Mislove theorem]) Let $X$ be a sober space, i.e., a $T_0$-space where every nonempty closed set is either the closure of a point or the union of two proper closed subsets. The intersection of a filtered family $\{ A_i \}_{i \in I}$ (i.e., the intersection of any two subsets is in the family) of nonempty compact saturated subsets is compact and nonempty. If such a filtered intersection is contained in an open set $U$, then $A_i \subseteq U$ for some $i \in I$. Specifically, continuous dcpos equipped with the Scott-topology and coherent dcpos equipped with the Lawson-topology are sober.

3.2 Probabilistic Powerdomains

Jones et al.’s pioneer work on probabilistic powerdomains [36,35] extends the complete partially ordered sets, which are pervasively used in computer science, to model probabilistic computations. Let $X$ be a nonempty countable set. The set of all distributions on $X$ is denoted by $\mathcal{D}(X)$, i.e., a probabilistic powerdomain over $X$. Recall that a distribution on $X$ is a function $\Delta : X \to [0, 1]$ such that $\sum_{x \in X} \Delta(x) \leq 1$, and the point distribution $\delta(x)$ for some $x \in X$ is defined as $\lambda x'. [x = x']$. Distributions are ordered pointwise, i.e., $\Delta_1 \subseteq_D \Delta_2 \triangleq \forall x \in X : \Delta_1(x) \leq \Delta_2(x)$. We define the probabilistic-choice of distributions $\Delta_1, \Delta_2$ with respect to a weight $p \in [0, 1]$, written $\Delta_1 \odot p \cdot \Delta_2$, as $p \cdot \Delta_1 + (1 - p) \cdot \Delta_2$.

The following theorems provide a characterization of the probabilistic powerdomains.
Proposition 3.3 ([36,35,47,62]) The poset \( \langle \mathcal{P}(X), \sqsubseteq_D \rangle \) forms a coherent dcpo with a countable basis \( \{ \sum_{i=1}^n r_i \cdot \delta(x) \mid n \in \mathbb{N} \wedge r_i \in \mathbb{Q}_0^+ \wedge \sum_{i=1}^n r_i \leq 1 \wedge x_i \in X \} \). It admits a least element \( \perp_D \overset{\text{def}}{=} \lambda x.0 \). Moreover, \( \rho^D \) is Scott-continuous for all \( p \in [0,1] \).

Proposition 3.4 ([35,62]) Every function \( f : X \to \mathcal{P}(X) \) can be lifted to a unique Scott-continuous linear (in the sense that it preserves probabilistic-choice) map \( \hat{f} : \mathcal{P}(X) \to \mathcal{P}(X) \).

3.3 Nondeterministic Powerdomains

When nondeterminism comes into the picture, as we discussed in §1, existing studies usually resolve program inputs prior to nondeterminism [37,18,50,47,51,48,62]. In §1, we call such a model nondeterminism-last, which interprets nondeterministic functions as maps from inputs to sets of outputs. Let \( X \) be a nonempty countable set. A subset \( A \) of \( \mathcal{P}(X) \) is called convex if for all \( \Delta_1, \Delta_2 \in A \) and for all \( p \in [0,1] \), we have \( \Delta_1 \oplus_p \Delta_2 \in A \). The convex hull of an arbitrary subset \( A \) is the smallest convex set containing \( A \) as a subset, denoted by \( \text{conv}(A) \).

The convexity condition ensures that from the perspective of programming, nondeterministic choices can always be refined by probabilistic choices. The convex powerdomain \( \mathcal{P}D(X) \) over the probabilistic powerdomain \( D(X) \) is then defined as convex lenses in \( D(X) \) with the Egli-Milner order \( A \sqsubseteq_P B \overset{\text{def}}{=} A \sqsubseteq B \wedge \uparrow A \sqsubseteq B \).

The following theorems provide a characterization of the convex powerdomains.

Proposition 3.5 ([47,62]) The poset \( \langle \mathcal{P}D(X), \sqsubseteq_P \rangle \) forms a coherent dcpo. It admits a least element \( \perp_P \overset{\text{def}}{=} \{ \perp_D \} \). For \( r_1, r_2 \in [0,1] \) satisfying \( r_1 + r_2 \leq 1 \), we define \( \Delta_1 + \Delta_2 \overset{\text{def}}{=} \overline{C \cap \uparrow C} \) where \( C = \{ r_1 \cdot \Delta_1 + r_2 \cdot \Delta_2 \mid \Delta_1 \in A \wedge \Delta_2 \in B \} \). Then the probabilistic-choice operation is lifted to a Scott-continuous operation as \( A \oplus_P B \overset{\text{def}}{=} p \cdot A + (1-p) \cdot B \). Moreover, it carries a Scott-continuous semilattice operation, defined as \( A \sqcup_P B \overset{\text{def}}{=} \overline{C \cap \uparrow C} \) where \( C = \text{conv}(A \cup B) \). Intuitively, the formal union operation stands for nondeterministic choices.

Proposition 3.6 ([62]) Every function \( g : X \to \mathcal{P}D(X) \) can be lifted to a unique Scott-continuous linear (in the sense that it preserves lifted probabilistic-choice) map \( \hat{g} : \mathcal{P}D(X) \to \mathcal{P}D(X) \) preserving formal unions.

Example 3.7 Consider the following program \( P \) where \( * \) can be refined by any deterministic condition involving the program variable \( t \):

\[
\text{if } * \text{ then } t := t + 1 \text{ else } t := t - 1 \fi
\]

and we want to assign a semantic object to it from \( X \to \mathcal{P}D(X) \), where the state space \( X = \mathbb{Q} \) represents the value of \( t \). Fix an input \( t \in \mathbb{Q} \). The data actions \( t := t + 1 \) and \( t := t - 1 \) then take the input to singletons \( \{ \delta(t+1) \} \) and \( \{ \delta(t-1) \} \), respectively, in the powerdomain \( \mathcal{P}D(\mathbb{Q}) \). Thus the nondeterministic-choice is interpreted as \( \{ \delta(t+1) \} \sqcup_P \{ \delta(t-1) \} \), which is \( \{ r \cdot \delta(t+1) + (1-r) \cdot \delta(t-1) \mid r \in [0,1] \} \), for a given \( t \in \mathbb{Q} \).

4 Nondeterminism-First

In this section, we develop a new model of nondeterminism—the nondeterminism-first approach, which resolves nondeterministic choices prior to program inputs—in a domain-theoretic way. This model is inspired by reasoning about a program’s behavior on different inputs (as mentioned in §1), which requires nondeterministic functions to be treated as a family of transformers (i.e., an element of \( \phi(X \to X) \)) instead of a set-valued map (i.e., an element of \( X \to \phi(X) \)). As will be shown in this section, with nondeterminism-first, \( t := t + 1 \) and \( t := t - 1 \) are assigned semantic objects \( \{ \lambda x.\delta(t+1) \} \) and \( \{ \lambda x.\delta(t-1) \} \), respectively.

We first introduce kernels, then propose a new notion of generalized convexity (g-convexity, for short), and finally develop a powerdomain for nondeterminism-first. Complete proofs are included in appendix A.

4.1 A Powerdomain for Sub-Probability Kernels

Let \( X \) be a nonempty countable set. A function \( \kappa : X \to \mathcal{P}(X) \) is called a (sub-probability) kernel. Intuitively, a kernel maps an input state to a distribution over output states. The set of all such kernels is denoted by \( \mathcal{K}(X) = X \to \mathcal{P}(X) \). Kernels are ordered pointwise, i.e., \( \kappa_1 \sqsubseteq_K \kappa_2 \overset{\text{def}}{=} \forall x \in X : \kappa_1(x) \sqsubseteq \kappa_2(x) \).

Theorem 4.1 The poset \( \langle \mathcal{K}(X), \sqsubseteq_K \rangle \) forms a coherent dcpo, with \( \perp_K \overset{\text{def}}{=} \lambda x.\perp_D \) as its least element.

Let \( \mathcal{W}(X) = X \to [0,1] \) be the set of functions from \( X \) to the interval \( [0,1] \). We denote the pointwise comparison by \( \leq \) and the constant function by \( r \) for any \( r \in [0,1] \). If \( \kappa \) is a kernel and \( \phi \in \mathcal{W}(X) \), we write \( \phi \cdot \kappa \) for
the kernel \( \langle x, \phi(x) \rangle \cdot (x) \). If \( \kappa_1, \kappa_2 \) are kernels and \( \phi_1, \phi_2 \in \mathcal{W}(X) \) such that \( \phi_1 + \phi_2 \leq 1 \), we write \( \phi_1 \cdot \kappa_1 + \phi_2 \cdot \kappa_2 \) for the kernel \( \lambda x. \phi_1(x) \cdot (x) + \phi_2(x) \cdot (x) \cdot \kappa_2 \). More generally, if \( \{ \kappa_i \}_{i \in \mathbb{N}}^+ \) is a sequence of kernels, and \( \{ \phi_i \}_{i \in \mathbb{N}}^+ \) is a sequence of functions in \( \mathcal{W}(X) \) such that \( \sum_{i=1}^{\infty} \phi_i \leq 1 \), we write \( \sum_{i=1}^{\infty} \phi_i \cdot \kappa_i \) for the kernel \( \bigcup_{i \in \mathbb{N}} \sum_{i=1}^{n} \phi_i \cdot \kappa_i \). Then we define \textit{conditional-choice} of kernels \( \{}_f \cdot \kappa_2 \) on a function \( \phi \in \mathcal{W}(X) \) as \( \kappa_1 \cdot \ placer_1 \cdot \kappa_2 = \phi \cdot \kappa_1 + (1 - \phi) \cdot \kappa_2 \).

We define the \textit{composition of kernels} \( \kappa_1, \kappa_2 \) as \( \kappa_1 \otimes \kappa_2 \triangleq \lambda x. \lambda x'' . \sum_{x' \in X} \kappa_1(x)(x') \cdot \kappa_2(x')(x'') \).

**Lemma 4.2**

\( \text{(i) The conditional-choice operation } \cdot \textit{ is Scott-continuous for all } \phi \in \mathcal{W}(X). \)

\( \text{(ii) The composition operation } \otimes \textit{ is Scott-continuous.} \)

### 4.2 Generalized Convexity

As shown in §3.3, nondeterminism-last is captured by convex sets of distributions. However, a more complicated notion of convexity is needed to develop nondeterminism-first semantics over kernels. Let \( X \) be a nonempty countable set. Every semantic object should be closed under the conditional-choice \( \cdot \) for every function \( \phi \in \mathcal{W}(X) \). Recall that the definition \( \kappa_1 \cdot \phi \cdot \kappa_2 \triangleq \phi \cdot \kappa_1 + (1 - \phi) \cdot \kappa_2 \) is similar to a convex combination, except that the coefficients might not only be constants, but can also depend on the state. We formalize the idea by defining a notion of \( g \)-\textit{convexity}.

**Definition 4.3** A subset \( A \) of \( \mathcal{K}(X) \) is called g-\textit{convex}, if for all sequences \( \{ \kappa_i \}_{i \in \mathbb{N}}^+ \subseteq \mathcal{K}(X) \) and \( \{ \phi_i \}_{i \in \mathbb{N}}^+ \subseteq \mathcal{W}(X) \) such that \( \sum_{i=1}^{\infty} \phi_i = 1 \), then \( \sum_{i=1}^{\infty} \phi_i \cdot \kappa_i \) is contained in \( A \).

We now show that some domain-theoretic operations preserve g-convexity.

**Lemma 4.4** Let \( A \) be a g-convex subset of \( \mathcal{K}(X) \). Then

\( \text{(i) The saturation } \uparrow A \text{ and the lower closure } \downarrow A \text{ are g-convex.} \)

\( \text{(ii) The closure } A \text{ is g-convex.} \)

The g-convex hull of a subset \( A \) of \( \mathcal{K}(X) \) is the smallest g-convex set containing \( A \) as a subset, denoted by \( gconv(A) \). Intuitively, \( gconv(A) \) enriches \( A \) to become a reasonable semantic object that is closed under arbitrary conditional-choice.

Following are some properties of the \( gconv(\cdot) \) operator.

**Lemma 4.5** Suppose that \( A \) and \( B \) are g-convex subsets of \( \mathcal{K}(X) \). Then \( \{ \kappa_1 \cdot \rho \mid \kappa \in A \land \rho \in B \} \) is g-convex for all functions \( \rho \in \mathcal{W}(X) \).

**Corollary 4.6** If \( A \) and \( B \) are g-convex, then \( gconv(A \cup B) \) is given by \( \{ \kappa_1 \cdot \phi \cdot \kappa_2 \mid \kappa_1 \in A \land \kappa_2 \in B \land \phi \in \mathcal{W}(X) \} \).

**Proof.** It is straightforward to show that \( gconv(A \cup B) \) is a superset of \( \{ \kappa_1 \cdot \phi \cdot \kappa_2 \mid \kappa_1 \in A \land \kappa_2 \in B \land \phi \in \mathcal{W}(X) \} \). Then it suffices to show this set is indeed g-convex. We conclude the proof by Lem. 4.5. \( \square \)

For a finite subset \( F \) of \( \mathcal{K}(X) \), as an immediate corollary of Cor. 4.6, by a simple induction we know that \( gconv(F) = \{ \sum_{\kappa \in F} \phi_\kappa \cdot \kappa \mid \{ \phi_\kappa \}_{\kappa \in F} \subseteq \mathcal{W}(X) \land \sum_{\kappa \in F} \phi_\kappa = 1 \} \).

**Lemma 4.7** For an arbitrary \( A \subseteq \mathcal{K}(X) \), we have

\[ gconv(A) = \left\{ \sum_{i=1}^{\infty} \phi_i \cdot \kappa_i \mid \{ \kappa_i \}_{i \in \mathbb{N}}^+ \subseteq A \land \{ \phi_i \}_{i \in \mathbb{N}}^+ \subseteq \mathcal{W}(X) \land \sum_{i=1}^{\infty} \phi_i = 1 \right\}. \]

**Lemma 4.8**

\( \text{(i) For an arbitrary } A \subseteq \mathcal{K}(X) \text{, we have } gconv(A) = gconv(A). \)

\( \text{(ii) If } \{ A_i \}_{i \in I} \text{ is a directed collection of Scott-closed subsets of } \mathcal{K}(X) \text{ ordered by set inclusion, then } gconv \left( \bigcup_{i \in I} A_i \right) = \bigcup_{i \in I} gconv(A_i). \)

**Lemma 4.9** Let \( A \) and \( B \) be Scott-compact g-convex subsets of \( \mathcal{K}(X) \). Then \( gconv(A \cup B) \) is also Scott-compact.

We now turn to discuss some separation properties for g-convexity.

**Lemma 4.10**

\( \text{(i) If } A \subseteq \mathcal{K}(X) \text{ is g-convex, then for all } x, \{ \kappa(x) \mid \kappa \in A \} \text{ is convex.} \)

\( \text{(ii) If } A \subseteq \mathcal{K}(X) \text{ is Scott-compact, then for all } x, \{ \kappa(x) \mid \kappa \in A \} \text{ is Scott-compact.} \)
(iii) If $A \subseteq K(X)$ is Scott-closed, then for all $x$, $\{\kappa(x) \mid \kappa \in A\}$ is Scott-closed.

**Lemma 4.11** Let us consider subsets of $K(X)$. Suppose that $K$ is a Scott-compact $g$-convex set and $A$ is a nonempty Scott-closed $g$-convex set that is disjoint from $K$. Then they can be separated by a $g$-convex $K$-

**Lemma 4.12** If $K \subseteq K(X)$ is nonempty and Scott-compact, then $gconv(K)$ is Scott-compact.

### 4.3 A $g$-convex Powerdomain for Nondeterminism-First

From the literature, a Plotkin powertheory [1] is defined by one binary operation $\cup$, called formal union, and the following laws: (i) $A \cup B = B \cup A$, (ii) $(A \cup B) \cup C = A \cup (B \cup C)$, and (iii) $A \cup A = A$, for all objects $A, B, C$ in the powerdomain. Intuitively, the formal union $\cup$ represents nondeterministic-choice. Moreover, the formal union induces a semilattice ordering: $A \leq B$ if $A \cup B = B$. The semilattice ordering is usually not interesting from the perspective of domain theory, however, it is instrumental to describe the relation between conditional-choice and nondeterministic-choice—$A \diamond B \leq A \cup B$ for all semantic objects $A, B$—a nondeterministic-choice should abstract an arbitrary (possibly probabilistic) conditional-choice.

Let $X$ be a nonempty countable set. As nondeterminism-first interprets programs as collections of input-output transformers, we hope to develop a powerdomain on $K(X)$, i.e., kernels on $X$. To achieve this goal, we need to (i) identify a collection of well-formed semantic objects in $K(X)$, (ii) lift conditional-choice $\diamond$ and composition $\otimes$ on kernels to the powerdomain properly, and (iii) prove the powerdomain is a dcpo and the operations are Scott-continuous.

Inspired by studies on convex powerdomains [1,47,62], we start with the following collection

$$G(K(X)) \overset{\text{def}}{=} \{S \subseteq K(X) \mid S \text{ a nonempty } g\text{-convex lens}\}$$

The following theorem establishes a characterization of $g$-convex powerdomains.

**Theorem 4.13** $(G(K(X)), \sqsubseteq)$ forms a dcpo, with a least element $\bot_G \overset{\text{def}}{=} \{\bot_K\}$.

We now lift conditional-choice $\diamond$ (where $\phi \in \mathbb{W}(X)$) and composition $\otimes$ for kernels to the powerdomain $G(K(X))$ as follows.

$$A \diamond_G B \overset{\text{def}}{=} \{a \diamond b \mid a \in A \land b \in B\} \cup \{a \diamond b \mid a \in A \land b \in B\}$$

$$A \otimes_G B \overset{\text{def}}{=} gconv(\{a \otimes b \mid a \in A \land b \in B\}) \cup gconv(\{a \otimes b \mid a \in A \land b \in B\})$$

The operations construct nonempty $g$-convex lenses by Lemmas 4.4 and 4.12. As conditional-choice and composition operations are Scott-continuous on kernels, the lifted operations are also Scott-continuous in the powerdomain.

**Lemma 4.14** The operations $\diamond_G$ and $\otimes_G$ are Scott-continuous for all $\phi \in \mathbb{W}(X)$.

Finally, we define a formal union operation $\sqcup_G$ as in Prop. 3.5 to interpret nondeterministic-choice as $A \sqcup_G B \overset{\text{def}}{=} C \cap \uparrow C$ where $C$ is $gconv(A \cup B)$.

**Lemma 4.15** The formal union $\sqcup_G$ is a Scott-continuous semilattice operation on $G(K(X))$.

**Example 4.16** Recall the probabilistic program $P$ in Ex. 3.7:

\[
\text{if } * \text{ then } t := t + 1 \text{ else } t := t - 1 \text{ fi}
\]

the state space $X$ is $\mathbb{Q}$, and we want to show that for any probabilistic refinement $P_r$ of $P$ (i.e., $*$ is refined by $\text{prob}(r)$), for input values $t_1, t_2$ of $t$, we have $\mathbb{E}_{t_1 \sim \Delta_1, t_2 \sim \Delta_2}[t_1 - t_2] = t_1 - t_2$, where the program $P_r$ ends up with a distribution $\Delta_1$ starting with $t = t_1$ and $\Delta_2$ with $t = t_2$.

With the $g$-convex powerdomain $G(K(X))$ for nondeterminism-first, $t := t + 1$ and $t := t - 1$ are assigned semantic objects $\{\lambda t.\delta(t + 1)\}$ and $\{\lambda t.\delta(t - 1)\}$, respectively. Thus the nondeterministic-choice is interpreted as a subset of $\{\lambda t.\delta(t + 1)\} \sqcup_G \{\lambda t.\delta(t - 1)\}$, which is $\{\kappa_r \mid r \in [0, 1]\}$, where $\kappa_r = \lambda t.\delta(t + 1) + (1 - r) \cdot \delta(t - 1)$ is the kernel for the deterministic refinement $P$, of $P$. Therefore for every $r \in [0, 1]$, we have $\mathbb{E}_{t_1 \sim \Delta_1, t_2 \sim \Delta_2}[t_1 - t_2] = \mathbb{E}_{t_1 \sim \kappa_r(t_1), t_2 \sim \kappa_r(t_2)}[t_1 - t_2] - \mathbb{E}_{t_1 \sim \kappa_r(t_1), t_2 \sim \kappa_r(t_2)}[t_2] = (r(t_1 + 1) + (1 - r)(t_1 - 1)) - (r(t_2 + 1) + (1 - r)(t_2 - 1)) = t_1 - t_2$.

In contrast, if we started with the convex powerdomain $\mathbb{P}(X)$ reviewed in §3.3 for nondeterminism-last, we would obtain the semantic object $\lambda t.\{r \cdot \delta(t + 1) + (1 - r) \cdot \delta(t - 1) \mid r \in [0, 1]\}$ for the program $P$, as shown.
in Ex. 3.7. Now the refinements of \( P \) include some \( \kappa \) such that \( \kappa(t_1) = 0.5 \cdot \delta(t_1 + 1) + 0.5 \cdot \delta(t_1 - 1) \) and \( \kappa(t_2) = 0.3 \cdot \delta(t_2 + 1) + 0.7 \cdot \delta(t_2 - 1) \), thus we are not able to prove the claim \( \mathbb{E}[t'_1 - t'_2] = t_1 - t_2 \).

5 An Algebraic Denotational Semantics

The operational semantics described in §2.2 presents a reasonable model for evaluating single-procedure probabilistic programs without nondeterminism. In this section, we develop a general denotational semantics for CFHGs (introduced in §2.1) of multi-procedure probabilistic programs with nondeterminism. The semantics is algebraic in the sense that it could be instantiated with different concrete models of nondeterminism, e.g., nondeterminism-last reviewed in §3.3, as well as nondeterminism-first developed in §3.4. We will show the denotational semantics is equivalent to the operational semantics in §2.2 if we suppress procedure calls and nondeterminism in the programming model.

5.1 A Fixpoint Semantics based on Markov Algebras

The algebraic denotational semantics is obtained by composing \( \text{Ctrl}(e) \) operations along hyper-edges. The semantics of programs is determined by an interpretation, which consists of two parts: (i) a semantic algebra, which defines a set of possible program meanings, and which is equipped with sequencing, conditional-choice, and nondeterministic-choice operators to compose these meanings, and (ii) a semantic function, which assigns a meaning to each data action \( act \in \text{Act} \). The semantic algebras that we use are Markov algebras introduced in [64]:

**Definition 5.1** A Markov algebra (MA) over a set \( \text{Cond} \) of deterministic conditions is a 7-tuple \( \mathcal{M} = \langle M, \sqsubseteq_M , \otimes_M , \diamond_M , \oplus_M , \perp_M , 1_M \rangle \), where \( (M, \sqsubseteq_M) \) forms a dcpo with \( \perp_M \) as its least element; \( (M, \otimes_M, 1_M) \) forms a monoid (i.e., \( \otimes_M \) is an associative binary operator with \( 1_M \) as its identity element); \( \diamond_M \), is a binary operator parametrized by a condition \( \varphi \in \text{Cond} \); \( \sqcup_M \) is idempotent, commutative, associative and for all \( a, b \in M \) and \( \varphi \in \text{Cond} \) we have \( a \sqcup_M b \leq_M a \sqcup_M b \) where \( \leq_M \) is the semilattice ordering induced by \( \sqcup_M \) (i.e., \( a \leq_M b \) if \( a \sqcup_M b = b \); and \( \otimes_M, \diamond_M, \oplus_M \) are Scott-continuous.

**Example 5.2** Let \( \Omega \) be a nonempty countable set of program states and \( \text{Cond} \) be a set of deterministic conditions, the definition and meaning of which are given in §2.1 and §2.2.

(i) The convex powerdomain \( \mathcal{PD}(\Omega) \) admits an MA \( \langle \Omega \rightarrow \mathcal{PD}(\Omega), \sqsubseteq_P , \otimes_P , \diamond_P , \oplus_P , \perp_P , 1_P \rangle \), where \( \sqsubseteq_P , \otimes_P , \perp_P \) are pointwise extensions of \( \sqsubseteq_M, \otimes_M , \perp_M \), defined in §3.3, and \( g \otimes_P h \triangleq \hat{h} \circ g \) where \( \hat{h} \) is given by Prop. 3.6, \( g \varphi \hat{P} h \triangleq \lambda \omega . g(\omega) \| P(\omega) \| P h(\omega) \), as well as \( 1_P \triangleq \lambda \omega . \{ \delta(\omega) \} \).

(ii) The g-convex powerdomain \( \mathcal{GK}(\Omega) \) admits an MA \( \langle \mathcal{GK}(\Omega), \sqsubseteq_G , \otimes_G , \diamond_G , \oplus_G , \perp_G , 1_G \rangle \), where \( \sqsubseteq_G , \otimes_G , \diamond_G , \oplus_G , \perp_G , 1_G \) come from §4.3, and \( 1_G \triangleq \{ \lambda \omega . \delta(\omega) \} \).

**Definition 5.3** An **interpretation** is a pair \( \mathcal{I} = \langle M, \llbracket \cdot \rrbracket^\mathcal{I} \rangle \), where \( M \) is an MA and \( \llbracket \cdot \rrbracket^\mathcal{I} : \text{Act} \rightarrow M \). We call \( M \) the **semantic algebra** of the interpretation and \( \llbracket \cdot \rrbracket^\mathcal{I} \) the **semantic function**.

**Example 5.4** We can lift the interpretation of data actions defined in Fig. 3 to semantic functions with respect to convex or g-convex powerdomains—\( \mathcal{I} = \langle \mathcal{P}(\Omega), \llbracket \cdot \rrbracket^\mathcal{I} \rangle \) with \( \llbracket \text{act} \rrbracket^\mathcal{I} \triangleq \lambda \omega . \{ \llbracket \text{act} \rrbracket (\omega) \} \) and \( \mathcal{I} = \langle \mathcal{GK}(\Omega), \llbracket \cdot \rrbracket^\mathcal{I} \rangle \) with \( \llbracket \text{act} \rrbracket^\mathcal{I} \triangleq \{ \llbracket \text{act} \rrbracket (\omega) \} \).

Given a probabilistic program \( P = \{ H_i \}_{1 \leq i \leq n} \) where each \( H_i = \langle V_i, E_i, v_i^{\text{entry}}, v_i^{\text{exit}} \rangle \) is a CFHG, and an interpretation \( \mathcal{I} = \langle M, \llbracket \cdot \rrbracket^\mathcal{I} \rangle \), we define \( \mathcal{I}[P] \) to be the interpretation of the probabilistic program, as the least fixpoint of the function \( \mathcal{F} \), which is defined as

\[
\lambda S . \lambda u . \bigcup_{1 \leq i \leq n} \left\{ \text{Ctrl}(e)(S(u_1), \ldots, S(u_k)) \mid e = \langle v, \{ u_1, \ldots, u_k \} \rangle \in E \right\}
\]

where \( \text{Ctrl}(e) \) for different kinds of control-flow actions is defined as follows:

\[
\text{seq}\llbracket \text{act} \rrbracket(S_1) \triangleq \llbracket \text{act} \rrbracket \otimes_M S_1, \quad \text{cond} \llbracket \varphi \rrbracket(S_1, S_2) \triangleq S_1 \varphi \diamond_M S_2, \quad \text{call} \llbracket i \rightarrow j \rrbracket(S_1) \triangleq S \llbracket \varphi_j \rrbracket \otimes_M S_1.
\]

---

3 The conditional-choice is actually interpreted as \( \llbracket \cdot \rrbracket \otimes_M \cdot \) in the powerdomain.
The least fixpoint of $F_P$ exists by Prop. 3.1 as well as the following lemma. Hence the semantics of the procedure $H_i$ is given by $\llbracket H_i \rrbracket_{ds} \overset{\text{def}}{=} (\text{lfp}_{\downarrow M}^{= M} F_P)(v_i^\text{entry})$.

**Lemma 5.5** The function $F_P$ is Scott-continuous on the dcpo $\langle V \to M, \subseteq_M \rangle$ with $\downarrow_M \overset{\text{def}}{=} \lambda v.\downarrow_M$ as the least element, where $\subseteq_M$ is the pointwise extension of $\subseteq_M$.

**Proof.** Appeal to the Scott-continuity of the operations $\otimes_M$, $\varphi_M$, and $\triangleright_M$. \hfill $\Box$

### 5.2 An Equivalence Result

To justify the denotational semantics proposed in §5.1, we go back to the restricted programming language used to define the operational semantics in §2.2. If we suppress the features of multi-procedure and nondeterminism, we should end up with a semantics that is equivalent to the operational semantics $\llbracket \cdot \rrbracket_{os}$.

**Lemma 5.6** Let $P = (V, E, v^\text{entry}, v^\text{exit})$ be a deterministic single-procedure probabilistic program.

(i) If we interpret $P$ using $\mathcal{P} = \langle P^D(\Omega), \llbracket \cdot \rrbracket_{\mathcal{P}} \rangle$, we will have $\llbracket P \rrbracket_{ds} = \lambda \omega.\llbracket P \rrbracket_{os}(\omega)$.

(ii) If we interpret $P$ using $\mathcal{G} = \langle G^\Delta(\Omega), \llbracket \cdot \rrbracket_{\mathcal{G}} \rangle$, we will have $\llbracket P \rrbracket_{ds} = (\llbracket P \rrbracket_{os})_{\Delta}$.

**Proof.** Recall the definition $\llbracket P \rrbracket \overset{\text{def}}{=} \lambda \omega.\sup_{n \in \mathbb{N}} \{\Delta \mid (\omega^\text{entry}, \omega) \to_n \Delta\}$. On the other hand, the fixpoint $(\text{lfp}_{\downarrow_M}^{= M} F_P)(v_i^\text{entry})$ is obtained by $\left\uparrow_{\downarrow_M}^{= M} F_P(v_i^\text{entry})$ by Prop. 3.1. The proof proceeds by induction on $n$. \hfill $\Box$

### 6 Application: Static Analysis for Probabilistic Programs with Nondeterminism

A lot of recent studies on probabilistic programming focus on rigorous reasoning about probabilistic programs (e.g., [52, 53, 39, 4, 17, 11, 57, 12, 10, 34, 28, 13, 38, 55, 5, 9]). In this section, we discuss an application of the new denotational semantics as the concrete semantics of a static-analysis framework for probabilistic programs. More details about the static analysis and its soundness proof can be found in a companion paper [64].

**Definition 6.1** A pre-Markov algebra (PMA) over a set $\text{Cond}$ of deterministic conditions is a 7-tuple $M^\sharp = \langle M, \subseteq_M, \otimes_M, \varphi_M, \triangleright_M, 1_M \rangle$, which is essentially an MA, except that $\langle M, \subseteq_M \rangle$ forms a complete lattice, and $\otimes_M$, $\varphi_M$, and $\triangleright_M$ are only required to be monotone.

Intuitively, PMAs specify abstract semantics used in static analyses. We can define interpretations with respect to PMAs in the same way, except that we obtain the least fixpoint $\mathcal{F}^\sharp[P]$ of the function $F_P$ by the Knaster-Tarski theorem, given a probabilistic program $P$ and an interpretation $\mathcal{F} = \langle M^\sharp, \llbracket \cdot \rrbracket_{\mathcal{F}} \rangle$.

**Definition 6.2** A probabilistic over-abstraction (resp., under-abstraction) from an MA $C$ (i.e., a concrete semantics such as $P^D(\Omega)$ and $G^\Delta(\Omega)$) to a PMA $\mathcal{Y}$ is a concretization mapping, $\gamma : Y \to C$, such that

- $\downarrow_C \subseteq C \gamma(\downarrow_Y)$ (resp., $\gamma(\downarrow_Y) \subseteq C \downarrow_C$),
- $1_C \subseteq C \gamma(1_Y)$ (resp., $\gamma(1_Y) \subseteq C 1_C$),
- for all $Q_1, Q_2 \in Y$, $\gamma(Q_1) \otimes C \gamma(Q_2) \subseteq C \gamma(Q_1 \otimes_Y Q_2)$ (resp., $\gamma(Q_1 \otimes_Y Q_2) \subseteq C \gamma(Q_1) \otimes C \gamma(Q_2)$),
- for all $Q_1, Q_2 \in Y$, $\gamma(Q_1) \varphi_C \gamma(Q_2) \subseteq C \gamma(Q_1 \varphi_Y Q_2)$ (resp., $\gamma(Q_1 \varphi_Y Q_2) \subseteq C \gamma(Q_1) \varphi_C \gamma(Q_2)$), and
- for all $Q_1, Q_2 \in Y$, $\gamma(Q_1) \triangleright_C \gamma(Q_2) \subseteq C \gamma(Q_1) \triangleright_C \gamma(Q_2)$.

A probabilistic abstraction leads to a sound analysis:

**Theorem 6.3** Let $\mathcal{C}$ and $\mathcal{Y}$ be interpretations over an MA $C$ and a PMA $\mathcal{Y}$; let $\gamma$ be a probabilistic over-abstraction (resp., under-abstraction) from $C$ to $\mathcal{Y}$; and let $P$ be an arbitrary program. If for all data actions $\text{act}$, $\llbracket \text{act} \rrbracket_{\mathcal{C}} \subseteq C \gamma(\llbracket \text{act} \rrbracket_{\mathcal{Y}})$ (resp., $\gamma(\llbracket \text{act} \rrbracket_{\mathcal{Y}}) \subseteq C \llbracket \text{act} \rrbracket_{\mathcal{C}}$), then we have $\mathcal{C}[P] \subseteq C \gamma(\mathcal{Y}[P])$ (resp., $\gamma(\mathcal{Y}[P]) \subseteq C \mathcal{C}[P]$).

### 7 Discussion

#### 7.1 Continuous Distributions

One of the most important features of probabilistic programming is continuous probability distributions over real numbers, such as Gaussian distributions. Notions from measure theory, such as measures and kernels, are
extensively used to model continuous distributions in probabilistic programming. Kozen studied the relation between deterministic probabilistic programs and continuous distributions via a metric on measures [43]. Many approaches use probability kernels [44,58], sub-probability kernels [8], and s-finite kernels [59,7]. A different approach uses measurable functions $A \rightarrow D(\mathbb{R}_{\geq 0} \times B)$ where $D(S)$ stands for the set of all probability measures on $S$ [60]. For higher-order languages, Jones and Plotkin [35,36] have developed a probabilistic powerdomain that consists of continuous evaluations, which are a reformulation of distributions in domain theory, on a state space. They show that the powerdomain can be used to solve recursive domain equations. Smolka et al. [58] study the semantics of probabilistic networks. Ehrhard et al. [20] provide a Cartesian-closed category on stable and measurable maps between cones, and use it to give a semantics for probabilistic PCF.

However, those measure-theoretic developments do not work properly when nondeterminism comes into the picture. To overcome this challenge, people have been adapting domain-theoretic results. McIver and Morgan build a Plotkin-style powerdomain over probability distributions on a discrete state space [47,48]. Mislove et al. [50,51] study powerdomain constructions for probabilistic CSP. Tix et al. [62] generalize McIver and Morgan’s results to continuous state spaces, and construct three powerdomains for the extended probabilistic powerdomains. Although there has been a lot of work on this direction, one has to keep in mind that the domain-theoretic notion of “continuous” distributions is different from the notion in measure theory—instead, the domain-theoretic studies are focused on computable distributions. In other words, real numbers are realized by some computable models, such as partial reals [21]. These models would become unsatisfactory when one wants to observe a random value drawn from a continuous distribution, e.g., the meaning of $x := \text{Normal}(0, 1)$: if $x = 0$ then $\cdots$ fi is not expressible. We leave the semantic development of combining nondeterminism and continuous distributions (from a measure-theoretic perspective) for future work.

7.2 Higher-Order Functions

In functional programming, higher-order functions are functions that can take functions as arguments, as well as return a function as a result. Some probabilistic programming languages, such as Church [30], are indeed functional programming languages and can express higher-order functions. While operational models for probabilistic functional programming have been proposed [8], developing a denotational semantics for higher-order probabilistic programming has been an open problem for years.

The major challenge is to propose a Cartesian-closed category for semantic objects of probabilistic programming. Intuitively, the Cartesian-closure property ensures that if type $A$ and type $B$ are two objects in the category, then the function space $B^A$ (i.e., an object for the arrow type $A \rightarrow B$) is also contained in the category. The category of measures is clearly not Cartesian-closed; a lot of probabilistic powerdomains also do not admit a Cartesian-closed category [37]. Recently, Heunen et al. [32] propose quasi-Borel measures for higher-order functions in probabilistic programming. The measure-theoretic approach is further extended by Vákár et al. [63] to support recursive types. However, it is unclear how to model nondeterminism in the framework of quasi-Borel measures. We leave the combination of nondeterminism and higher-order functions for future work.

8 Conclusion

We have developed a framework for denotational semantics of low-level probabilistic programs with unstructured control-flow, general recursion, and nondeterminism, represented by control-flow hyper-graphs. The semantics is algebraic and it can be instantiated with different models of nondeterminism. We have demonstrated two instantiations with nondeterminism-first and nondeterminism-last, respectively. We have proposed a powerdomain for nondeterminism-first that consists of collections of kernels and enjoys generalized convexity. As an application, we have reviewed a static-analysis framework for probabilistic programs, which has been proposed in a companion paper.

In the future, we plan to combine continuous distributions and higher-order functions with nondeterminism in our semantics framework. We will also work on models of nondeterminism, especially nondeterminism-first, and investigate its connection with relational reasoning. Another research direction is to develop more formal reasoning techniques based on the denotational semantics.

Acknowledgement

This work was supported, in part, by a gift from Rajiv and Ritu Batra; by AFRL under DARPA MUSE award FA8750-14-2-0270, DARPA STAC award FA8750-15-C-0082 and DARPA AA award FA8750-18-C-0092; by ONR under grant N00014-17-1-2889; by NSF under SaTC award 1801369, SHF grant 1812876, and CAREER award 1845514; and by the UW-Madison OVRGE with funding from WARF.
References


A Proofs

A.1 Thm. 4.1

Proof. We equip $X$ with the discrete topology. We define $X_\perp = X \cup \{\perp\}$ with a distinguished least element $\perp$ and thus $X_\perp$ is a flat domain. Then $X_\perp$ is a bounded-complete domain. The Scott-compact subsets of $X_\perp$ are precisely finite subsets of $X$ and all subsets that contain $\perp$. Thus $X_\perp$ is coherent. By [1, Ex. 4.3.11.14], we know that $X_\perp$ is an FS-domain.

By Prop. 3.3 we know that $\mathcal{D}(X)$ is coherent. Moreover, $\mathcal{D}(X)$ is also bounded-complete. Thus $\mathcal{D}(X)$ is an FS-domain. By [1, Thm. 4.2.11], we know that $[X_\perp \to \mathcal{D}(X)]$ is an FS-domain.

Let $s \overset{\text{def}}{=} \lambda f.f$ and $r \overset{\text{def}}{=} \lambda g.\lambda x.\text{if } x = \perp \text{ then } \perp \text{ else } g(x)$. Then $s : [X_\perp \to \mathcal{D}(X)] \to [X_\perp \to \mathcal{D}(X)]$, $r : [X_\perp \to \mathcal{D}(X)] \to [X_\perp \to \mathcal{D}(X)]$, and $r \circ s$ is the identity on $[X_\perp \to \mathcal{D}(X)]$, where $[A \to B]$ stands for continuous functions from a dcpo $A$ to a dcpo $B$ that preserve the least element. Hence $[X_\perp \to \mathcal{D}(X)]$ is a retract of $[X_\perp \to \mathcal{D}(X)]$. By [1, Prop. 4.2.12], we know that $[X_\perp \to \mathcal{D}(X)]$ is also an FS-domain.

For any $f$ in $[X \to \mathcal{D}(X)]$, we could define a function $g \overset{\text{def}}{=} \lambda x.\text{if } x = \perp \text{ then } \perp \text{ else } f(x)$. For any $g$ in $[X_\perp \to \mathcal{D}(X)]$, we could define a function $f \overset{\text{def}}{=} \lambda x.g(x)$. Thus $[X \to \mathcal{D}(X)]$ is homeomorphic to $[X_\perp \to \mathcal{D}(X)]$, and we know that $[X \to \mathcal{D}(X)]$ is also an FS-domain. By [1, Thm. 4.2.18], we know that $[X \to \mathcal{D}(X)]$ is coherent. Because the topology on $X$ is discrete, $[X \to \mathcal{D}(X)]$ is precisely $X \to \mathcal{D}(X)$. Thus we conclude that $\mathcal{K}(X)$ is coherent. \hfill \Box

A.2 Lem. 4.2

Proof.

(i) Monotonicity is trivial. It then suffices to show that for all directed set $A \subseteq \mathcal{K}(X)$, $\phi \cdot (\bigcup A) = \bigcup_{\kappa \in A} \phi \cdot \kappa$.

Let $\kappa' \overset{\text{def}}{=} \bigcup A$. We conclude the proof by $\bigcup_{\kappa \in A} \phi(x) \cdot \kappa(x) = \phi(x) \cdot \bigcup_{\kappa \in A} \kappa(x) = \phi(x) \cdot (\bigcup A)(x) = \phi(x) \cdot \kappa'(x)$ for any $x$.

(ii) Monotonicity is trivial.

Left-Scott-continuity. For all directed set $A \subseteq \mathcal{K}(X)$ and all $\rho \in \mathcal{K}(X)$, we want to show that $(\bigcup A) \otimes \rho = \bigcup_{\kappa \in A} \kappa \otimes \rho$. Let $\kappa' \overset{\text{def}}{=} \bigcup A$. Then it is sufficient to show that for all $x$ and $x''$, $\int \kappa'(x)(dx')\rho(x')(x'') = \bigcup_{\kappa \in A} \kappa(x)(dx')\rho(x')(x'')$. Because $A$ is directed and $\mathcal{K}(X)$ is ordered pointwise, $\{\kappa(x) \mid \kappa \in A\}$ is also directed in $\mathcal{D}(X)$. By [36, Thm. 3.3], the right-hand-side is equal to $\int \bigcup_{\kappa \in A} \kappa(x)(dx')\rho(x')(x'')$. We conclude the proof by $\kappa'(x) = \bigcup_{\kappa \in A} \kappa(x)$ by the definition of $\kappa'$.

Right-Scott-continuity. For all directed set $A \subseteq \mathcal{K}(X)$ and all $\rho \in \mathcal{K}(X)$, we want to show that $\rho \otimes (\bigcup A) = \bigcup_{\kappa \in A} \rho \otimes \kappa$. Let $\kappa' \overset{\text{def}}{=} \bigcup A$. Then it is sufficient to show that for all $x$ and $x''$, $\int \rho(x)(dx')\kappa'(x')(x'') = \bigcup_{\kappa \in A} \int \rho(x)(dx')\kappa(x'(x''))$. Because $A$ is directed and $\mathcal{K}(X)$ as well as $\mathcal{D}(X)$ are ordered pointwise, $\{\lambda x'.\kappa'(x'') \mid x' \in A\}$ is directed and bounded. By [36, Thm. 3.1], the right-hand-side is equal to $\int \rho(x)(dx')\bigcup_{\kappa \in A} \lambda x'.\kappa(x'(x''))(x'')$. We conclude the proof by $\lambda x'.\kappa'(x')(x'') = \bigcup_{\kappa \in A} \lambda x'.\kappa(x'(x''))$ by the definition of $\kappa'$.

\hfill \Box

A.3 Lem. 4.4

Proof.

(i) Straightforward by the fact that if $\kappa_i \subseteq K \rho_i$ for all $i \in \mathbb{N}$, then $\sum_{i=0}^{\infty} \phi_i \cdot \kappa_i \subseteq K \sum_{i=0}^{\infty} \phi_i \cdot \rho_i$.

(ii) The Scott-closure of $A$ can be obtained by $\overline{A} = \{\bigcup B \mid B \subseteq \downarrow A, B \text{ directed}\}$ [62]. For any $\{\kappa_i\}_{i \in \mathbb{N}^+} \subseteq \overline{A}$, there are directed subsets $B_i$ of $\downarrow A$ such that $\kappa_i = \bigcup B_i$ for all $i \in \mathbb{N}^+$. For any $\{\phi_i\}_{i \in \mathbb{N}^+} \subseteq \mathcal{W}(X)$ such
that $\sum_{i=1}^{\infty} \phi_i = 1$, we have
\[
\sum_{i=1}^{\infty} \phi_i \cdot \kappa_i = \bigcup_{n \in \mathbb{N}} \sum_{i=1}^{n} \phi_i \cdot \kappa_i \\
= \bigcup_{n \in \mathbb{N}} \sum_{i=1}^{n} \phi_i \cdot \left( \bigcup_{B_i} B_i \right) \\
= \bigcup_{n \in \mathbb{N}} \sum_{i=1}^{n} \bigcup_{\rho_i \in B_i} \phi_i \cdot \rho_i \\
= \bigcup_{n \in \mathbb{N}} \bigcup_{\eta \in A} \sum_{i=1}^{n} \phi_i \cdot \rho_i \\
= \bigcup_{n \in \mathbb{N}} \bigcup_{\eta \in A} \sum_{i=1}^{\infty} \phi_i \cdot \rho_i
\]
where $\sum_{i=1}^{\infty} \phi_i \cdot \rho_i$ is indeed contained in $\downarrow A$ by its g-convexity and hence $\{\sum_{i=1}^{\infty} \phi_i \cdot \rho_i \mid \forall i: \rho_i \in B_i\}$ is a directed subset of $\downarrow A$, thus $\sum_{i=1}^{\infty} \phi_i \cdot \kappa_i$ is contained in $\overline{A}$.

A.4 Lem. 4.5

**Proof.** Let $\{\eta_i\}_{i \in \mathbb{N}^+}$ be any sequence in $\{\kappa \diamond \rho \mid \kappa \in A \land \rho \in B\}$, and $\eta_i = \kappa_i \diamond \rho_i$ such that $\kappa_i \in A, \rho_i \in B$ for all $i \in \mathbb{N}^+$. For any $\{\psi_i\}_{i \in \mathbb{N}^+} \subseteq \mathcal{W}(X)$ such that $\sum_{i=1}^{\infty} \psi_i = 1$, we have
\[
\sum_{i=1}^{\infty} \psi_i \cdot \eta_i = \bigcup_{n \in \mathbb{N}} \sum_{i=1}^{n} \psi_i \cdot \eta_i \\
= \bigcup_{n \in \mathbb{N}} \sum_{i=1}^{n} \psi_i \cdot (\kappa_i \diamond \rho_i) \\
= \bigcup_{n \in \mathbb{N}} \sum_{i=1}^{n} \psi_i \cdot (\phi \cdot \kappa_i + (1 - \phi) \cdot \rho_i) \\
= \bigcup_{n \in \mathbb{N}} \sum_{i=1}^{n} ((\psi_i \phi) \cdot \kappa_i + (\psi_i - \psi_i \phi) \cdot \rho_i) \\
= \bigcup_{n \in \mathbb{N}} \sum_{i=1}^{n} (\psi_i \phi) \cdot \kappa_i + \bigcup_{n \in \mathbb{N}} \sum_{i=1}^{n} (\psi_i - \psi_i \phi) \cdot \rho_i \\
= \phi \cdot \bigcup_{n \in \mathbb{N}} \sum_{i=1}^{n} \psi_i \cdot \kappa_i + (1 - \phi) \cdot \bigcup_{n \in \mathbb{N}} \sum_{i=1}^{n} \psi_i \cdot \rho_i \\
= \left( \sum_{i=1}^{\infty} \psi_i \cdot \kappa_i \right) \cdot \phi + \left( \sum_{i=1}^{\infty} \psi_i \cdot \rho_i \right) .
\]
Because $A$ and $B$ are g-convex, we know that $\sum_{i=0}^{\infty} \psi_i \cdot \kappa_i \in A$ and $\sum_{i=1}^{\infty} \psi_i \cdot \rho_i \in B$. Hence $\sum_{i=1}^{\infty} \psi_i \cdot \eta_i$ is contained in $\{\kappa \diamond \rho \mid \kappa \in A \land \rho \in B\}$. □

A.5 Lem. 4.7

**Proof.** It is straightforward to show that $gconv(A)$ is a superset of the right-hand-side. Then we want to show the right-hand-side is indeed g-convex, which indicates the desired equality by the definition of $gconv(A)$. 

17
Suppose \( \{\kappa_i\}_{i \in \mathbb{N}^+} \) are contained in the right-hand-side. Then for all \( i \in \mathbb{N}^+ \), there exists \( \{\kappa_{i,j}\}_{j \in \mathbb{N}^+} \subseteq A \) and \( \{\phi_{i,j}\}_{j \in \mathbb{N}^+} \) such that \( \sum_{j=1}^{\infty} \phi_{i,j} = 1 \) and \( \kappa_i = \sum_{j=1}^{\infty} \phi_{i,j} \cdot \kappa_{i,j} \). It is sufficient to show that for all \( \{\phi_i\}_{i \in \mathbb{N}^+} \), \( \sum_{i=1}^{\infty} \phi_i \cdot \kappa_i \) is contained in the right-hand-side. We have

\[
\sum_{i=1}^{\infty} \phi_i \cdot \kappa_i = \bigcup_{n \in \mathbb{N}} \sum_{i=1}^{n} \phi_i \cdot \kappa_i
\]

\[
= \bigcup_{n \in \mathbb{N}} \sum_{i=1}^{n} \sum_{j=1}^{\infty} \phi_{i,j} \cdot \kappa_{i,j}
\]

\[
= \bigcup_{n \in \mathbb{N}} \sum_{i=1}^{n} \sum_{j=1}^{m} \phi_{i,j} \cdot \kappa_{i,j}
\]

\[
= \bigcup_{n \in \mathbb{N}, m \in \mathbb{N}} \sum_{1 \leq i \leq n, 0 \leq j \leq m} (\phi_i \phi_{i,j}) \cdot \kappa_{i,j}
\]

that is indeed contained in the right-hand-side. The second last equation is established as follows:

- To show \( \bigcup_{n \in \mathbb{N}, m \in \mathbb{N}} \sum_{1 \leq i \leq n, 0 \leq j \leq m} \psi_{\theta(i,j)} \cdot \rho_{\theta(i,j)} \subseteq K \bigcup_{n \in \mathbb{N}} \sum_{k=1}^{n} \psi_k \cdot \rho_k \): Fix \( n_o \in \mathbb{N} \) and \( m_o \in \mathbb{N} \). Let \( l_o \defeq \max_{1 \leq i \leq n_o} \sum_{j=1}^{i} \psi_{\theta(i,j)} \cdot \rho_{\theta(i,j)} \subseteq K \sum_{k=1}^{l_o} \psi_k \cdot \rho_k \).

- To show \( \bigcup_{n \in \mathbb{N}, m \in \mathbb{N}} \sum_{1 \leq i \leq n, 0 \leq j \leq m} \psi_{\theta(i,j)} \cdot \rho_{\theta(i,j)} \subseteq K \sum_{k=1}^{l_o} \psi_k \cdot \rho_k \): Fix \( l_o \in \mathbb{N} \). Let \( n_o \defeq \max_{1 \leq k \leq l_o} \sum_{j=1}^{k} \psi_{\theta^{-1}(k)} \cdot \rho_{\theta^{-1}(k)} \) and \( m_o \defeq \max_{1 \leq k \leq l_o} \sum_{j=1}^{k} \psi_{\theta^{-1}(k)} \cdot \rho_{\theta^{-1}(k)} \).

\( \square \)

A.6 Lem. 4.8

Proof.

(i) The \( \subseteq \) direction is straightforward. For the \( \supseteq \) direction, we have

\[
gconv(A) = \left\{ \sum_{i=1}^{\infty} \phi_i \cdot \kappa_i \mid \{\kappa_i\}_{i \in \mathbb{N}^+} \subseteq \overline{A} \land \{\phi_i\}_{i \in \mathbb{N}^+} \subseteq \mathcal{W}(X) \land \sum_{i=1}^{\infty} \phi_i = 1 \right\}
\]

by Lem. 4.7 and \( \overline{A} = \left\{ \bigcup B \mid \langle \subseteq A, B \text{ directed} \rangle \right\} \). Let \( \kappa \defeq \sum_{i=1}^{\infty} \phi_i \cdot \kappa_i \) be an element of \( gconv(A) \) where \( \{\kappa_i\}_{i \in \mathbb{N}^+} \subseteq \overline{A} \). Then for all \( i \in \mathbb{N}^+ \), there exists a directed \( B_i \subseteq \downarrow A \) satisfying \( \kappa_i = \bigcup B_i \). Then
we have

\[
\sum_{i=1}^{\infty} \phi_i \cdot \kappa_i = \bigcup_{n \in \mathbb{N}} \sum_{i=1}^{n} \phi_i \cdot \kappa_i = \bigcup_{n \in \mathbb{N}} \sum_{i=1}^{n} \phi_i \cdot \bigcup_{1}^{n} B_i = \bigcup_{n \in \mathbb{N}} \sum_{i=1}^{n} \sum_{i=1}^{n} \phi_i \cdot \rho_i = \bigcup_{n \in \mathbb{N}} \sum_{i=1}^{n} \sum_{i=1}^{n} \phi_i \cdot \rho_i = \bigcup_{n \in \mathbb{N}} \sum_{i=1}^{n} \sum_{i=1}^{n} \phi_i \cdot \rho_i.
\]

Because \( \rho_i \in B_i \subseteq \downarrow A \), there exists \( \eta_i \in A \) satisfying \( \rho_i \subseteq \kappa \eta_i \) for all \( i \in \mathbb{N}^+ \), and thus \( \sum_{i=1}^{\infty} \phi_i \cdot \eta_i \in \text{gconv}(A) \). We also know that \( \sum_{i=1}^{\infty} \phi_i \cdot \rho_i \subseteq \sum_{i=1}^{\infty} \phi_i \cdot \eta_i \), thus \( \sum_{i=1}^{\infty} \phi_i \cdot \rho_i \in \downarrow \text{gconv}(A) \). Therefore \( \sum_{i=1}^{\infty} \phi_i \cdot \kappa_i \in \text{gconv}(A) \). By \( \text{gconv}(\bar{A}) \subseteq \text{gconv}(A) \) we conclude that \( \text{gconv}(\bar{A}) \subseteq \text{gconv}(A) \).

(ii) For the \( \supseteq \)-direction, we have

\[
\frac{\text{gconv}(\bigcup A_i) \supseteq \text{gconv}(A_i)}{\Rightarrow \text{gconv}(\bigcup A_i) \supseteq \bigcup \text{gconv}(A_i)} \Rightarrow \text{gconv}(\bigcup A_i) \supseteq \bigcup \text{gconv}(A_i).
\]

For the \( \subseteq \)-direction, we know that

\[
\text{gconv}(\bigcup A_i) = \left\{ \sum_{j=1}^{\infty} \phi_j \cdot \kappa_j \mid \{ \kappa_j \}_{j \in \mathbb{N}^+} \subseteq \bigcup A_i \land \{ \phi_j \}_{j \in \mathbb{N}^+} \subseteq \mathbb{W}(X) \land \sum_{j=1}^{\infty} \phi_j = 1 \right\}
\]

by Lem. 4.7. Let \( \kappa \equiv \sum_{j=1}^{\infty} \phi_j \cdot \kappa_j \) be an element of \( \text{gconv}(\bigcup A_i) \) where \( \{ \kappa_j \}_{j \in \mathbb{N}^+} \subseteq \bigcup A_i \). For all \( n \in \mathbb{N} \), because \( \{ A_i \}_{i \in I} \) is directed, there exists \( A_{\phi(n)} \) satisfying \( \{ \kappa_1, \ldots, \kappa_n \} \subseteq A_{\phi(n)} \). Thus \( \sum_{j=1}^{n} \phi_j \cdot \kappa_j \in \text{gconv}(A_{\phi(n)}) \). By the definition of Scott-closure, we know that \( \bigcup_{n \in \mathbb{N}} \sum_{j=1}^{n} \phi_j \cdot \kappa_j \in \bigcup \text{gconv}(A_i) \).

Thus \( \kappa \) is contained in the right-hand-side and \( \text{gconv}(\bigcup A_i) \subseteq \bigcup \text{gconv}(A_i) \). Hence we conclude that \( \text{gconv}(\bigcup A_i) \subseteq \bigcup \text{gconv}(A_i) \).

\[
\square
\]

A.7 Lem. 4.9

Proof. \([0,1]\) equipped with its usual linear order forms a Scott-compact topology. By Tychonoff’s theorem we know that \( X \rightarrow [0,1] \) with the product topology is a Scott-compact space. Hence \( \Gamma \equiv \{ (\phi, 1 - \phi) \mid \phi \in \mathbb{W}(X) \} \) is also a Scott-compact space. The map from \( \Gamma \times \mathbb{K}(X) \times \mathbb{K}(X) \) to \( \mathbb{K}(X) \) defined by \( ((\phi, 1 - \phi), \kappa_1, \kappa_2) \mapsto \kappa_1 \circ \kappa_2 \) is Scott-continuous. By Cor. 4.6 we know that \( \text{gconv}(A \cup B) \) is precisely the image of the Scott-compact set \( \Gamma \times A \times B \). Because Scott-continuous functions preserve Scott-compactness, we conclude that \( \text{gconv}(A \cup B) \) is also Scott-compact.

A.8 Lem. 4.10

Proof.

19
(i) Let \( x \in X, \kappa_1, \kappa_2 \in A, \) and \( p \in [0,1] \). We want to show that \( p \cdot \kappa_1(x) + (1 - p) \cdot \kappa_2(x) \in \{ \kappa(x) \mid \kappa \in A \} \).

Let \( \phi \overset{\text{def}}{=} \lambda x.p \). Then \( \kappa_1 \circ \kappa_2 \in A \) because of \( g \)-convexity. We conclude the proof by \( (\kappa_1 \circ \kappa_2)(x) = \phi(x) \cdot \kappa_1(x) + (1 - \phi(x)) \cdot \kappa_2(x) = p \cdot \kappa_1(x) + (1 - p) \cdot \kappa_2(x) \).

(ii) Let \( x \in X \). Let \( F(\kappa) \overset{\text{def}}{=} \kappa(x) \) be a map from \( K(X) \) to \( D(X) \). Because \( F \) is \( g \)-continuous and \( g \)-continuous functions preserve \( SC \)-compactness, we conclude that \( F(A) \) is \( SC \)-compact because \( A \) is \( SC \)-compact.

(iii) Straightforward by the fact that \( K(X) = X \to D(X) \) and \( K(X) \) is ordered pointwise.

\[ \square \]

A.9 Lem. 4.11

Proof. We claim that there exists \( x \in X \) such that \( K(x) \cap A(x) = \emptyset \).

If not, then for all \( x \in X \) there is \( K(x) \cap A(x) \neq \emptyset \). Hence we can define a kernel \( \kappa \) such that \( \kappa(x) \in K(x) \cap A(x) \) for every \( x \). We want to show that \( \kappa \in A \) and \( \kappa \in K \). This follows from \( g \)-convexity of \( A \) and \( K \): suppose \( \kappa(x) = \kappa_x(x) \) such that \( \kappa_x \in K \) for all \( x \), then \( \kappa = \sum_{x \in X} (\lambda x'. [x = x']) \cdot \kappa_x \). This contradicts the fact that \( K \) and \( A \) are disjoint.

Let \( x \in X \) such that \( K(x) \cap A(x) = \emptyset \). By Lem. 4.10(ii)(iii) we know that \( K(x) \) is \( SC \)-compact and \( A(x) \) is \( SC \)-closed. By [62, Thm. 3.8] we know that there exist a \( SC \)-continuous linear map \( F \) and an \( a \in \mathbb{R}_0^\uparrow \) such that \( F(\mu) > a > 1 \geq F(\nu) \) for all \( \mu \in K(x) \) and \( \nu \in A(x) \). Let \( V \overset{\text{def}}{=} \{ x \mid F(\kappa(x)) > a \} \) be a \( SC \)-open subset of \( K(X) \). Then we know that \( K \subseteq V \) and \( A \cap V = \emptyset \). Then it suffices to show that \( V \) is \( g \)-convex. For any \( \{ \kappa_i \}_{i \in \mathbb{N}^+} \subseteq V \) and \( \{ \phi_i \}_{i \in \mathbb{N}^+} \subseteq \mathbb{W}(X) \) such that \( \sum_{i=1}^\infty \phi_i = 1 \). Then

\[
F \left( \sum_{i=1}^\infty \phi_i \cdot \kappa_i(x) \right) = F \left( \sum_{i=1}^\infty \phi_i(x) \cdot \kappa_i(x) \right)
\]

\[
= F \left( \bigcup_{n \in \mathbb{N}} \sum_{i=1}^n \phi_i(x) \cdot \kappa_i(x) \right)
\]

\[
= \bigcup_{n \in \mathbb{N}} F \left( \sum_{i=1}^n \phi_i(x) \cdot \kappa_i(x) \right)
\]

\[
= \bigcup_{n \in \mathbb{N}} \phi_i(x) \cdot F(\kappa(x))
\]

\[
> a
\]

hence \( \sum_{i=1}^\infty \phi_i \cdot \kappa_i \in V \).

\[ \square \]

A.10 Lem. 4.12

Proof. It suffices to show that if any \( SC \)-open of \( K \) is an \( SC \)-open of \( gconv(K) \). Let \( C \) be an \( SC \)-open of \( K \). Let \( U = \bigcup C \) be an \( SC \)-open of \( gconv(K) \). Similar to the proof of Lem. 4.11, we claim that there exist \( x \in X \) and a \( SC \)-continuous linear map \( F \) and an \( a \in \mathbb{R}_0^\uparrow \) such that \( F(\mu) > a > 1 \geq F(\nu) \) for all \( \mu \in K(x) \) and \( \nu \in A(x) \). Then \( F(\kappa(x)) = F((\sum_{i=1}^\infty \phi_i(x) \cdot \kappa_i(x))) = F(\sum_{i=1}^\infty \phi_i(x) \cdot \kappa_i(x)) = \bigcup_{n \in \mathbb{N}} F(\sum_{i=1}^n \phi_i(x) \cdot \kappa_i(x)) = \bigcup_{n \in \mathbb{N}} \phi_i(x) \cdot F(\kappa_i(x)) > a > 1 \), but because \( \kappa \in A \) we also know that \( F(\kappa(x)) \leq 1 \). We then conclude the proof by contradiction.

\[ \square \]

A.11 Thm. 4.13

Proof. It is straightforward to show that \( \langle G(K(X)), \subseteq G \rangle \) forms a poset and \( \perp G \) is the least element. Then it suffices to show the powerdomain admits directed suprema. For a directed collection \( A = \{ A_i \}_{i \in I} \subseteq G(K(X)) \), we define \( \bigcup_{i \in I} A_i \overset{\text{def}}{=} \bigcup_{i \in I} \downarrow A_i \cap \bigcap_{i \in I} A_i \). We now show \( \bigcup_{i \in I} A_i \) is indeed the least upper bound of \( A \).

We already know \( K(X) \) is coherent by Thm. 4.1. Observe that \( \bigcup_{i \in I} A_i = \bigcup_{i \in I} \downarrow A_i \cap \bigcap_{i \in I} A_i = \bigcap_{i \in I} (\bigcup_{i \in I} \downarrow A_i \cap \bigcap_{i \in I} A_i) \) and \( \{ \bigcup_{A_i \in I} \downarrow A_i \}_{i \in I} \) is a filtered family of nonempty lenses, or more generally, nonempty Lawson-closed subsets thus nonempty Lawson-compact subsets because of the coherence of \( K(X) \). By Prop. 3.2 we know the filtered
family admits a nonempty intersection. Thus $\bigsqcup \mathcal{A}_i$ is a nonempty lens that is indeed g-convex by Lem. 4.4 and the g-convexity of $\mathcal{A}_i$’s. In this way we show that $\bigsqcup \mathcal{A}_i \in \mathcal{GK}(X)$.

Let $B \defeq \bigsqcup \mathcal{A}_i$. To show that $B$ is the least upper bound of $\mathcal{A}$, we claim that $\downarrow B = \bigcup_i \downarrow \mathcal{A}_i$ and $\uparrow B = \bigcap_i \uparrow \mathcal{A}_i$. If so, then $B$ is obviously an upper bound of $\mathcal{A}$ and if $A_i \subseteq B'$ for all $i \in I$, then $\downarrow \mathcal{A}_i \subseteq \bigcup B'$ and $\uparrow \mathcal{A}_i \supseteq \bigcap B'$ for all $i \in I$, thus $\downarrow B = \bigcup_i \downarrow \mathcal{A}_i \subseteq B'$ and $\uparrow B = \bigcap_i \uparrow \mathcal{A}_i \supseteq B'$, or equivalently, $B \subseteq \bigcup B'$. Since $B'$ is arbitrarily chosen, we can conclude that $B$ is the least upper bound of $\mathcal{A}$. We adapt proofs from [62] as follows.

- To show $\downarrow B = \bigcup_i \downarrow \mathcal{A}_i$: Inclusion is trivial. For the reverse inclusion, it is sufficient to show $\downarrow B \supseteq \bigcup_i \downarrow \mathcal{A}_i$ since $\downarrow B$ is Scott-closed. Fix $x \in \downarrow \mathcal{A}_i$ for some $i \in I$. Then there exists $y \in \mathcal{A}_i$ such that $x \leq_K y$. For all $j \in I$ satisfying $A_j \subseteq \mathcal{A}_j$, there exists $z \in \mathcal{A}_j$ such that $y \leq_K z$. Therefore $\uparrow x \cap \bigcup_i \downarrow \mathcal{A}_i \cap \uparrow \mathcal{A}_i \neq \emptyset$. Again a filtered family of nonempty Lawson-compact subsets admits a nonempty intersection by Prop. 3.2, we have $\uparrow x \cap \bigcup_i \downarrow \mathcal{A}_i \cap \uparrow \mathcal{A}_i \neq \emptyset$, or equivalently, $\downarrow x \subseteq B'$, thus $x \in \downarrow B$.

- To show $\uparrow B = \bigcap_i \uparrow \mathcal{A}_i$: Inclusion is trivial. For the reverse inclusion, fix $x \in \bigcap_i \uparrow \mathcal{A}_i$. Then we have $\downarrow x \cap \bigcup_i \downarrow \mathcal{A}_i \cap \uparrow \mathcal{A}_i \neq \emptyset$ for all $i \in I$. By a similar reasoning to the previous case we have $\downarrow x \cap \bigcup_i \downarrow \mathcal{A}_i \cap \uparrow \mathcal{A}_i \neq \emptyset$, i.e., $\downarrow x \subseteq B$.

\[ \square \]

A.12 Lem. 4.14

**Proof.** The only nontrivial part of the proof is to show $\otimes_G$ preserves directed suprema. Firstly we claim that $\downarrow (A \otimes_G B) = \{ a \otimes b \mid a \in \downarrow A \land b \in \downarrow B \}$ and $\uparrow (A \otimes_G B) = \{ a \otimes b \mid a \in \uparrow A \land b \in \uparrow B \}$. Let’s write $A \otimes B$ for $\{ a \otimes b \mid a \in \downarrow A \land b \in \downarrow B \}$.

- To show $\downarrow (A \otimes_G B) = \{ a \otimes b \mid a \in \downarrow A \land b \in \downarrow B \}$: Inclusion is trivial. For the reverse inclusion, we have $\downarrow (A \otimes_G B) \subseteq \downarrow (A \otimes B) = \{ a \otimes b \mid a \in \downarrow A \land b \in \downarrow B \}$ and $\downarrow (A \otimes_G B) \subseteq \downarrow (A \otimes_G B)$.

- To show $\uparrow (A \otimes_G B) = \{ a \otimes b \mid a \in \uparrow A \land b \in \uparrow B \}$: Inclusion is trivial. For the reverse inclusion, we have $\uparrow (A \otimes_G B) \subseteq \uparrow (A \otimes_G B) \subseteq \uparrow (A \otimes_G B)$.

Then it suffices to show that $\otimes_G$ is Scott-continuous in the space of down-closures (i.e., $\{ \downarrow A \mid A \in \mathcal{GK}(X) \}$), as well as in the space of up-closures (i.e., $\{ \uparrow A \mid A \in \mathcal{GK}(X) \}$).

- Let a directed family $\{ \mathcal{A}_i \}_{i \in I}$ (ordered by inclusion) and $B$ be nonempty Scott-closed g-convex subsets of $\mathcal{K}(X)$. We want to show that $\downarrow (\bigcup \mathcal{A}_i \otimes B) = \bigcup \downarrow (\mathcal{A}_i \otimes B)$, i.e., the left-Scott-continuity. Indeed, we have $\downarrow (\bigcup \mathcal{A}_i \otimes B) = \downarrow (\bigcup \mathcal{A}_i \otimes B)$ and $\downarrow (\bigcup \mathcal{A}_i \otimes B) = \downarrow (\bigcup \mathcal{A}_i \otimes B) = \downarrow (\bigcup \mathcal{A}_i \otimes B) = \downarrow (\bigcup \mathcal{A}_i \otimes B) = \downarrow (\bigcup \mathcal{A}_i \otimes B)$ by Lem. 4.8 and Scott-continuity of $\otimes$ from Lem. 4.2(ii). The right-Scott-continuity is proved in a similar way.

- Let a directed family $\{ \mathcal{A}_i \}_{i \in I}$ (ordered by reverse inclusion) and $B$ be nonempty Scott-compact saturated g-convex subsets of $\mathcal{K}(X)$. We want to show that $\uparrow gconv((\bigcap \mathcal{A}_i) \otimes B) = \bigcap \uparrow gconv(\mathcal{A}_i \otimes B)$, Inclusion is trivial. For the reverse inclusion, choose any g-convex Scott-open set $U$ containing $\uparrow gconv((\bigcap \mathcal{A}_i) \otimes B)$. As every g-convex Scott-compact saturated subset of a dcpo is the intersection of its g-convex Scott-open neighborhoods (by Lem. 4.11), it suffices to prove that the right-hand-side is contained in $U$. Observe that $gconv((\bigcap \mathcal{A}_i) \otimes B) \subseteq U$ and also $\bigcap \mathcal{A}_i \otimes B \subseteq U$, as $\otimes$ is Scott-continuous by Lem. 4.2(ii) and $\bigcap \mathcal{A}_i$ and $B$ are Scott-compact saturated, we know that $\bigcap \mathcal{A}_i$ and $B$ have Scott-open neighborhoods $V$ and $W$ respectively such that $V \otimes W \subseteq U$. Because $\bigcap \mathcal{A}_i \subseteq V$, by Prop. 3.2 we know there is an $i$ such that $\mathcal{A}_i \subseteq V$. Therefore $\mathcal{A}_i \otimes B \subseteq \bigcup V \otimes W \subseteq U$, and because $U$ is g-convex, we know $gconv(\mathcal{A}_i \otimes B) \subseteq U$. Recall that $U$ is Scott-open, we conclude that $\uparrow gconv(\mathcal{A}_i \otimes B) \subseteq U$. The right-Scott-continuity is proved in a similar way.

\[ \square \]

A.13 Lem. 4.15

**Proof.** It is straightforward to show that $\otimes_G$ is idempotent, commutative, and associative, i.e., $\otimes_G$ is a semilattice operation. Similar to the argument in the proof of Lem. 4.14, it suffices to show the Scott-continuity of $\otimes_G$ with respect to lower closures as well as upper closures.

- Let a directed family $\{ \mathcal{A}_i \}_{i \in I}$ (ordered by inclusion) and $B$ be nonempty Scott-closed g-convex subsets of $\mathcal{K}(X)$. We want to show $gconv(\bigcup \mathcal{A}_i \cup B) = \bigcup gconv(\mathcal{A}_i \cup B)$. Indeed, we have $gconv(\bigcup \mathcal{A}_i \cup B) = \bigcup \mathcal{A}_i \cup B$. Observe that $gconv(\bigcup \mathcal{A}_i \cup B) \subseteq \bigcup gconv(\mathcal{A}_i \cup B)$ and also $\bigcup \mathcal{A}_i \cup B \subseteq \bigcup gconv(\mathcal{A}_i \cup B)$, as $\otimes$ is Scott-continuous by Lem. 4.2(ii) and $\bigcup \mathcal{A}_i$ and $B$ are Scott-compact saturated, we know that $\bigcup \mathcal{A}_i$ and $B$ have Scott-open neighborhoods $V$ and $W$ respectively such that $V \otimes W \subseteq U$. Because $\bigcup \mathcal{A}_i \subseteq V$, by Prop. 3.2 we know there is an $i$ such that $\mathcal{A}_i \subseteq V$. Therefore $\mathcal{A}_i \otimes B \subseteq \bigcup V \otimes W \subseteq U$, and because $U$ is g-convex, we know $gconv(\mathcal{A}_i \otimes B) \subseteq U$. Recall that $U$ is Scott-open, we conclude that $\bigcup gconv(\mathcal{A}_i \otimes B) \subseteq U$. The right-Scott-continuity is proved in a similar way.

\[ \square \]
\[ \text{lem.} \ A.4.8 \]

Let a directed family \( \{A_i\}_{i \in I} \) (ordered by reverse inclusion) and \( B \) be nonempty Scott-compact saturated \( g \)-convex subsets of \( K(X) \). We want to show that \( \text{lem.} \ A.4.8 \). For reverse inclusion, it suffices to show that for every open set \( U \) that is a neighborhood of \( \text{lem.} \ A.4.8 \), we have \( U \) contains the right-hand-side as a subset by Lem. 4.11. Observe that \( g \text{conv}(\bigcap A_i) \cup B \subseteq U \) since \( \bigcap A_i \subseteq V \) by the fact that \( \bigcap A_i \subseteq V \). Thus \( A_i \cup B \subseteq V \cup W \subseteq U \). Recall that \( U \) is \( g \)-convex, we have \( g \text{conv}(A_i \cup B) \subseteq U \). Moreover, \( U \) is Scott-open, thus saturated, hence we conclude that \( \bigcap g \text{conv}(A_i \cup B) \subseteq U \).

\[ \boxed{A.14 \quad \text{lem.} \ A.5.6} \]

Lemma A.1 For any configuration \( \langle v, \omega \rangle \), there is at most one \( \Delta \) such that \( \langle v, \omega \rangle \rightarrow \Delta \).

Proof. Straightforward.

Lemma A.2 \( \rightarrow \) is a kernel.

Proof. Lem. A.2 tells us that \( \rightarrow \) can be seen as a function \( \rightarrow \) defined as follows:

\[
\rightarrow (x)(y) = \begin{cases} \Delta(y) & \text{if } x \rightarrow \Delta \\ 0 & \text{otherwise} \end{cases}
\]

For any \( x \), it is straightforward to show that \( \rightarrow \) is a distribution.

Lemma A.3 \( \rightarrow_n \) is a kernel for all \( n \in \mathbb{N} \).

Proof. By induction on \( n \):

- \( \rightarrow_0 \) can be seen as the everywhere-zero function \( \rightarrow_0 \) which is trivially a kernel.
- \( \rightarrow_{n+1} \) can be seen as the function defined as follows:

\[
\rightarrow_{n+1}(\langle v, \omega \rangle)(\omega') = \sum_{\tau \in \text{supp}(\Delta)} \Delta(\tau) \cdot \rightarrow_n(\tau)(\omega') \quad \langle v, \omega \rangle \rightarrow \Delta.
\]

For any \( x \), it is straightforward to show that \( \rightarrow_{n+1}(x) \) is a distribution given that \( \rightarrow_n \) is a kernel.

Now we prove Lem. 5.6.

Proof. It is sufficient to show that

\[
\omega \sup_n \rightarrow_n(\langle v_{\text{entry}}, \omega \rangle) = (\text{lfp}_{K \cup \perp K} F_P)(v_{\text{entry}})
\]

and we are instead going to show for all \( n \in \mathbb{N} \) and \( v \in V \) the following holds

\[
\omega \sup_n \rightarrow_n(\langle v, \omega \rangle) = F^n_P(\lambda v, \perp K)(v).
\]

By induction on \( n \), the base case is trivial because both sides compute to \( \perp K \). Suppose that for some \( n \), the equality holds for all \( v \in V \). Then for all \( v \in V \), we want to show that

\[
\omega \sup_n \rightarrow_n(\langle v, \omega \rangle) = F^{n+1}_P(\lambda v, \perp K)(v).
\]

- If \( v \) is not associated with any edges, then \( \rightarrow_{n+1}(\langle v, \omega \rangle)(\omega') = [\omega = \omega'] \) for all \( \omega \) and \( \omega' \). The right-hand-side computes to \( F_P(F^n_P(\lambda v, \perp K))(v) \) and by the definition of \( F_P \) we know it is equal to \( \omega \sup_n \rightarrow_n(\langle u_i, \omega \rangle) = F^n_P(\lambda v, \perp K)(u_i) \) for all \( i \) by induction hypothesis.
• If $\text{Ctrl}(e) = \text{seq}[\text{act}]$, then the right-hand-side is equal to $\llbracket \text{act} \rrbracket \otimes F^p_\lambda(\lambda v. \bot_K)(u_1)$. The left-hand-side is
\[
\lambda \omega. \lambda \omega' \cdot \sum_{\tau} \xrightarrow{\langle \nu, \omega \rangle} (\langle \nu, \omega \rangle)(\tau) \cdot \xrightarrow{n}(\tau)(\omega')
\]
\[
= \lambda \omega. \lambda \omega' \cdot \sum_{\omega''} \llbracket \text{act} \rrbracket (\omega'' \cdot \xrightarrow{n}(\langle u_1, \omega'' \rangle)(\omega'))
\]
\[
= \llbracket \text{act} \rrbracket \otimes F^p_\lambda(\lambda v. \bot_K)(u_1).
\]

• If $\text{Ctrl}(e) = \text{cond}[\varphi]$, then the right-hand-side is equal to $F^p_\lambda(\lambda v. \bot_K)(u_1) \llbracket \varphi \rrbracket \otimes F^p_\lambda(\lambda v. \bot_K)(u_2)$. The left-hand-side is
\[
\lambda \omega. \lambda \omega' \cdot \sum_{\tau} \xrightarrow{\langle \nu, \omega \rangle} (\langle \nu, \omega \rangle)(\tau) \cdot \xrightarrow{n}(\tau)(\omega')
\]
\[
= \lambda \omega. \lambda \omega' \cdot \left( \sum_{\omega''} \llbracket \varphi \rrbracket (\omega) \cdot \delta(\omega)(\omega'') \cdot \xrightarrow{n}(\langle u_1, \omega'' \rangle)(\omega') + \sum_{\omega''} (1 - \llbracket \varphi \rrbracket (\omega)) \cdot \delta(\omega)(\omega'') \cdot \xrightarrow{n}(\langle u_2, \omega'' \rangle)(\omega') \right)
\]
\[
= \lambda \omega. \lambda \omega' \cdot \llbracket \varphi \rrbracket (\omega) \cdot \xrightarrow{n}(\langle u_1, \omega \rangle)(\omega') + (1 - \llbracket \varphi \rrbracket (\omega)) \cdot \xrightarrow{n}(\langle u_2, \omega \rangle)(\omega')
\]
\[
= \lambda \omega. \lambda \omega' \cdot \llbracket \varphi \rrbracket (\omega) \cdot F^p_\lambda(\lambda v. \bot_K)(u_1)(\omega')(\omega') + (1 - \llbracket \varphi \rrbracket (\omega)) \cdot F^p_\lambda(\lambda v. \bot_K)(u_2)(\omega')(\omega')
\]
\[
= F^p_\lambda(\lambda v. \bot_K)(u_1) \llbracket \varphi \rrbracket \otimes F^p_\lambda(\lambda v. \bot_K)(u_2).
\]

Thus we conclude the proof.

\[\square\]

A.15 Thm. 6.3

Proof. We only show the proof for the over-approximations. By definition, we have $\mathcal{C}[P] = \text{lfp}_{C}^{\mathcal{C}} F^p_{\mathcal{C}} =$ $\bigsqcup_{n \in \mathbb{N}} (F^p_{\mathcal{C}})^n(\bot_{\mathcal{C}})$, and $\mathcal{Y}[P] = \text{lfp}_{Y}^{\mathcal{Y}} F^p_{Y}$ obtained by Knaster-Tarski. Then it suffices to show that for every $n \in \mathbb{N}$, we have $(F^p_{\mathcal{C}})^n(\bot_{\mathcal{C}}) \subseteq C \gamma(\mathcal{Y}[P])$. Now we proceed by induction on $n$.

• If $n = 0$, the result follows immediately because $\bot_{\mathcal{C}}$ is the least element in $\mathcal{C}$.

• Suppose that $(F^p_{\mathcal{C}})^k(\bot_{\mathcal{C}}) \subseteq C \gamma(\mathcal{Y}[P])$ for some $k \in \mathbb{N}$. Let’s denote the left hand side by $LHS$ and $\mathcal{Y}[P]$ by $\text{SOL}$. We want to show that $F^p_{\mathcal{C}}(LHS) \subseteq C \gamma(\text{SOL}(v))$. This expands to $F^p_{\mathcal{C}}(LHS)(v) \subseteq C \gamma(\text{SOL}(v))$ for all $v \in V$. We proceed by a case analysis on the kind of edges leaving $v$.

(i) If $v = v^i_{e_{i1}}$ for some $i$, then $F^p_{\mathcal{C}}(LHS)(v) = 1_{\mathcal{C}}$. Then we can conclude this case by showing that $\text{SOL}(v) = 1_{Y}$. By definition of $\text{SOL}$, we know that $F^p_{Y}(\text{SOL}) = \text{SOL}$, thus $F^p_{\mathcal{C}}(\text{SOL})(v) = \text{SOL}(v)$. By definition of $F^p_{\mathcal{C}}$, we know that $F^p_{\mathcal{C}}(\text{SOL})(v) = 1_{Y}$.

(ii) If $v \neq v^i_{e_{i1}}$ for all $i$, we have
\[
F^p_{\mathcal{C}}(LHS)(v) = \bigcup_{C} \left\{ \text{Ctrl}(e)(LHS(u_1), \ldots, LHS(u_k)) \mid e = \langle v, \{u_1, \ldots, u_k\} \rangle \in E \right\}
\]
\[
= \bigcup_{C} \left\{ \text{Ctrl}(e)(\gamma(\text{SOL}(u_1), \ldots, \gamma(\text{SOL}(u_k))) \mid e = \langle v, \{u_1, \ldots, u_k\} \rangle \in E \right\}
\]
\[
= \gamma(\bigcup_{C} \left\{ \text{Ctrl}(e)(\gamma(\text{SOL}(u_1), \ldots, \gamma(\text{SOL}(u_k))) \mid e = \langle v, \{u_1, \ldots, u_k\} \rangle \in E \right\})
\]
\[
= \gamma(F^p_{\mathcal{C}}(\text{SOL})(v))
\]

Now consider the form of $\text{Ctrl}(e)$.

• $\text{Ctrl}(e) = \text{seq}[\text{act}]$: We want to show that $\text{seq}[\text{act}(\gamma(x_1))] \subseteq C \gamma(\text{seq}[\text{act}(x_1)])$. It is equivalent to $\llbracket \text{act} \rrbracket \otimes C \gamma(\langle x_1 \rangle) \subseteq C \gamma(\llbracket \text{act} \rrbracket \otimes Y x_1)$. Indeed, we have
\[
\llbracket \text{act} \rrbracket \otimes C \gamma(\langle x_1 \rangle) \subseteq C \gamma(\llbracket \text{act} \rrbracket \otimes Y x_1)
\]
by assumption, monotonicity of $\otimes_C$, and properties of $\gamma$. 23
\[ \text{Ctrl}(e) = \text{cond}[\varphi]: \text{We want to show that } \text{cond}[\varphi](\gamma(x_1), \gamma(x_2)) \sqsubseteq_C \gamma(\text{cond}[\varphi](x_1, x_2)). \text{ It is equivalent to } \gamma(x_1) \varphi^C \gamma(x_2) \sqsubseteq_C \gamma(x_1) \varphi_{Y}^C \gamma(x_2). \text{ Appeal to properties of } \gamma. \]

\[ \text{Ctrl}(e) = \text{call}[i \rightarrow j]: \text{We want to show that } \text{call}[i \rightarrow j](\gamma(x_1)) \sqsubseteq_C \gamma(\text{call}[i \rightarrow j](x_1)). \text{ It is equivalent to } \text{LHS}(v_j^{\text{entry}}) \otimes_C \gamma(x_1) \sqsubseteq_C \gamma(\text{SOL}(v_j^{\text{entry}}) \otimes_Y x_1). \text{ Indeed, we have } \]

\[ \text{LHS}(v_j^{\text{entry}}) \otimes_C \gamma(x_1) \sqsubseteq_C \gamma(\text{SOL}(v_j^{\text{entry}}) \otimes_C \gamma(x_1) \sqsubseteq_C \gamma(\text{SOL}(v_j^{\text{entry}}) \otimes_Y x_1) \]

by induction hypothesis, monotonicity of \( \otimes_C \), and properties of \( \gamma \). \qed