# Constructive Logic (15-317), Fall 2020
# Assignment 3: Proof Terms, Verification and Quantification

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Due: Submit to Gradescope by Friday, September 25, 2020, 11:59 pm

## 1 Hype for *hyp*s

Consider the following notations for a proofs of $A \land B \supset B \land A$.

Floating Hypothesis Notation

$$\dfrac{\dfrac{\dfrac{[A\ \textit{true}]_x \quad [B\ \textit{true}]_y}{A \land B\ \textit{true}}\ \land I}{B \supset (A \land B)\ \textit{true}}\ \supset I_y}{A \supset (B \supset (A \land B))\ \textit{true}}\ \supset I_x$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A\ \textit{true}, B\ \textit{true} \vdash A\ \textit{true}}\ \textit{hyp} \quad \dfrac{}{A\ \textit{true}, B\ \textit{true} \vdash B\ \textit{true}}\ \textit{hyp}}{A\ \textit{true}, B\ \textit{true} \vdash A \land B\ \textit{true}}\ \land I}{A\ \textit{true} \vdash B \supset (A \land B)\ \textit{true}}\ \supset I}{\vdash A \supset (B \supset (A \land B))\ \textit{true}}\ \supset I$$

Context Notation

The $\Gamma$ notation of hypotheses can be created by slightly modifying the rules of Natural Deduction to carry a context.

$$\dfrac{\Gamma \vdash A\ \textit{true} \quad \Gamma \vdash B\ \textit{true}}{\Gamma \vdash A \land B\ \textit{true}}\ \land I \qquad \dfrac{\Gamma \vdash A \land B\ \textit{true}}{\Gamma \vdash A\ \textit{true}}\ \land E_1 \qquad \dfrac{\Gamma \vdash A \land B\ \textit{true}}{\Gamma \vdash B\ \textit{true}}\ \land E_2$$

$$\dfrac{\Gamma \vdash A\ \textit{true}}{\Gamma \vdash A \lor B\ \textit{true}}\ \lor I_1 \qquad \dfrac{\Gamma \vdash B\ \textit{true}}{\Gamma \vdash A \lor B\ \textit{true}}\ \lor I_2 \qquad \dfrac{\Gamma \vdash A \lor B\ \textit{true} \quad \Gamma, A\ \textit{true} \vdash C\ \textit{true} \quad \Gamma, B\ \textit{true} \vdash C\ \textit{true}}{\Gamma \vdash C\ \textit{true}}\ \lor E$$

$$\dfrac{\Gamma, A\ \textit{true} \vdash B\ \textit{true}}{\Gamma \vdash A \supset B\ \textit{true}}\ \supset I \qquad \dfrac{\Gamma \vdash A \supset B\ \textit{true} \quad \Gamma \vdash A\ \textit{true}}{\Gamma \vdash B\ \textit{true}}\ \supset E$$

$$\dfrac{}{\Gamma \vdash \top\ \textit{true}}\ \top I \qquad \dfrac{\Gamma \vdash \bot\ \textit{true}}{\Gamma \vdash C\ \textit{true}}\ \bot E$$

Finally, the hypothesis rule, which allows you to conclude a hypothesis.

$$\dfrac{J \in \Gamma}{\Gamma \vdash J}\ \textit{hyp}$$

**Task 1** (4 points)**.** Consider this proof (LaTeX) and write the same proof using context notation.

$$\dfrac{\dfrac{\dfrac{\dfrac{[(A \supset A) \land (A \supset A)\ \textit{true}]_u}{A \supset A\ \textit{true}}\ \land E_2 \quad \dfrac{\dfrac{[(A \supset A) \land (A \supset A)\ \textit{true}]_u}{A \supset A\ \textit{true}}\ \land E_1 \quad [A\ \textit{true}]_w}{A\ \textit{true}}\ \supset E}{A\ \textit{true}}\ \supset E}{(A \supset A)\ \textit{true}}\ \supset I_w}{(A \supset A) \land (A \supset A) \supset (A \supset A)\ \textit{true}}\ \supset I_u$$

## 2 All the things you can do with a $\Diamond$

Consider the $\Diamond$ connective.

$$\frac{\begin{array}{c}\overline{A\ \mathit{true}}^{\,u}\\ \vdots\\ B\ \mathit{true}\end{array}}{\Diamond(A,B,C)\ \mathit{true}}\ \Diamond I_1^u \qquad \frac{\begin{array}{c}\overline{A\ \mathit{true}}^{\,u}\\ \vdots\\ C\ \mathit{true}\end{array}}{\Diamond(A,B,C)\ \mathit{true}}\ \Diamond I_2^u \qquad \frac{\Diamond(A,B,C)\ \mathit{true} \quad A\ \mathit{true} \quad \begin{array}{c}\overline{B\ \mathit{true}}^{\,u}\\ \vdots\\ D\ \mathit{true}\end{array} \quad \begin{array}{c}\overline{C\ \mathit{true}}^{\,v}\\ \vdots\\ D\ \mathit{true}\end{array}}{D\ \mathit{true}}\ \Diamond E^{u,v}$$

**Task 2** (3 points)**.** Invent a proof term assignment for the rules, it should be unique from any other proof terms we've used.

**Task 3** (8 points)**.** Show all the local reduction(s) and expansion(s) for these rules (proving local soundness and completeness) in proof term notation (no trees needed) if possible. If the connective is not locally sound or complete, explain why. Be sure to indicate which are reductions and which are expansions.

## 3 Verifications

Recall the $\Diamond$ connective from Section 2:

**Task 4** (7 points)**.** Give rules for forming the judgments that $\Diamond(A,B,C)$ has a verification and that $\Diamond(A,B,C)$ can be used.

**Task 5** (4 points)**.** On assignment 2, you showed

$$(\neg A \wedge B) \supset ((A \supset B) \supset (\neg A \supset \neg B)) \supset \bot\ \mathit{true}.$$

Give a verification for this proposition.

**Task 6** (14 points)**.** For each of the following propositions, give a verification and its corresponding proof term. (The proof term for a verifications-and-uses derivation is the same as the proof term for the corresponding natural-deduction derivation.)

a. $\bot \supset \top$

b. $\bot \supset \top$ **(Do not use the same verification/proof term as part a. Use a new one.)**

c. $(A \supset B) \supset (\neg B \supset \neg A)$

d. $(A \supset B) \supset (B \supset C) \supset (A \supset C)$