

Constructive Logic (15-317), Fall 2018

Assignment 3: Proof Terms, Verification and Quantification

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Due: Friday, September 21, 2018, 11:59 pm

This assignment must be submitted electronically at autolab. Submit your homework as a tar archive containing the following files:

- hw3.pdf (your written solutions); and
- hw3_8a.tut, hw3_8b.tut, hw3_8c.tut (your Tutch solutions for task 8)

1 All the things you can do with a \Diamond

Consider the \Diamond connective.

$$\begin{array}{c}
 \overline{A \text{ true}}^u \\
 \vdots \\
 \overline{B \text{ true}} \\
 \hline
 \Diamond(A, B, C) \text{ true} \quad \Diamond I_1^u
 \end{array}
 \quad
 \begin{array}{c}
 \overline{A \text{ true}}^u \\
 \vdots \\
 \overline{C \text{ true}} \\
 \hline
 \Diamond(A, B, C) \text{ true} \quad \Diamond I_2^u
 \end{array}
 \quad
 \frac{\Diamond(A, B, C) \text{ true} \quad \overline{B \text{ true}}^u \quad \overline{C \text{ true}}^v \quad \overline{D \text{ true}}}{\overline{A \text{ true}} \quad \overline{D \text{ true}}} \Diamond E^{u,v}$$

Task 1 (4 points). Invent a proof term assignment for the rules.

Task 2 (4 points). Show all the local reduction(s) and expansion(s) for these rules (proving local soundness and completeness) in proof term notation. Be sure to indicate which are reductions and which are expansions.

2 Verifications

Recall the \Diamond connective from Section 1:

Task 3 (9 points). Give rules for forming the judgments that $\Diamond(A, B, C)$ has a verification and that $\Diamond(A, B, C)$ can be used.

Task 4 (5 points). On assignment 2, you showed

$$(\neg A \wedge B) \supset ((A \supset B) \supset (\neg A \supset \neg B)) \supset \perp \text{ true}.$$

Give a verification for this proposition.

Task 5 (14 points). For each of the following propositions, give a verification and its corresponding proof term:

- a. $\perp \supset \top$
- b. $(A \supset B) \supset (B \supset C) \supset (A \supset C)$

3 Quantification

It is important to note that quantification extends as far to the right as syntactically possible. For example, the proposition $\exists x : \tau. A(x) \supset \forall x : \tau. A(x)$ should be interpreted as $\exists x : \tau. (A(x) \supset \forall x : \tau. A(x))$ and not as $(\exists x : \tau. A(x)) \supset (\forall x : \tau. A(x))$. Tutch implements the same convention.

3.1 Distributivity properties

In class, we saw that universal quantification distributes over conjunction, that is,

$$(\forall x : \tau. A(x) \wedge B(x)) \equiv (\forall x : \tau. A(x)) \wedge \forall x : \tau. B(x) \text{ true.}$$

In this section, we will explore various other distributivity properties.

Task 6 (10 points). Dually, existential quantification distributes over disjunction, that is,

$$(\exists x : \tau. A(x) \vee B(x)) \equiv (\exists x : \tau. A(x)) \vee \exists x : \tau. B(x) \text{ true.}$$

In this task, you will show this equivalence by giving a natural deduction proof of each of the following directions:

- a. $(\exists x : \tau. A(x) \vee B(x)) \supset (\exists x : \tau. A(x)) \vee \exists x : \tau. B(x) \text{ true}$
- b. $(\exists x : \tau. A(x)) \vee (\exists x : \tau. B(x)) \supset \exists x : \tau. A(x) \vee B(x) \text{ true}$

3.2 Constructive and classical quantification

Task 7 (9 points). For each of the following judgments, give a constructive natural deduction proof and the corresponding proof term if it is constructively valid. If it is not constructively valid, state this. *N.B. The following judgments are all classically valid.*

- a. $(\neg \forall x : \tau. \neg A(x)) \supset \exists x : \tau. A(x) \text{ true}$
- b. $(\exists x : \tau. A(x)) \supset \neg \forall x : \tau. \neg A(x) \text{ true}$

3.3 Tutch, Quantified

Tutch uses the concrete syntax `?x:t.A(x)` and `!x:t.A(x)` for $\exists x : \tau. A(x)$ and $\forall x : \tau. A(x)$, respectively. We encourage you to review the scoping rules for quantifiers described at the beginning of Section 3 of this assignment before starting this portion of the assignment. Please see the Tutch manual for more information on how to use quantifiers in Tutch.

Task 8 (6 points). Prove each of the following propositions using Tutch. Place the proof for part a (and only the proof for part a) in `hw3_8a.tut`, ..., and the proof for part c (and only the proof for part c) in `hw3_8c.tut`.

- a. proof apply : $(!x:t.A(x) \Rightarrow B(x)) \Rightarrow (!x:t.A(x)) \Rightarrow (!x:t.B(x));$
- b. proof instance : $(!x:t.A(x)) \ \& \ (?y:t.B(y)) \Rightarrow ?z:t.A(z);$
- c. proof frobenius : $(R \ \& \ ?x:t.Q(x)) \Leftrightarrow ?x:t.(R \ \& \ Q(x));$