Constructive Logic (15-317), Fall 2018
Assignment 10: Linear Logic

Course Staff

Due: Monday, November 26, 11:59 pm

Submit your homework via autolab as a file named **hw10.pdf**.

1 Practicing Linear Logic

For your convenience, we supply a self-contained presentation of the rules of linear logic.

Structural Rules

In the presentation of linear logic that we use in this course, contexts Δ should be taken as unordered lists; therefore, the principle of *exchange* is automatic. Weakening and contraction are *not* included in linear logic.

$$\frac{}{u : A \Vdash A}\ u$$

Multiplicative Conjunction

$$\frac{\Delta \Vdash A \quad \Delta' \Vdash B}{\Delta, \Delta' \Vdash A \otimes B}\ \otimes\mathsf{I} \qquad \frac{\Delta \Vdash A \otimes B \quad \Delta', u : A, v : B \Vdash C}{\Delta, \Delta' \Vdash C}\ \otimes\mathsf{E}^{u,v}$$

$$\frac{}{\cdot \Vdash \mathbf{1}}\ \mathbf{1}\mathsf{I} \qquad \frac{\Delta \Vdash \mathbf{1} \quad \Delta' \Vdash C}{\Delta, \Delta' \Vdash C}\ \mathbf{1}\mathsf{E}$$

Additive Conjunction

$$\frac{\Delta \Vdash A \quad \Delta \Vdash B}{\Delta \Vdash A \mathbin{\&} B}\ \&\mathsf{I} \qquad \frac{\Delta \Vdash A \mathbin{\&} B}{\Delta \Vdash A}\ \&\mathsf{E}_1 \qquad \frac{\Delta \Vdash A \mathbin{\&} B}{\Delta \Vdash B}\ \&\mathsf{E}_2$$

$$\frac{}{\Delta \Vdash \top}\ \top\mathsf{I}$$

There is no elimination rule for the unit $\top$.

Additive Disjunction

$$\frac{\Delta \Vdash A}{\Delta \Vdash A \oplus B}\ \oplus\mathsf{I}_1 \qquad \frac{\Delta \Vdash B}{\Delta \Vdash A \oplus B}\ \oplus\mathsf{I}_2 \qquad \frac{\Delta \Vdash A \oplus B \quad \Delta', u : A \Vdash C \quad \Delta', v : B \Vdash C}{\Delta, \Delta' \Vdash C}\ \oplus\mathsf{E}^{u,v}$$

$$\frac{\Delta \Vdash \mathbf{0}}{\Delta, \Delta' \Vdash C}\ \mathbf{0}\mathsf{E}$$

There is no introduction rule for the unit $\mathbf{0}$.

Implication

$$\frac{\Delta, u : A \Vdash B}{\Delta \Vdash A \multimap B}\ \multimap\mathsf{I}^u \qquad \frac{\Delta \Vdash A \multimap B \quad \Delta' \Vdash A}{\Delta, \Delta' \Vdash B}\ \multimap\mathsf{E}$$

Task 1 (15 pts)**.** Prove the following judgments in linear natural deduction, or state that they do not hold.

1. $f : A \multimap B \multimap C \Vdash A \otimes B \multimap C$
2. $f : (A \& B) \multimap C \Vdash A \multimap (B \multimap C)$
3. $f : ((A \otimes \top) \& (B \otimes \top)) \multimap C \Vdash A \multimap (B \multimap C)$

2 Proof-Theoretic Harmony

Just like we did in the beginning of the course, we can check a local correctness condition for the rules of linear natural deduction: proof-theoretic harmony.[1] Hint: exhibiting local reductions and expansions in linear logic is subtle: you must be sure to not constrain the contexts Δ in your local reductions and expansions any more than is warranted by the rules of linear logic.

Remark 1 (Linear substitution principle)**.** When exhibiting local reductions and expansions, you will need to use substitutions $[\mathcal{D}/u]\mathcal{E}$. These are governed by the *linear substitution principle*, which states:

$$\text{If } \Delta \overset{\mathcal{D}}{\Vdash} A \text{ and } \Delta', u : A \overset{\mathcal{E}}{\Vdash} B\text{, then } \Delta, \Delta' \overset{[\mathcal{D}/u]\mathcal{E}}{\Vdash} B.$$

You *must* ensure that your resulting derivations have the correct contexts.

Task 2 (10 pts)**.** Verify that the rules for the tensor $\otimes$ are harmonious.

Task 3 (10 pts)**.** Verify that the rules for the unit $\mathbf{1}$ are harmonious.

Task 4 (10 pts)**.** Verify that the rules for the unit $\top$ are harmonious.

[1] Harmony is a necessary condition for the correctness of rules, but not a sufficient condition.

3 Applications

In class, we looked *Blocks World*, an example of encoding *state* in linear logic; Blocks World is a class of scenarios in which there is a table, some number of blocks which can be stacked on top of each other, and a robotic arm which can pick up and move blocks. The following atomic predicates are used:

1. empty means that the robotic arm's hand is empty.
2. $\mathsf{clear}(x)$ means that the block x does not have anything on top of it.
3. $\mathsf{on}(x, y)$ means that the block x is directly on top of the block y.
4. $\mathsf{on\_table}(x)$ means that the block x is sitting directly on the table.

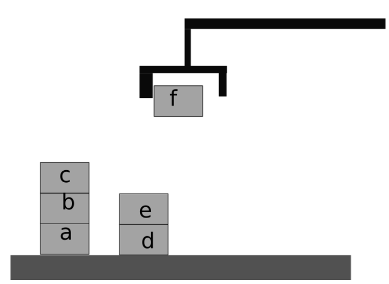
The possible state transitions for Blocks World are given by the following axioms in linear logic:

1. $\forall x, y.\ \mathsf{empty} \otimes \mathsf{clear}(x) \otimes \mathsf{on}(x, y) \multimap \mathsf{holds}(x) \otimes \mathsf{clear}(y)$
2. $\forall x.\mathsf{empty} \otimes \mathsf{clear}(x) \otimes \mathsf{on\_table}(x) \multimap \mathsf{holds}(x)$
3. $\forall x, y.\ \mathsf{holds}(x) \otimes \mathsf{clear}(y) \multimap \mathsf{empty} \otimes \mathsf{on}(x, y) \otimes \mathsf{clear}(x)$
4. $\forall x.\ \mathsf{holds}(x) \multimap \mathsf{empty} \otimes \mathsf{on\_table}(x) \otimes \mathsf{clear}(x)$

Task 5 (10 pts)**.** We have assumed that the table is infinitely broad and can therefore accomodate any number of blocks. **Consider the case that the table in fact only has a finite number of spaces for blocks on it, and modify the axioms of Blocks World above in order to preserve this invariant.**

You should use an atomic predicate space, which will mean that there is an available space on the table. Your solution to this task will not depend on how large the table is in the Blocks World configuration.

Task 6. Consider the following Blocks World scenario:



Write a formula in linear logic which expresses this configuration, assuming that the table can fit **four** blocks total directly on it.