A THEORY OF THE LEARNABLE (L.G. VALIANT)

Theory Lunch Presentation

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05/20/10

HOW DO YOU KNOW THAT?

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```
bool is_a_duck() {
    return (walks_like_a_duck() &&
                   quacks_like_a_duck());
}
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HOW DID YOU LEARN THAT?

"A program for performing a task [like recognizing ducks – Ed.] has been acquired by learning if it has been acquired by any means other than explicit programming."

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- Introduce the idea of Probably Approximately Learnable (PAL) problems, or problems that are learnable in polynomial time, with high correctness.
- Prove 3 classes of programs to be PAL.

Outline

1. General framework for defining Learning Machines, or programs that can learn/write/produce other programs of a particular type.
   - A Learning Machine for animal recognition, for example, might learn to write a program that recognizes whether a given animal is a duck.
2. Definition of a particular learning protocol.
3. Definition of when a program class is reasonably-learnable.
4. Definition/proofs of reasonably-learnable program classes.

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walks like a duck = true
purple = false
fluffy = true
yellow = true
beak = true
big = false
quacks like a duck=true
angry = false
...

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- Variables: {walks like a duck, beak, purple, …}
- Vector ***v***: {walks_like_a_duck=0, beak=1, purple=*, …}
- $\mathcal{F}(v)$ = ***is_a_duck(v)*** = ***false***

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Variables determined in is_a_duck: {walks_like_a_duck, quacks_like_a_duck}

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  - A learning protocol, or the method by which information is gathered from the world.
  - A deduction procedure, or the mechanism for learning new concepts from gathered information.

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2. ORACLE: takes as input a vector v, returns $\mathcal{F}(v)$.

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EXAMPLE()

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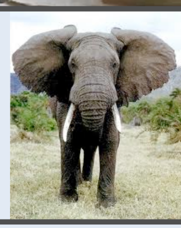
ORACLE()	

24

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25

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  - Uses this protocol.
  - Runs in reasonable time: polynomial by adjustable parameter h, size of learned program, and number of variables determined in the learned formula.
  - Produces a program that says something is false when it's true with probability no greater than $(1-h^{-1})$; never says that something is true when it's false.

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- We are trying to make a program (learning machine) that can learn, in polynomial time, another program (the learned program) that recognizes whether a boolean formula (concept) is true for any set of boolean data.
- The learning program has access to a function that will give it a bunch of examples, as well as a function that will check its work.
- The learning machine can learn a program that is sometimes wrong, so long as the probability that the learned program is ever wrong is adjustable.

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29

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- This means the next 3 slides are mathy.

A Combinatorial Bound

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- Returns smallest integer n such that in n independent Bernoulli trials, each with probability at least h^{-1} of success, P(< S successes) < h^{-1}
  - Bernoulli trial: an experiment whose outcomes are either "success" or "failure"; randomly distributed by some probability function.

Upper Bound on `L`(h, S)

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Upper Bound on $L(h, S)$

$$L(h,S) \leq 2h(S + \log_e h)$$

Proof by algebraic substitution of well-known inequalities:

1. $\forall x > 0, \ (1 + x^{-1})^{x} < e$
2. $\forall x > 0, \ (1 - x^{-1})^{x} < e^{-1}$
3. In m independent trials, each with success probability $\geq p$:

$P(\text{successes at most } k) \leq \left(\frac{m-mp}{m-k}\right)^{m-k} \left(\frac{mp}{k}\right)^{k}$

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- $\mathbf{L}(\boldsymbol{h}, \boldsymbol{S})$ is basically linear in both $\boldsymbol{h}$ and $\boldsymbol{S}$.
- Applies to using EXAMPLEs and ORACLE to determine vectors.
- An algorithm can approximate the set of determined variables in natural EXAMPLEs of $\mathcal{F}$ in runtime independent of *total* number of variables in the world.

So?

- $\mathbf{L}(\boldsymbol{h}, \boldsymbol{S})$ is basically linear in both $\boldsymbol{h}$ and $\boldsymbol{S}$.
- Applies to using EXAMPLEs and ORACLE to determine vectors.
- An algorithm can approximate the set of determined variables in natural EXAMPLEs of $\mathcal{F}$ in runtime independent of *total* number of variables in the world.
  - Dependent only the number of variables that are determined in $\mathcal{F}$.

Remaining Question

Given that learning protocol, what classes of tasks are learnable in polynomial time?

Answer: At Least 3 Classes of Programs

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- *k*-CNF expression: a CNF expression where each internal clause is composed of $\leq k$ literals.
- Learnable with an algorithm that does not call ORACLE, and calls EXAMPLE $\leq L(h, 2t^{k+1})$ times. *(t is the number of variables)*

Monotone DNF Expressions

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- Disjunctive Normal Form (DNF):

  $(a_1 \wedge a_2 \wedge a_3) \vee (a_1 \wedge a_4) \ldots$

Monotone DNF Expressions

- Disjunctive Normal Form (DNF):

  $(a_1 \land a_2 \land a_3) \lor (a_1 \land a_4) \ldots$

- An expression is monotone if it contains no negated literals.

Monotone DNF Expressions

- Disjunctive Normal Form (DNF):

  $(a_1 \wedge a_2 \wedge a_3) \vee (a_1 \wedge a_4) \ldots$

- An expression is monotone if it contains no negated literals.
- Learnable with an algorithm that calls EXAMPLEs $L = L(h,d)$ times and ORACLEs $d*t$ times, where d is the degree of the expression and t is the number of variables.

μ-expressions

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- General expression over $\{p_1, \ldots, p_t\}$ defined recursively $(1 \le i \le t)$:

$$f := p_i \mid \sim p_i \mid f_1 \wedge f_2 \mid f_1 \vee f_2$$

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- A μ-expression is an expression in which each p appears at most once.
- Learnable with an exactly correct algorithm that calls two slightly more powerful ORACLE functions $O(t^3)$ times total.

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- Learnability theory is concerned with what programs can be learned automatically.
- We should reason about what is programmatically learnable in the same way we reason about what is computable.
- A class of programs is Probably Approximately Learnable when, using a particular type of teacher, a given algorithm can learn a program that can recognize instances of that class with a certain probability.
- 3 examples of such learnable program types are k-CNF expressions, monotone DNF expressions, and μ-expressions.

Interesting Concluding Questions

- What else is learnable by these definitions?
- Is the definition of “learnable” reasonable?
  - How powerful should the teachers be?
  - What about if we use negative in addition to positive examples?
- How do humans learn?

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- Definition of `L(h,S)` implies:

  `|X| = L(100,S) ⇒ P(succeeded overall) > 99%`

- "Success" for each trial is defined as picking a marble we haven't picked before. Success clearly depends on previous choices, but the probability of each success will always be at least 1%, **independent of previous choices.**