CMSC 451: More NP-completeness Results

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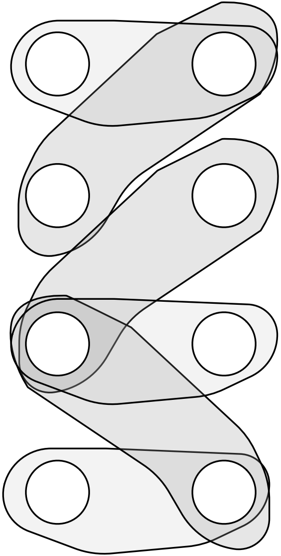
Based on Sect. 8.5,8.7,8.9 of *Algorithm Design* by Kleinberg & Tardos.

Three-Dimensional Matching

Two-Dimensional Matching



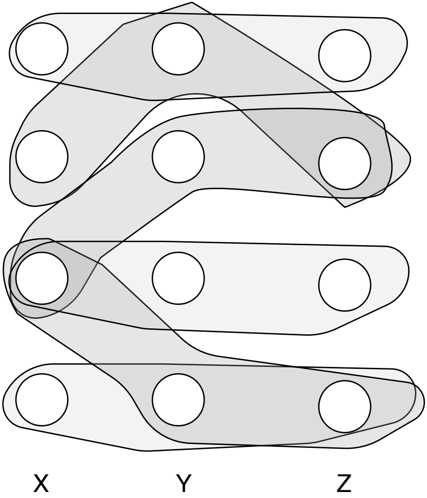
Recall '2-d matching':

Given sets X and Y, each with n elements, and a set E of pairs $\{x, y\}$,

Question: is there a choice of pairs such that every element in $X \cup Y$ is paired with some other element?

Usually, we thought of edges instead of pairs: $\{x, y\}$, but they are really the same thing.

Three-Dimensional Matching



Given: Sets X, Y, Z, each of size n, and a set $T \subset X \times Y \times Z$ of order triplets.

Question: is there a set of n triplets in T such that each element is contained in exactly one triplet?

3DM Is NP-Complete

Theorem

Three-dimensional matching (aka 3DM) is NP-complete

Proof. 3DM is in NP: a collection of n sets that cover every element exactly once is a certificate that can be checked in polynomial time.

Reduction from 3-SAT. We show that:

$$\text{3-SAT} \leq_P \text{3DM}$$

In other words, if we could solve 3DM, we could solve 3-SAT.

3-SAT $\leq_P$ 3DM

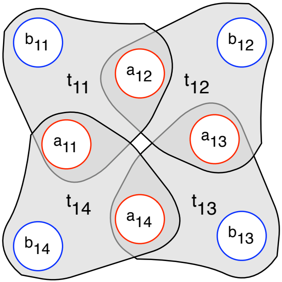
3SAT instance: $x_1, \ldots, x_n$ be n boolean variables, and $C_1, \ldots, C_k$ clauses.

We create a gadget for each variable x_i:

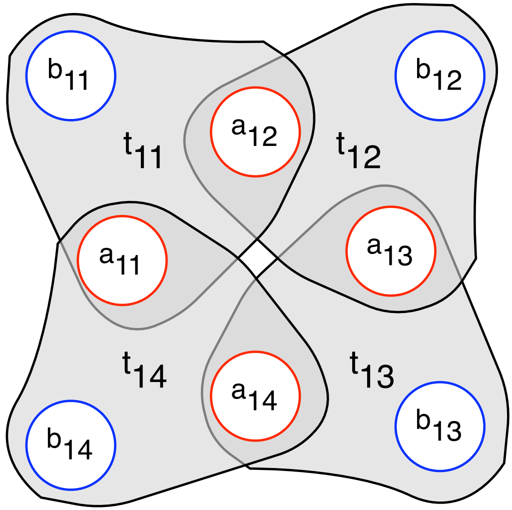
$$A_i = \{a_{i1}, \ldots, a_{i,2k}\} \quad \textit{core}$$

$$B_i = \{a_{i1}, \ldots, a_{i,2k}\} \quad \textit{tips}$$

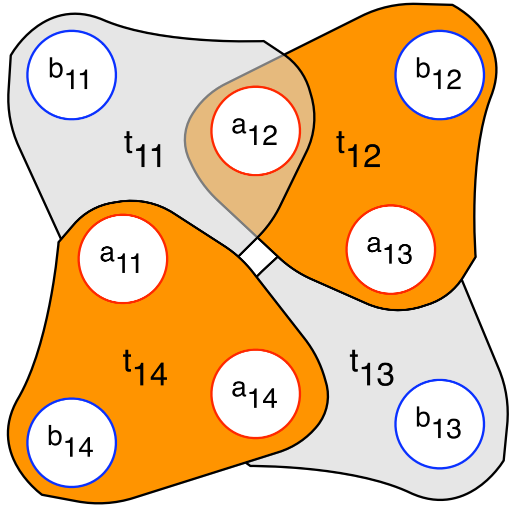
$$t_{ij} = (a_{ij}, a_{i,j+1}, b_{ij}) \quad \textit{TF triples}$$



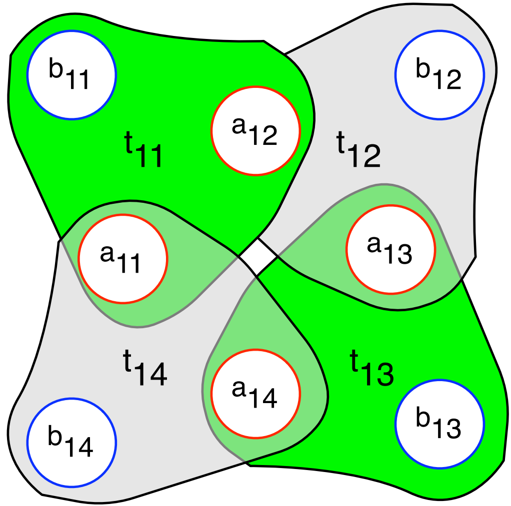
Gadget Encodes True and False



Gadget Encodes True and False

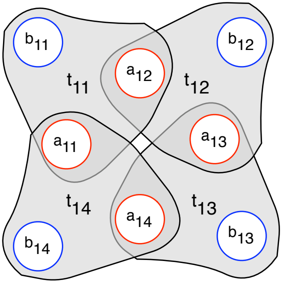


Gadget Encodes True and False



How “choice” is encoded

- We can only either use the even or odd “wings” of the gadget.
- In other words, if we use the even wings, we leave the odd tips uncovered (and vice versa).
- Leaving the odd tips free for gadget i means setting x_i to **false**.
- Leaving the odd tips free for gadget i means setting x_i to **true**.



Clause Gadgets

Need to encode constraints between the tips that ensure we satisfy all the clauses.

We create a gadget for each clause $C_j = \{t_1, t_2, t_3\}$

$$P_j = \{c_j, c_j'\} \quad \textit{Clause core}$$

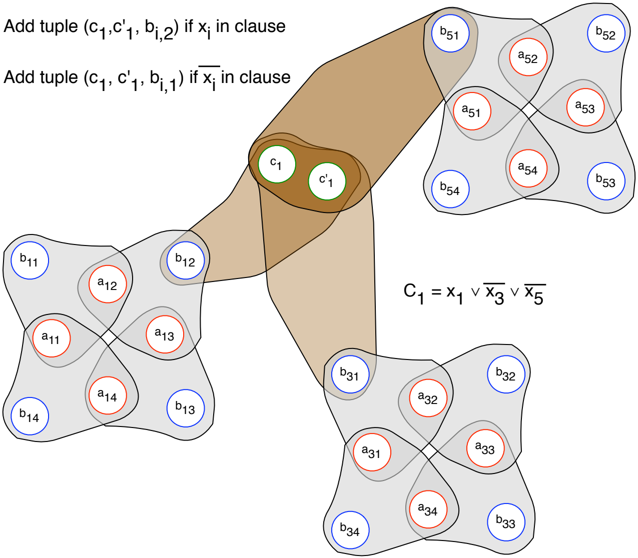
We will hook up these two clause core nodes with some tip nodes depending on whether the clause asks for a variable to be true or false.

See the next slide.

Clause Gadget Hookup

Add tuple $(c_1, c'_1, b_{i,2})$ if x_i in clause

Add tuple $(c_1, c'_1, b_{i,1})$ if $\overline{x_i}$ in clause



$C_1 = x_1 \vee \overline{x_3} \vee \overline{x_5}$

Clause Gadgets

Since only clause tuples (brown) cover c_j, c'_j, we have to choose exactly one of them for every clause.

We can only choose a clause tuple (c_j, c'_j, b_{ij}) if we haven't chosen a TF tuple that already covers b_{ij}.

Hence, we can satisfy (cover) the clause (c_j, c'_j) with the term represented by b_{ij} only if we "set" x_i to the appropriate value.

That's the basic idea. Two technical points left...

Details

Need to cover all the tips:

Even if we satisfy all the clauses, we might have extra tips left over. We add a clean up gadget (q_i, q_i', b) for every tip b.

Can we partition the sets?

$$X = \{a_{ij} : j \text{ even}\} \cup \{c_j\} \cup \{q_i\}$$
$$Y = \{a_{ij} : j \text{ odd}\} \cup \{c_j'\} \cup \{q_i'\}$$
$$Z = \{b_{ij}\}$$

Every set we defined uses 1 element from each of X, Y, Z.

Proof

If there is a satisfying assignment,

We choose the odd / even wings depending on whether we set a variable to **true** or **false**. At least 1 free tip for a term will be available to use to cover each clause gadget. We then use the clean up gadgets to cover all the rest of the tips.

If there is a 3D matching,

We can set variable x_i to **true** or **false** depending on whether it's even or odd wings were chosen. Because $\{c_j, c'_j\}$ were covered, we must have correctly chosen one even/odd wing that will satisfy this clause.