SAT, Coloring, Hamiltonian Cycle, TSP

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Sects. 8.2, 8.7, 8.5

Boolean Formulas

Boolean Formulas:

Variables: x_1, x_2, x_3 (can be either **true** or **false**)

Terms: $t_1, t_2, \ldots, t_\ell$: t_j is either x_i or $\bar{x}_i$ (meaning either x_i or **not** x_i).

Clauses: $t_1 \vee t_2 \vee \cdots \vee t_\ell$ ($\vee$ stands for "OR")
A clause is **true** if any term in it is **true**.

Example 1: $(x_1 \vee \bar{x}_2), (\bar{x}_1 \vee \bar{x}_3), (x_2 \vee \bar{v}_3)$

Example 2: $(x_1 \vee x_2 \vee \bar{x}_3), (\bar{x}_2 \vee x_1)$

Boolean Formulas

Def. A truth assignment is a choice of **true** or **false** for each variable, ie, a function $v : X \to \{\textbf{true}, \textbf{false}\}$.

Def. A CNF formula is a conjunction of clauses:

$$C_1 \wedge C_2, \wedge \cdots \wedge C_k$$

Example: $(x_1 \vee \bar{x_2}) \wedge (\bar{x_1} \vee \bar{x_3}) \wedge (x_2 \vee \bar{v_3})$

Def. A truth assignment is a satisfying assignment for such a formula if it makes every clause **true**.

SAT and 3-SAT

Problem (Satisfiability (SAT)). *Given a set of clauses $C_1, \ldots, C_k$ over variables $X = \{x_1, \ldots, x_n\}$ is there a satisfying assignment?*

Problem (Satisfiability (3-SAT)). *Given a set of clauses $C_1, \ldots, C_k$, each of length 3, over variables $X = \{x_1, \ldots, x_n\}$ is there a satisfying assignment?*

Cook-Levin Theorem

Theorem (Cook-Levin). *3-SAT is NP-complete.*

Proven in early 1970s by Cook. Slightly different proof by Levin independently.

Idea of the proof: encode the workings of a Nondeterministic Turing machine for an instance I of problem $X \in \mathbf{NP}$ as a SAT formula so that the formula is satisfiable if and only if the nondeterministic Turing machine would accept instance I.

We won't have time to prove this, but it gives us our first hard problem.

Reducing 3-SAT to Independent Set

Thm. 3-SAT $\leq_P$ Independent Set

Proof. Suppose we have an algorithm to solve Independent Set, how can we use it to solve 3-SAT?

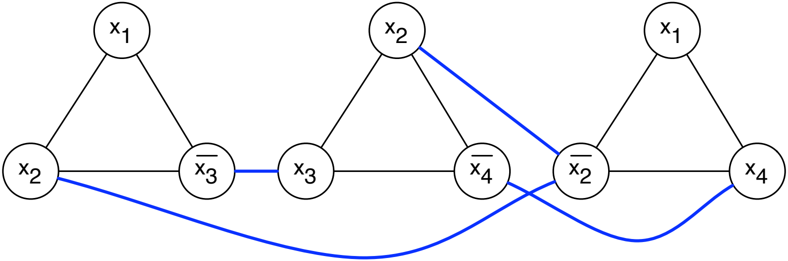
To solve 3-SAT:

- you have to choose a term from each clause to set to **true**,
- but you can't set both x_i and $\bar{x}_i$ to **true**.

How do we do the reduction?

3-SAT $\leq_P$ Independent Set

$$(x_1 \vee x_2 \vee \overline{x_3}) \wedge (x_2 \vee x_3 \vee \overline{x_4}) \wedge (x_1 \vee \overline{x_2} \vee x_4)$$



Proof

Theorem. *This graph has an independent set of size k iff the formula is satisfiable.*

Proof. $\Longrightarrow$ If the formula is satisfiable, there is at least one true literal in each clause. Let S be a set of one such true literal from each clause. $|S| = k$ and no two nodes in S are connected by an edge.

$\Longrightarrow$ If the graph has an independent set S of size k, we know that it has one node from each "clause triangle." Set those terms to **true**. This is possible because no 2 are negations of each other. $\square$

General Proof Strategy

General Strategy for Proving Something is NP-complete:

1. Must show that $X \in \mathbf{NP}$. Do this by showing there is an certificate that can be efficiently checked.

2. Look at some problems that are known to be NP-complete (there are thousands), and choose one Y that seems "similar" to your problem in some way.

3. Show that $Y \leq_P X$.

Strategy for Showing $Y \leq_P X$

One strategy for showing that $Y \leq_P X$ often works:

1. Let I_Y be any instance of problem Y.
2. Show how to construct an instance I_X of problem X in polynomial time such that:
   - If $I_Y \in Y$, then $I_X \in X$
   - If $I_X \in X$, then $I_Y \in Y$

Hamiltonian Cycle

Hamiltonian Cycle Problem

Problem (Hamiltonian Cycle). *Given a directed graph G, is there a cycle that visits every vertex exactly once?*

Such a cycle is called a Hamiltonian cycle.

Hamiltonian Cycle is NP-complete

Theorem. *Hamiltonian Cycle is NP-complete.*

Proof. First, HamCycle $\in$ **NP**. Why?

Second, we show 3-SAT $\leq_P$ Hamiltonian Cycle.

Suppose we have a black box to solve Hamiltonian Cycle, how do we solve 3-SAT?

In other words: how do we encode an instance I of 3-SAT as a graph G such that I is satisfiable exactly when G has a Hamiltonian cycle.

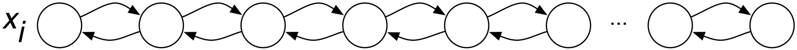
Consider an instance I of 3-SAT, with variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_k$.

Reduction Idea

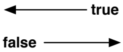
Reduction Idea (very high level):

- Create some graph structure (a "gadget") that represents the variables
- And some graph structure that represents the clauses
- Hook them up in some way that encodes the formula
- Show that this graph has a Ham. cycle iff the formula is satisfiable.

Gadget Representing the Variables

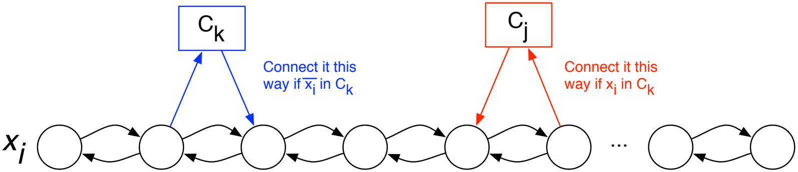


Direction we travel along this chain represents whether to set the variable to **true** or **false**.



Hooking in the Clauses

Add a new node for each clause:

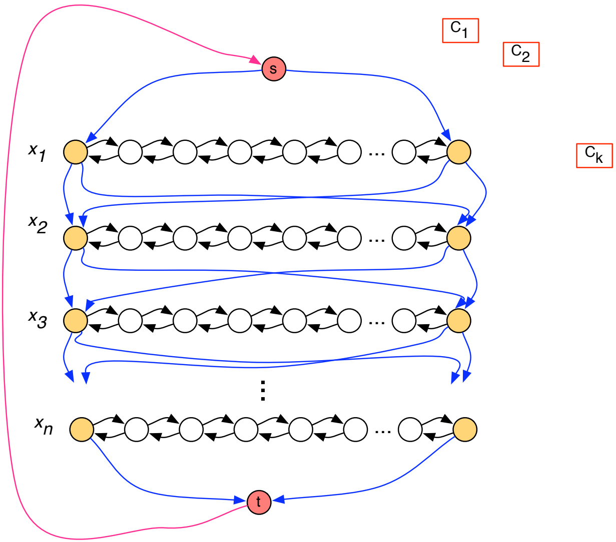


Direction we travel along this chain represents whether to set the variable to **true** or **false**.

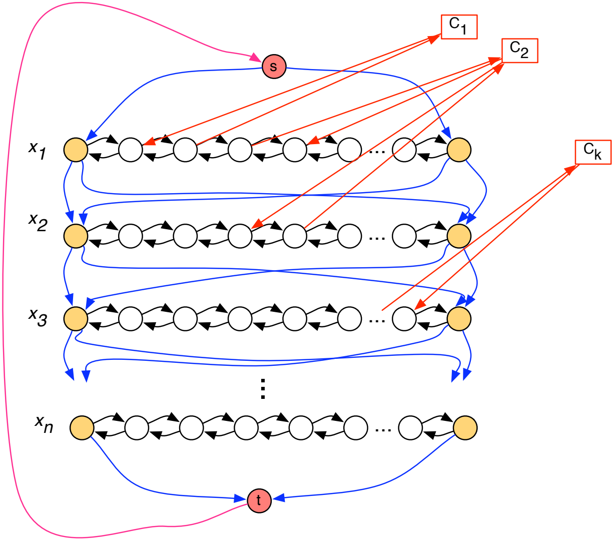
← **true**

false →

Connecting up the paths



Connecting up the paths



Hamiltonian Cycle is NP-complete

- A Hamiltonian path encodes a truth assignment for the variables (depending on which direction each chain is traversed)
- For there to be a Hamiltonian cycle, we have to visit every clause node
- We can only visit a clause if we satisfy it (by setting one of its terms to true)
- Hence, if there is a Hamiltonian cycle, there is a satisfying assignment

Hamiltonian Path

Hamiltonian Path: Does G contain a path that visits every node exactly once?

How could you prove this problem is NP-complete?

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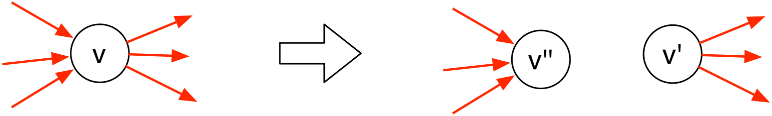
Reduce Hamiltonian Cycle to Hamiltonian Path.

Given instance of Hamiltonian Cycle G, choose an arbitrary node v and split it into two nodes to get graph G':



Now any Hamiltonian Path must start at v' and end at v''.

Hamiltonian Path

G'' has a Hamiltonian Path $\iff$ G has a Hamiltonian Cycle.

$\Longrightarrow$ If G'' has a Hamiltonian Path, then the same ordering of nodes (after we glue v' and v'' back together) is a Hamiltonian cycle in G.

$\Longleftarrow$ If G has a Hamiltonian Cycle, then the same ordering of nodes is a Hamiltonian path of G' if we split up v into v' and v''. $\square$

Hence, Hamiltonian Path is NP-complete.

Traveling Salesman Problem

Problem (Traveling Salesman Problem). *Given n cities, and distances $d(i,j)$ between each pair of cities, does there exist a path of length $\leq k$ that visits each city?*

Notes:

- We have a distance between every pair of cities.
- In this version, $d(i,j)$ doesn't have to equal $d(j,i)$.
- And the distances don't have to obey the triangle inequality ($d(i,j) \leq d(i,k) + d(k,j)$ for all i,j,k).

Traveling Salesman is NP-complete

Thm. Traveling Salesman is NP-complete.

TSP seems a lot like Hamiltonian Cycle. We will show that

$$\textsc{Hamiltonian Cycle} \leq_P \mathit{TSP}$$

To do that:

Given: a graph $G = (V, E)$ that we want to test for a Hamiltonian cycle,

Create: an instance of TSP.

Creating a TSP instance

A TSP instance D consists of n cities, and $n(n-1)$ distances.

Cities We have a city c_i for every node v_i.

Distances Let $d(c_i, c_j) = \begin{cases} 1 & \text{if edge } (v_i, v_j) \in E \\ 2 & \text{otherwise} \end{cases}$

TSP Reduction

Theorem. *G has a Hamiltonian cycle* $\iff$ *D has a tour of length* $\leq n$.

Proof. If G has a Ham. Cycle, then this ordering of cities gives a tour of length $\leq n$ in D (only distances of length 1 are used).

Suppose D has a tour of length $\leq n$. The tour length is the sum of n terms, meaning each term must equal 1, and hence cities that are visited consecutively must be connected by an edge in G. □

Also, TSP $\in$ **NP**: a certificate is simply an ordering of the n cities.

TSP is NP-complete

Hence, TSP is NP-complete.

Even TSP restricted to the case when the $d(i, j)$ values come from actual distances on a map is NP-complete.

Graph Coloring

Graph Coloring Problem

Problem (Graph Coloring Problem). *Given a graph G, can you color the nodes with $\leq k$ colors such that the endpoints of every edge are colored differently?*

Notation: A k-coloring is a function $f : V \to \{1, \ldots, k\}$ such that for every edge $\{u, v\}$ we have $f(u) \neq f(v)$.

If such a function exists for a given graph G, then G is k-colorable.

Special case of $k = 2$

How can we test if a graph has a 2-coloring?

Special case of $k = 2$

How can we test if a graph has a 2-coloring?

Check if the graph is bipartite.

Unfortunately, for $k \geq 3$, the problem is NP-complete.

Theorem. *3-Coloring is NP-complete.*

Graph Coloring is NP-complete

3-Coloring $\in$ **NP**: A valid coloring gives a certificate.

We will show that:

$$\text{3-SAT} \leq_P \text{3-Coloring}$$

Let $x_1, \ldots, x_n, C_1, \ldots, C_k$ be an instance of 3-SAT.

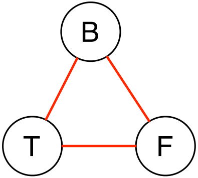
We show how to use 3-Coloring to solve it.

Reduction from 3-SAT

We construct a graph G that will be 3-colorable iff the 3-SAT instance is satisfiable.

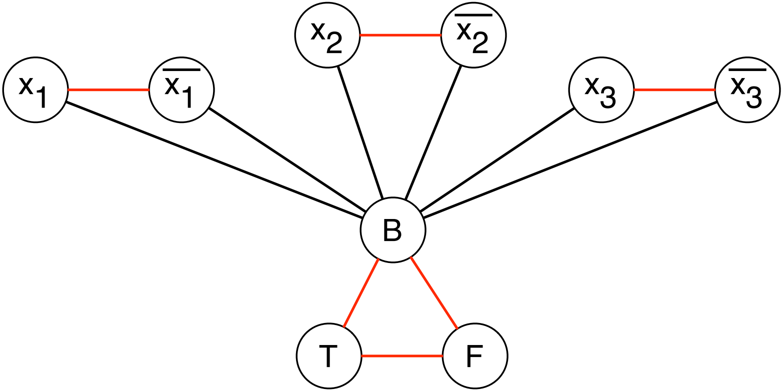
For every variable x_i, create 2 nodes in G, one for x_i and one for $\bar{x}_i$. Connect these nodes by an edge:

Create 3 *special nodes* T, F, and B, joined in a triangle:



Connecting them up

Connect every variable node to B:



Properties

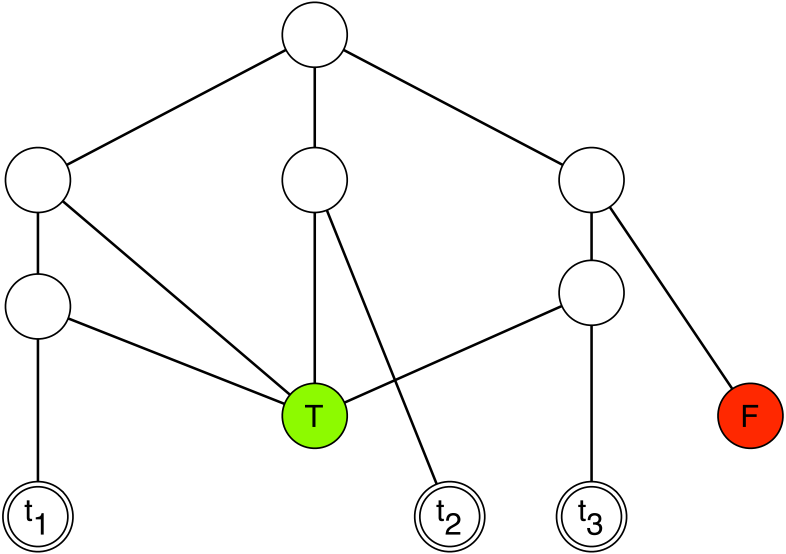
Properties:

- Each of x_i and $\bar{x}_i$ must get different colors
- Each must be different than the color of B.
- B, T, and F must get different colors.

Hence, any 3-coloring of this graph defines a valid truth assignment!

Still have to constrain the truth assignments to satisfy the given clauses, however.

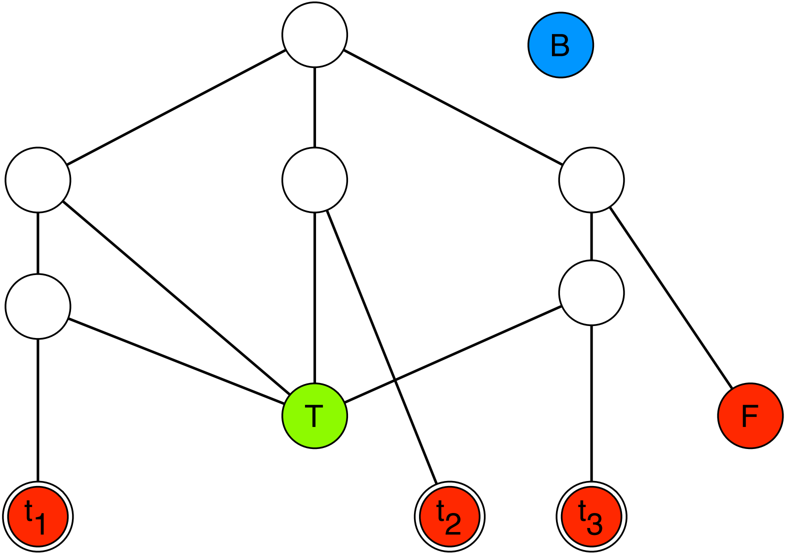
Connect Clause (t_1, t_2, t_3) up like this:



Suppose Every Term Was False

What if every term in the clause was assigned the **false** color?

Connect Clause (t_1, t_2, t_3) up like this:



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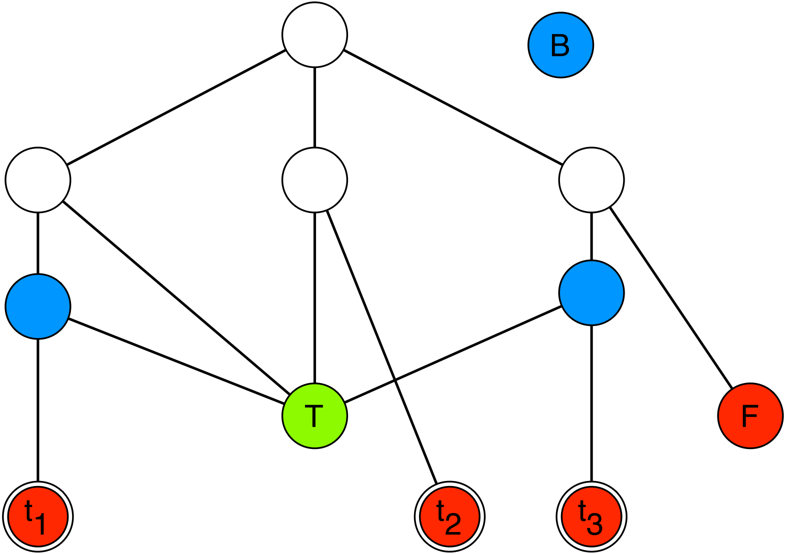

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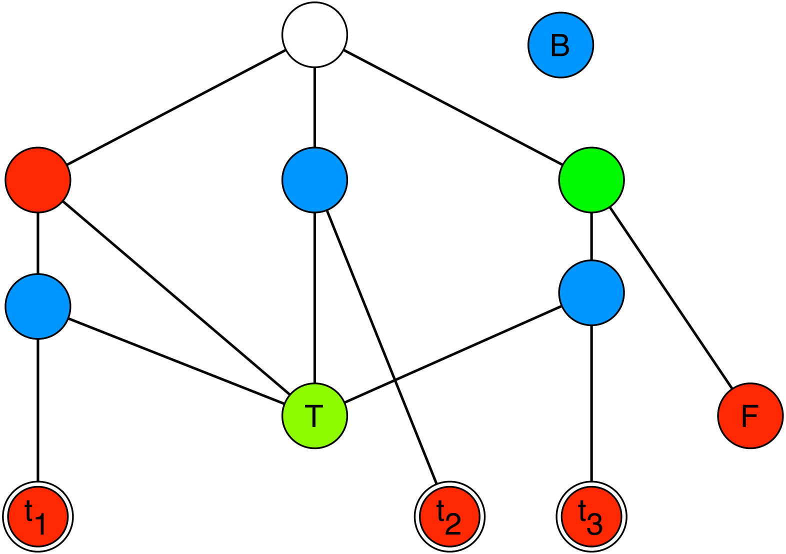

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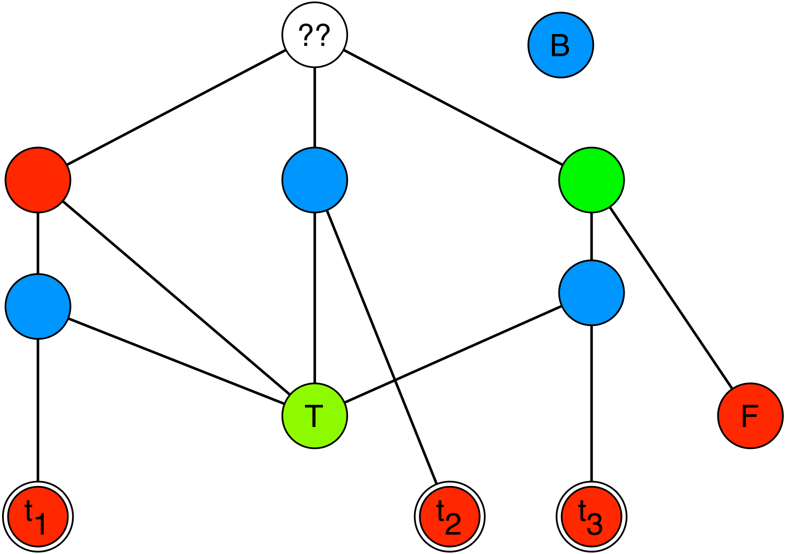

Suppose there is a 3-coloring

Top node is colorable iff one of its terms gets the **true** color.

Suppose there is a 3-coloring.

We get a satisfying assignment by:

- Setting $x_i =$ **true** iff v_i is colored the same as T

Let C be any clause in the formula. At least 1 of its terms must be true, because if they were all false, we couldn't complete the coloring (as shown above).

Suppose there is a satisfying assignment

Suppose there is a satisfying assignment.

We get a 3-coloring of G by:

- Coloring T, F, B arbitrarily with 3 different colors
- If $x_i =$ **true**, color v_i with the same color as T and $\bar{v}_i$ with the color of F.
- If $x_i =$ **false**, do the opposite.
- Extend this coloring into the clause gadgets.

Hence: the graph is 3-colorable iff the formula it is derived from is satisfiable.