Splay Trees

02-713

Dictionary Abstract Data Type (ADT)

- Most basic and most useful ADT:
  - `insert(key, value)`
  - `delete(key)`
  - `value = find(key)`

- Many languages have it built in:

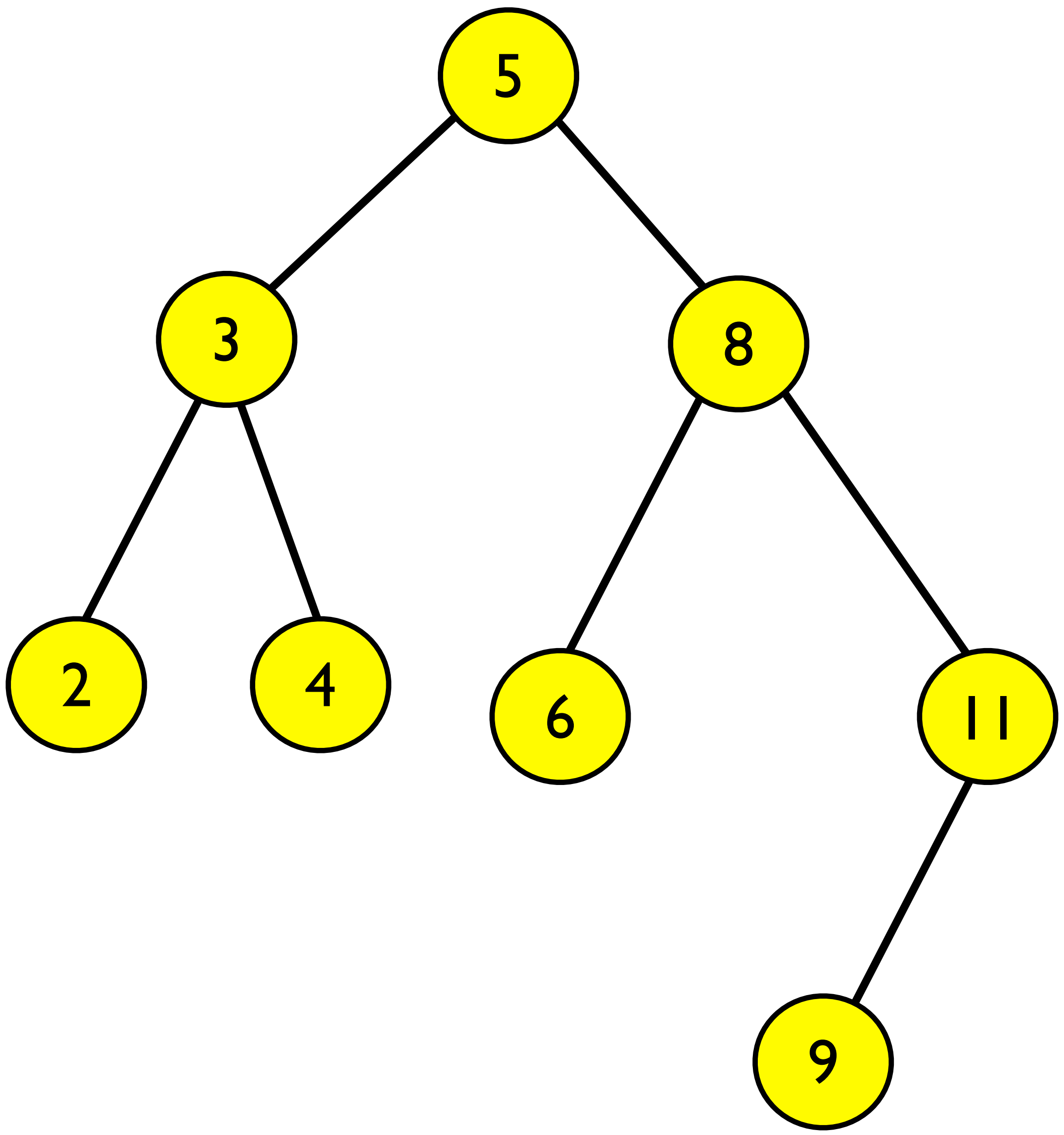
awk:	D["AAPL"] = 130	# associative array
perl:	`my %D; $D["AAPL"] = 130;`	# hash
python:	`D = {}; D["AAPL"] = 130`	# dictionary
C++:	`map<string,string> D = new map<string, string>();`	
	`D["AAPL"] = 130;`	// map

- **Insert**, **delete, find** each either O(log n) [C++] or expected constant [perl, python]

- How can such dictionaries are implemented?

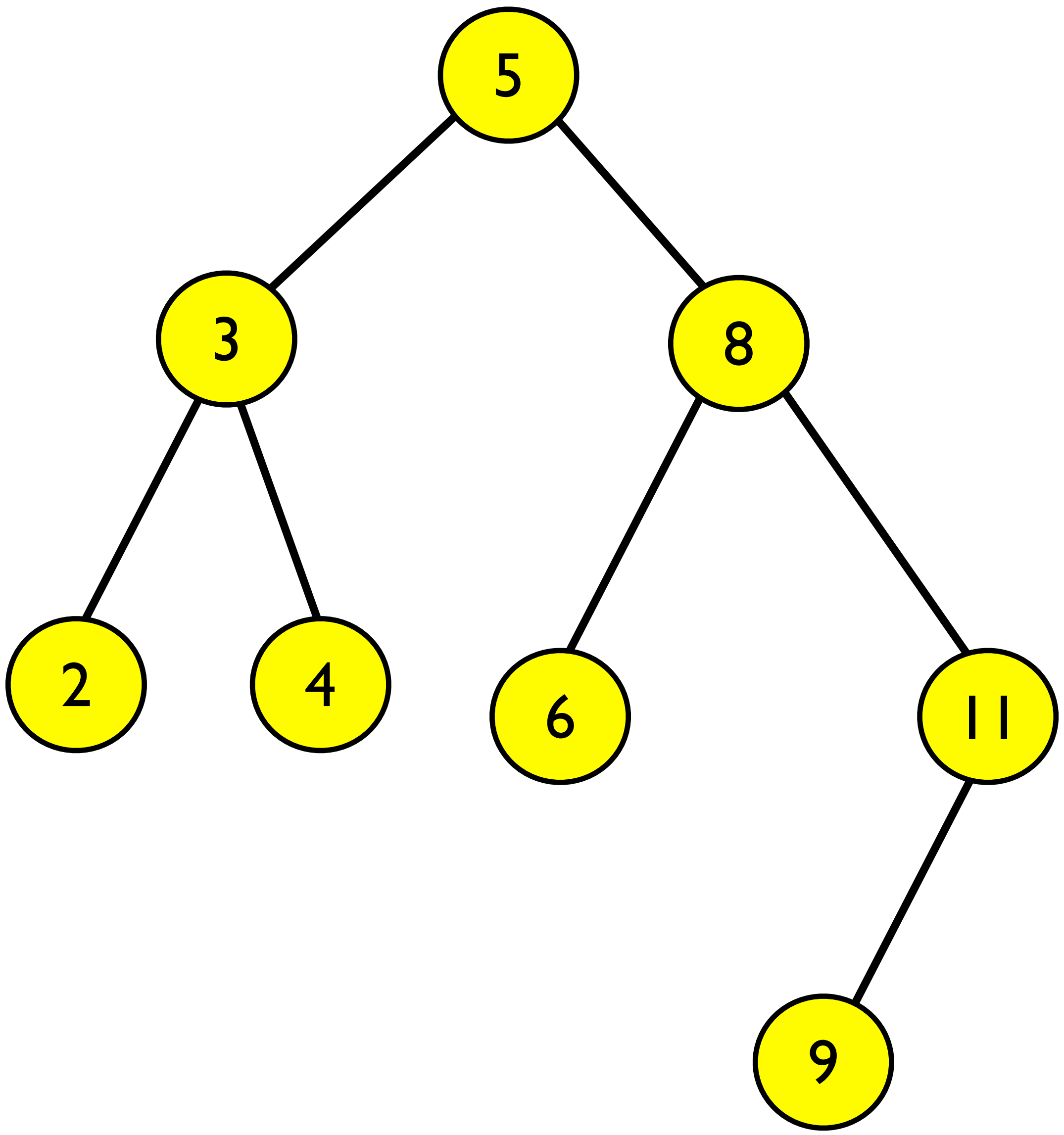
Binary Search Trees

- **BST Property:** If a node has key k then keys in the **left** subtree are $< k$ and keys in the **right** subtree are $> k$.
- For convenience, we disallow duplicate keys.
- Good for implementing the *dictionary ADT* we've already seen: insert, delete, find.



BST Find

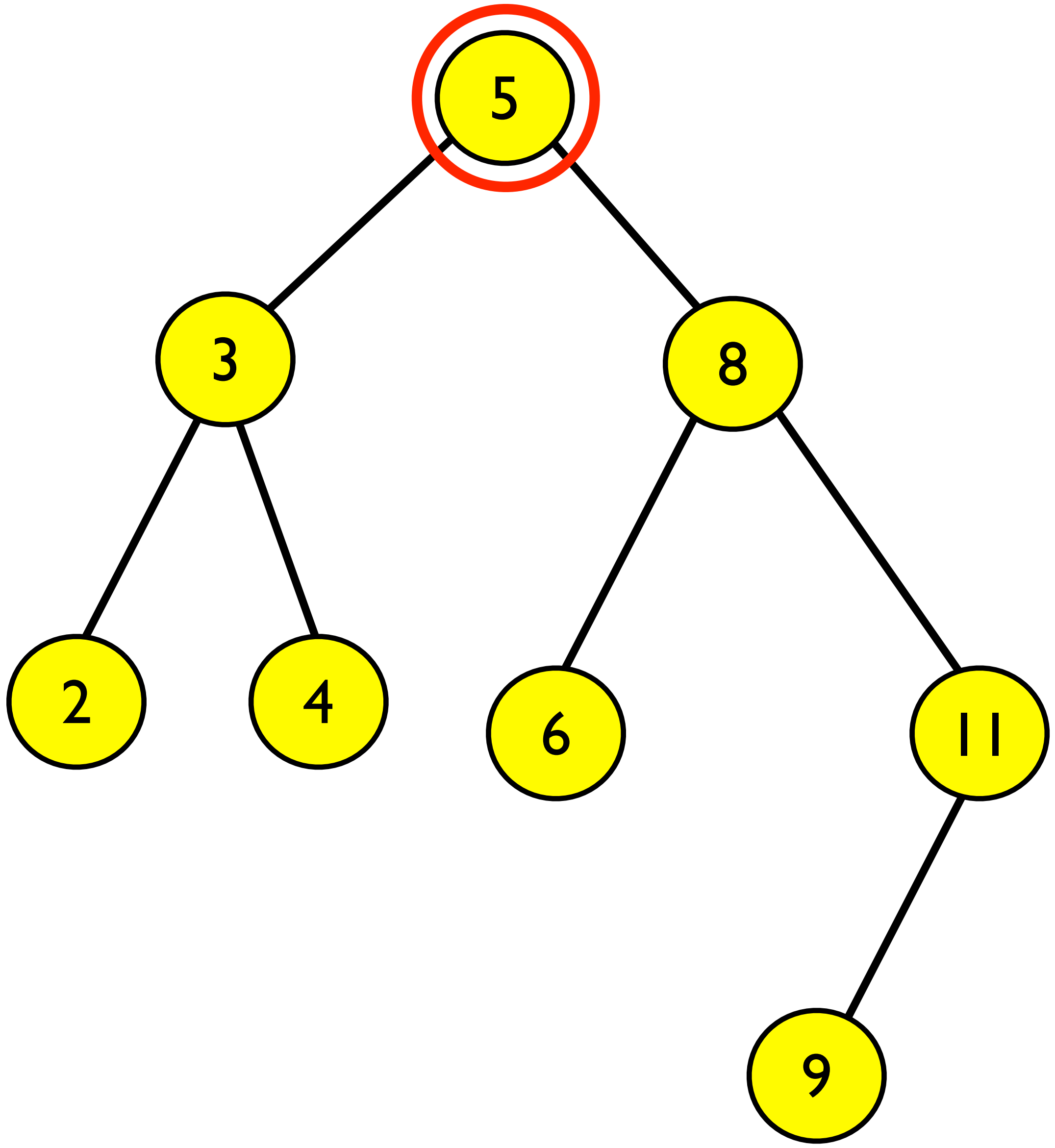
Find $k = 6$:



BST Find

Find $k = 6$:

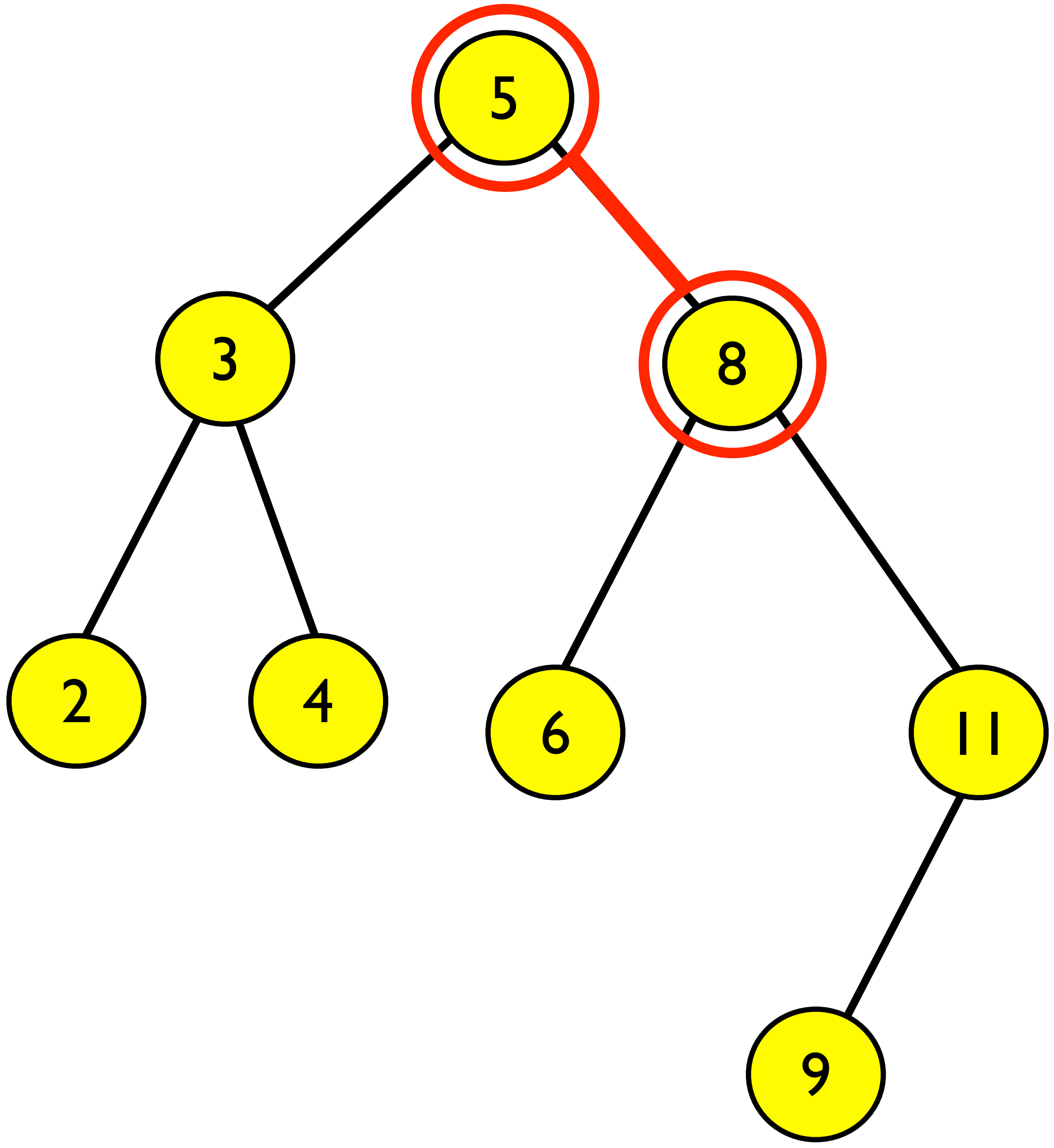
Is $k < 5$?



BST Find

Find $k = 6$:

Is $k < 5$? No, go right

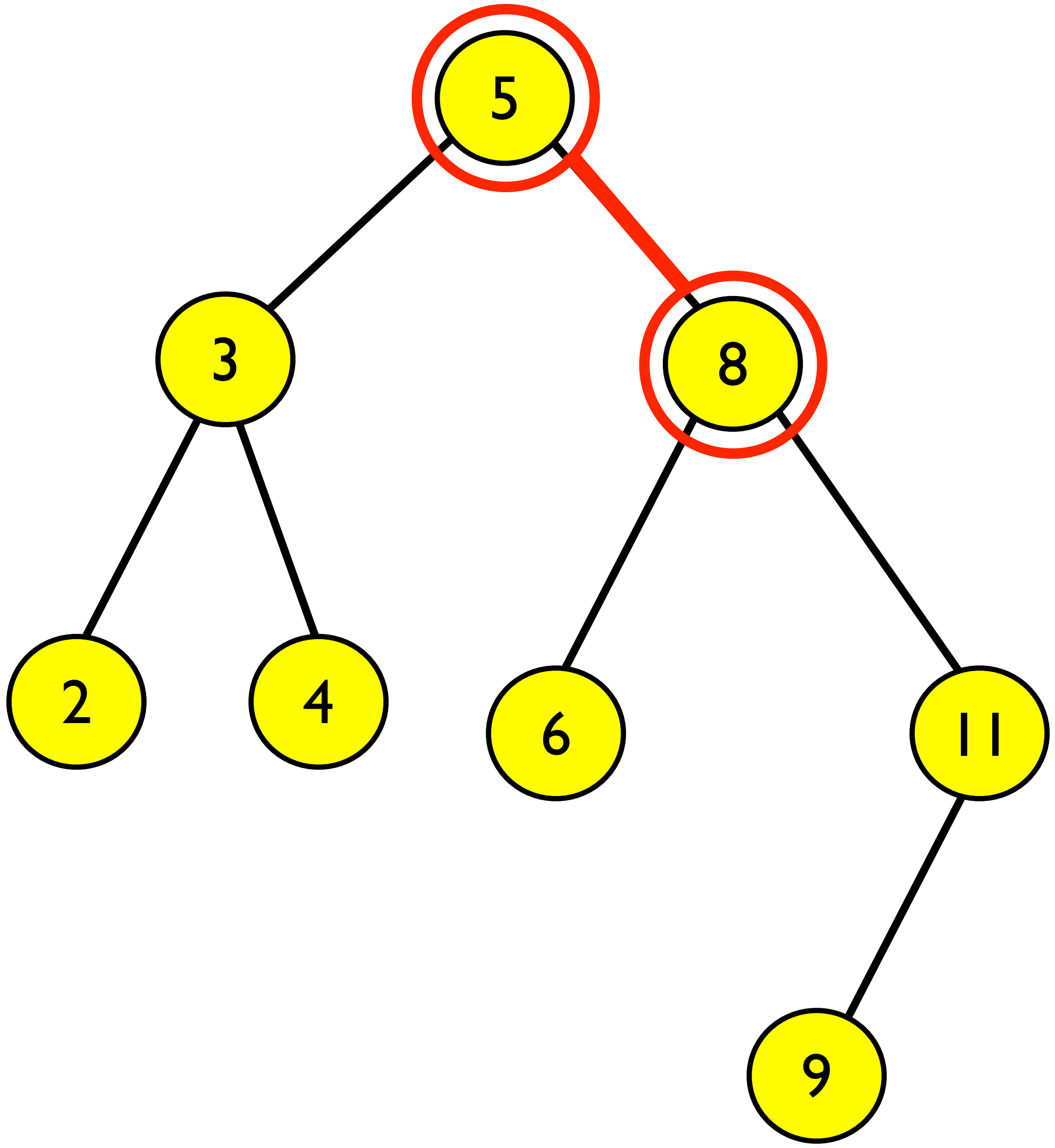


BST Find

Find $k = 6$:

Is $k < 5$? No, go right

Is $k < 8$?

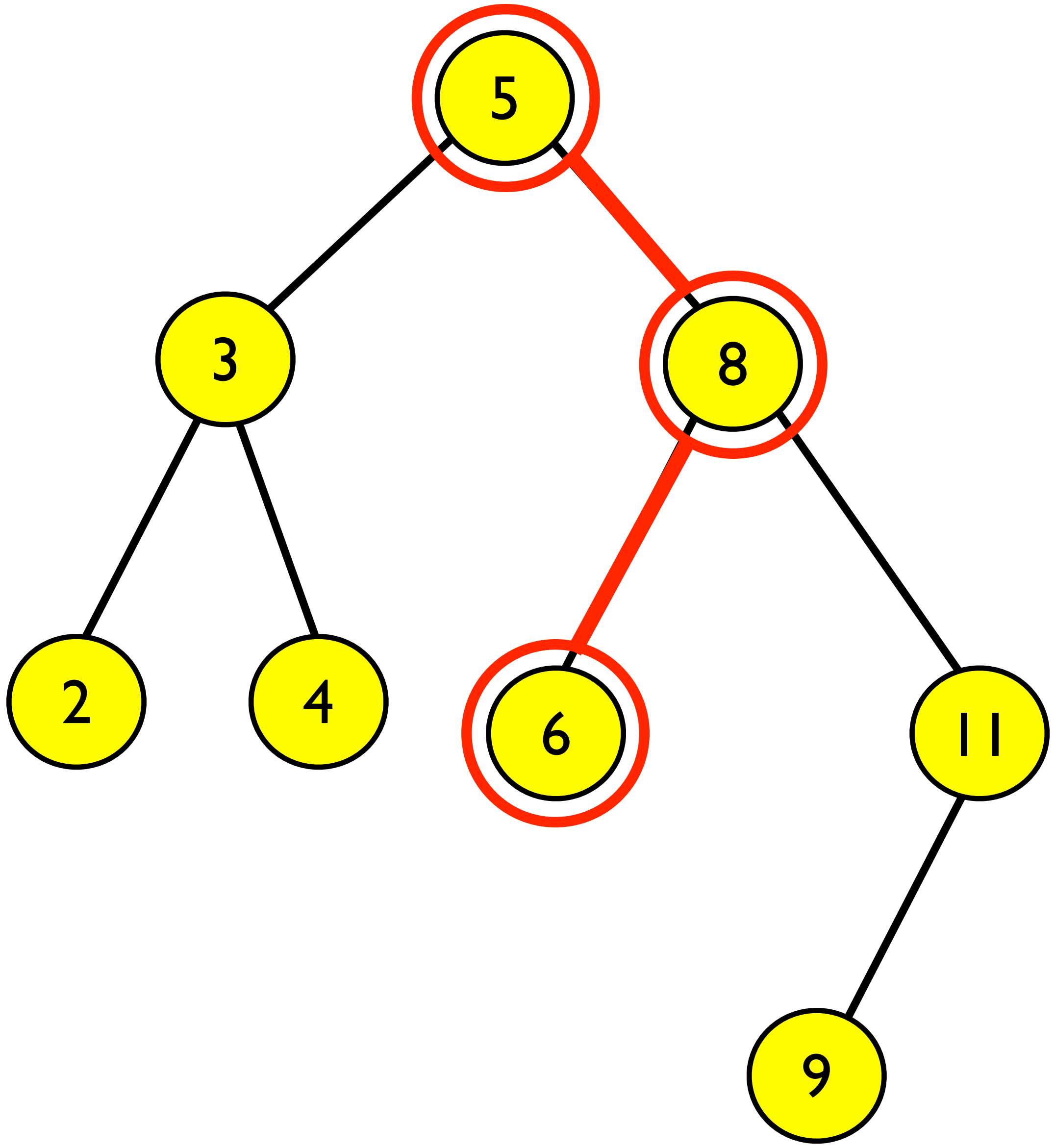


BST Find

Find $k = 6$:

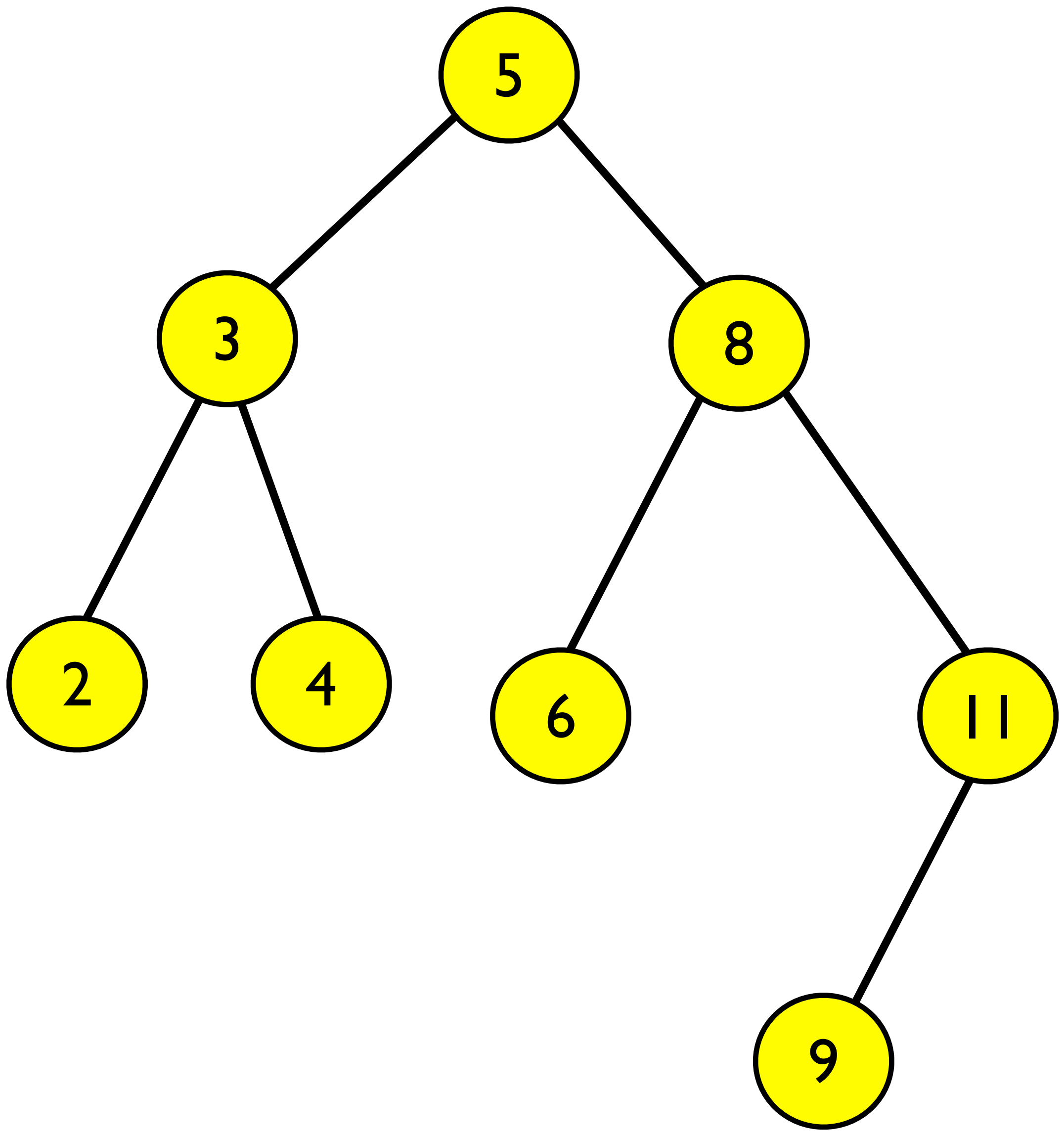
Is $k < 5$? No, go right

Is $k < 8$? Yes, go left



BST Find

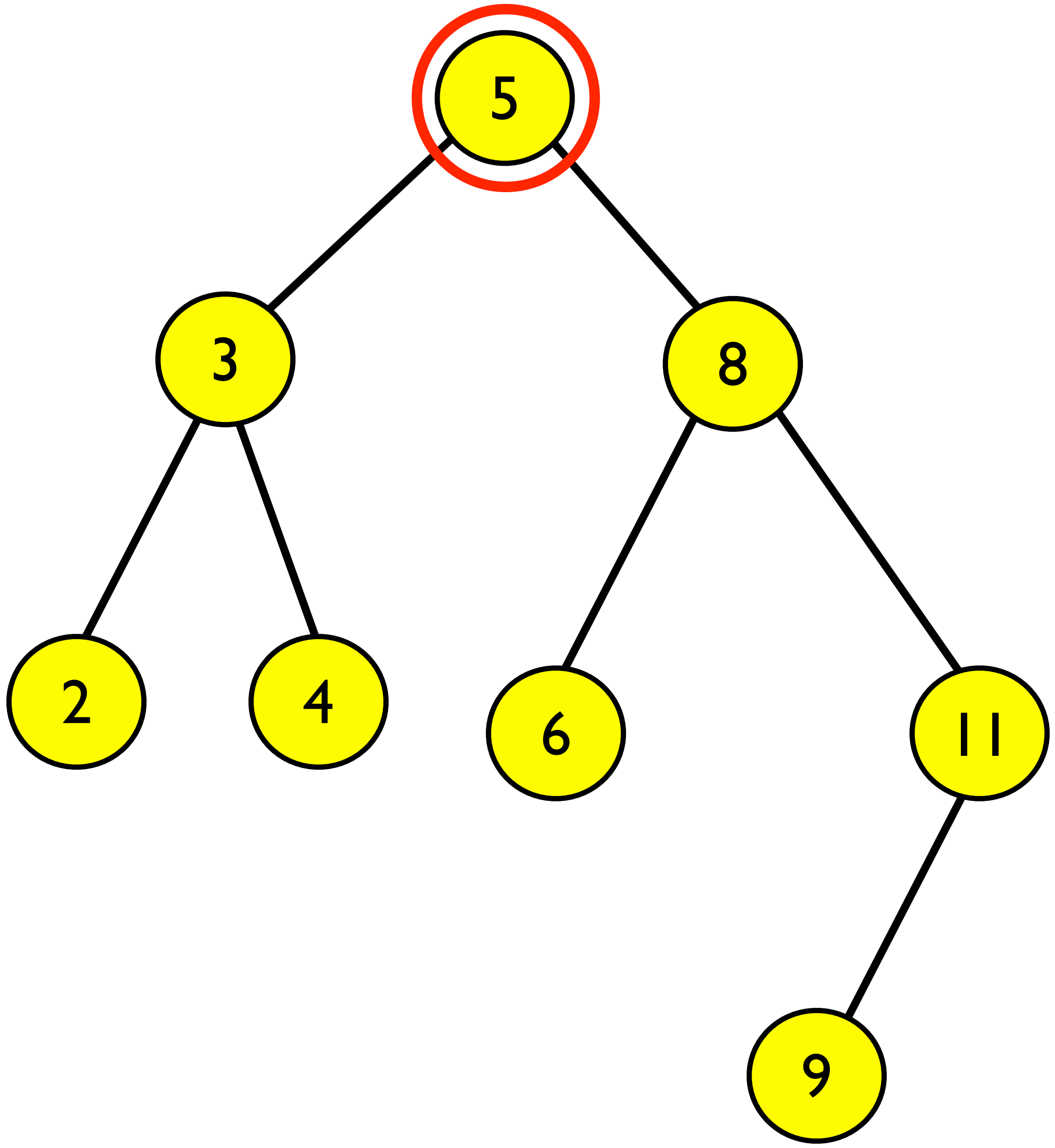
Find $k = 9$:



BST Find

Find $k = 9$:

Is $k < 5$?



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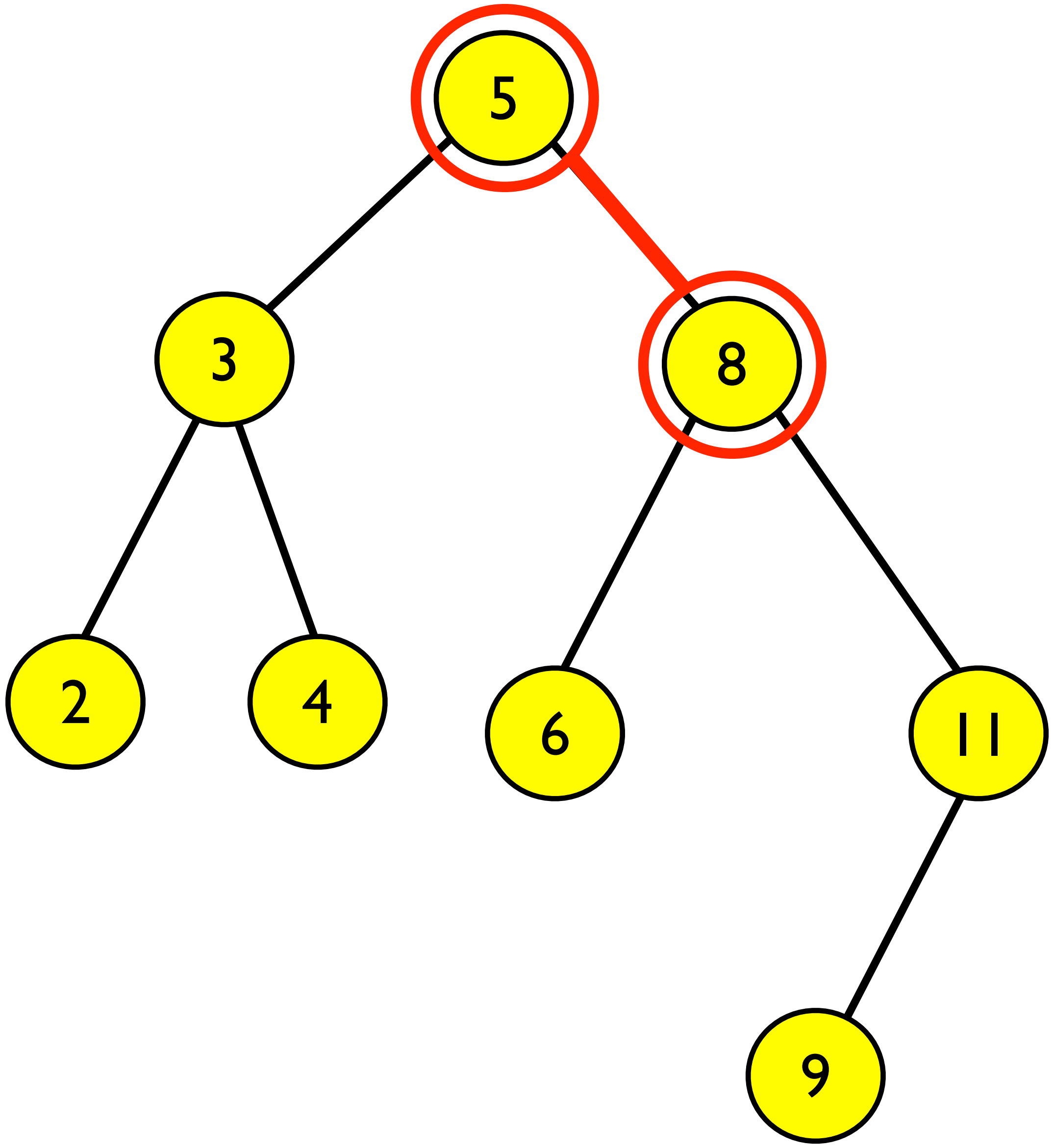
Find $k = 9$:

Is $k < 5$? No, go right

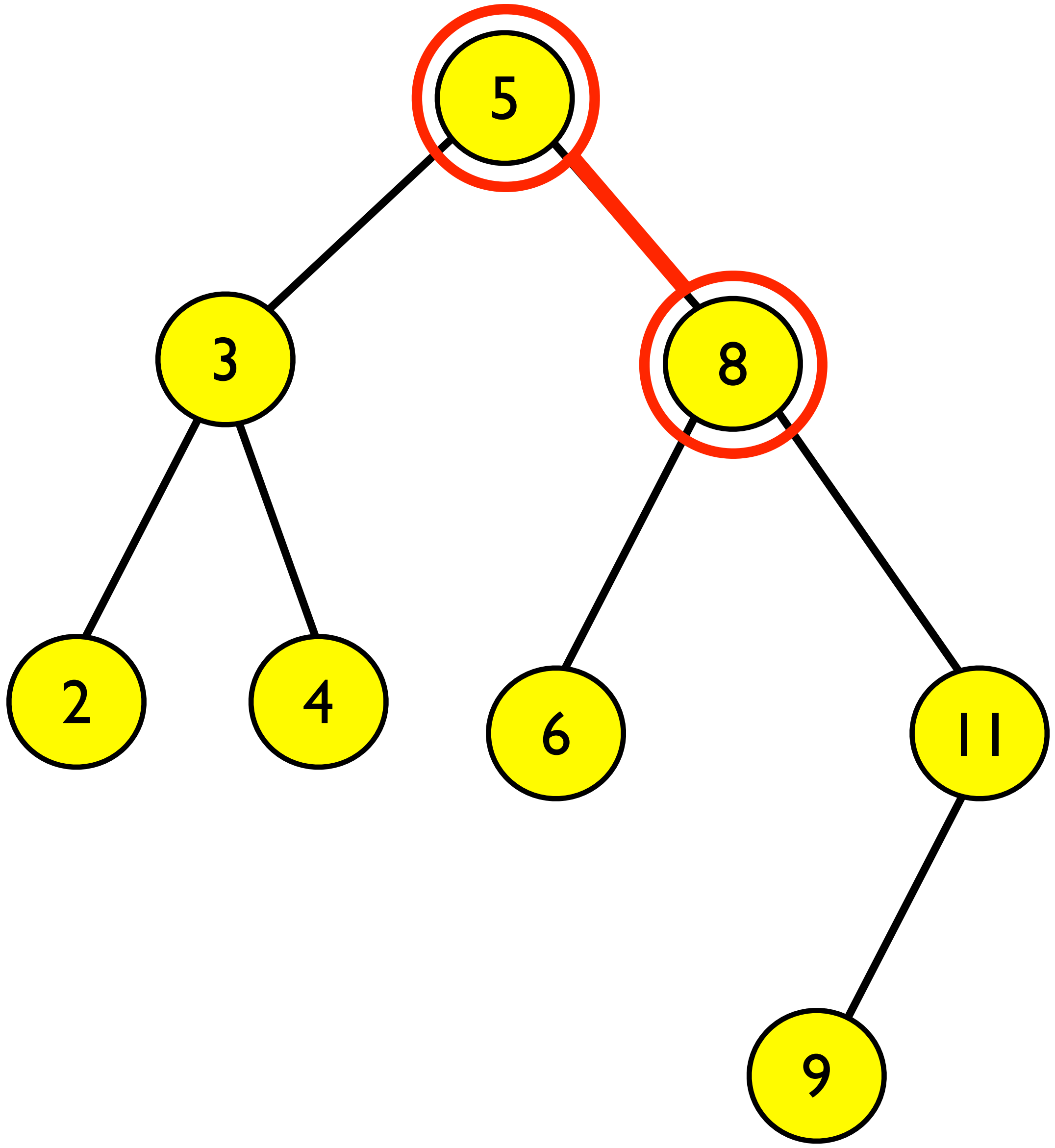


BST Find

Find $k = 9$:

Is $k < 5$? No, go right

Is $k < 8$?



BST Find

Find $k = 9$:

Is $k < 5$? No, go right

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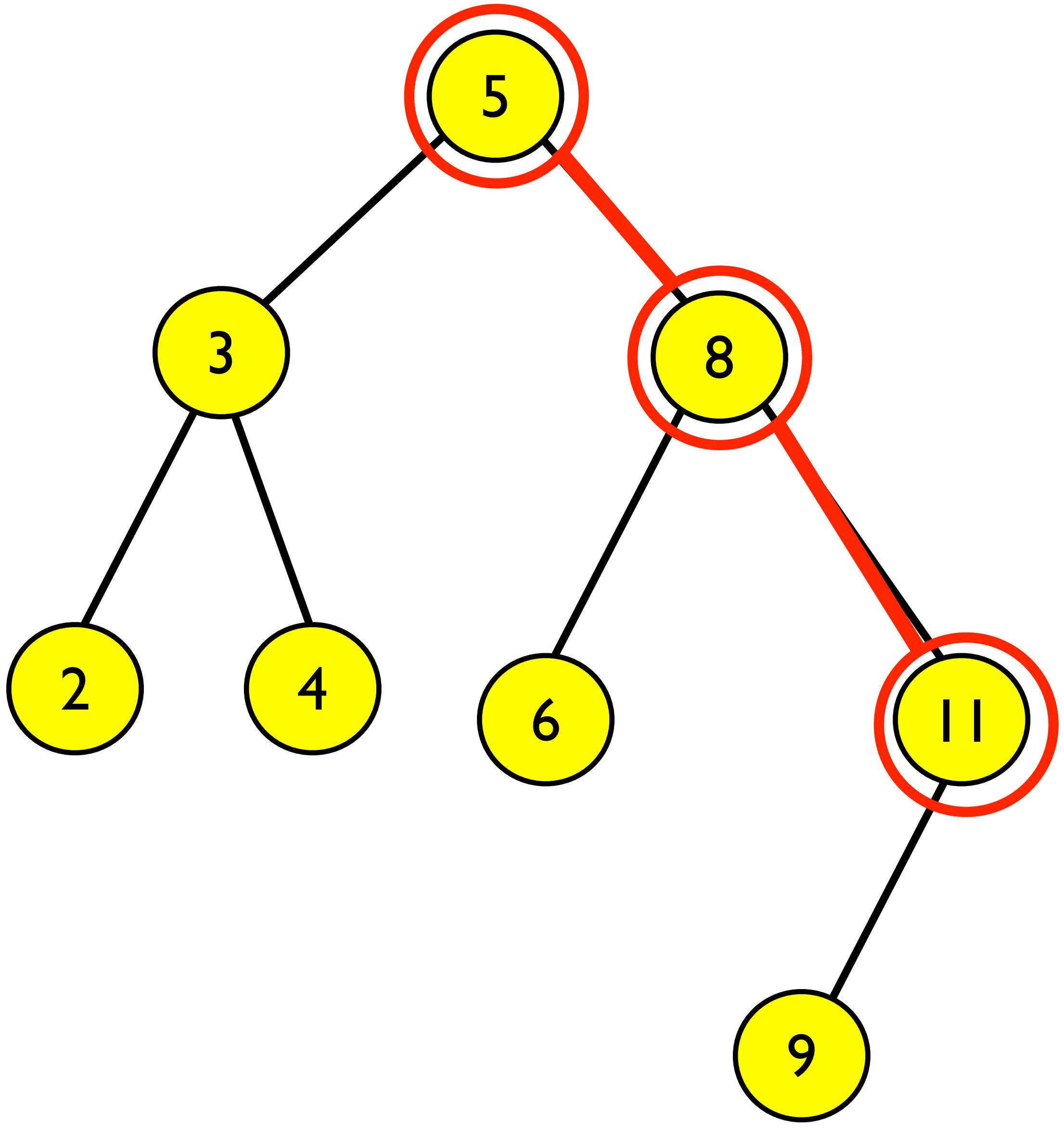


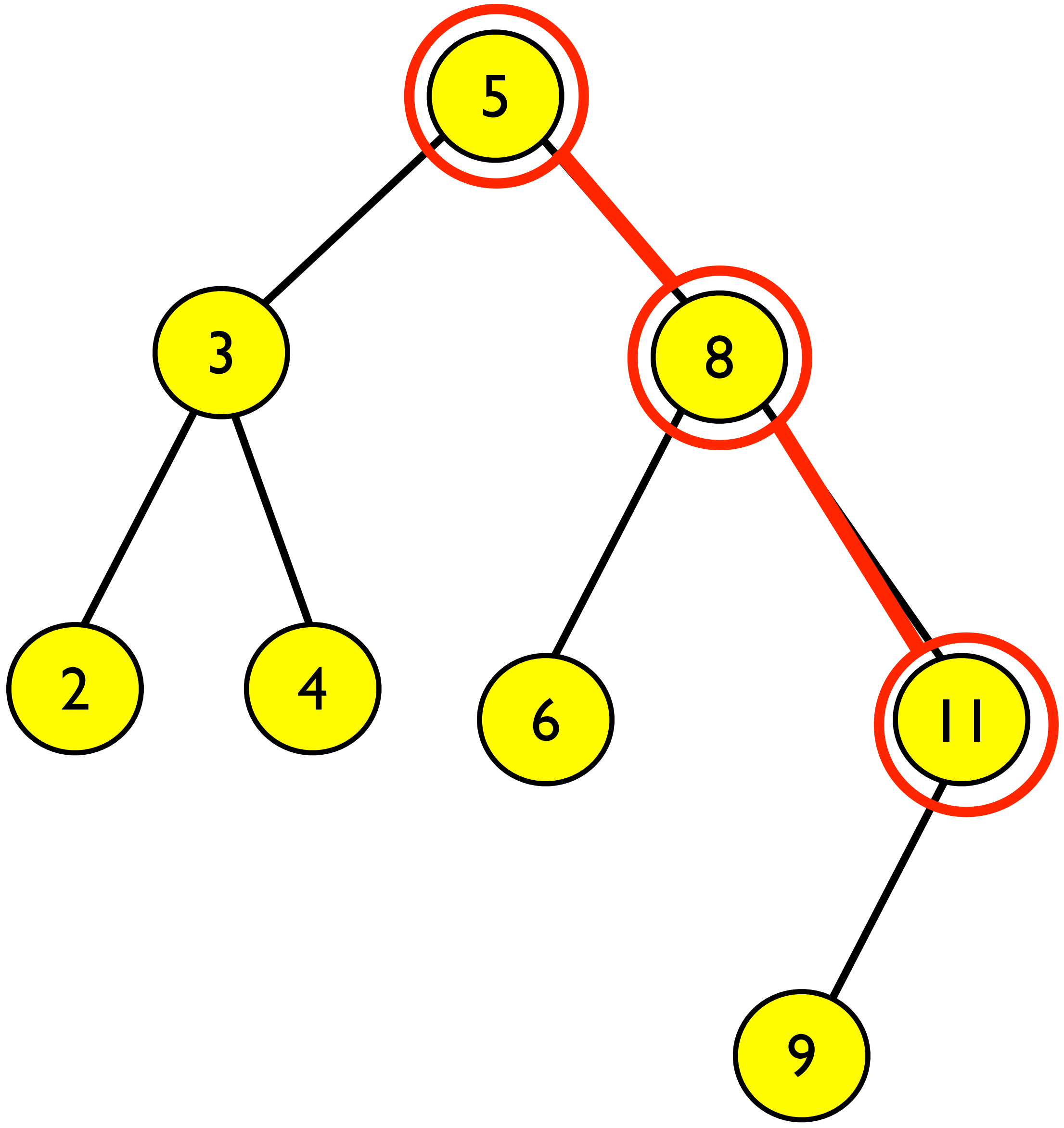
BST Find

Find $k = 9$:

Is $k < 5$? No, go right

Is $k < 8$? No, go right

Is $k < 11$?



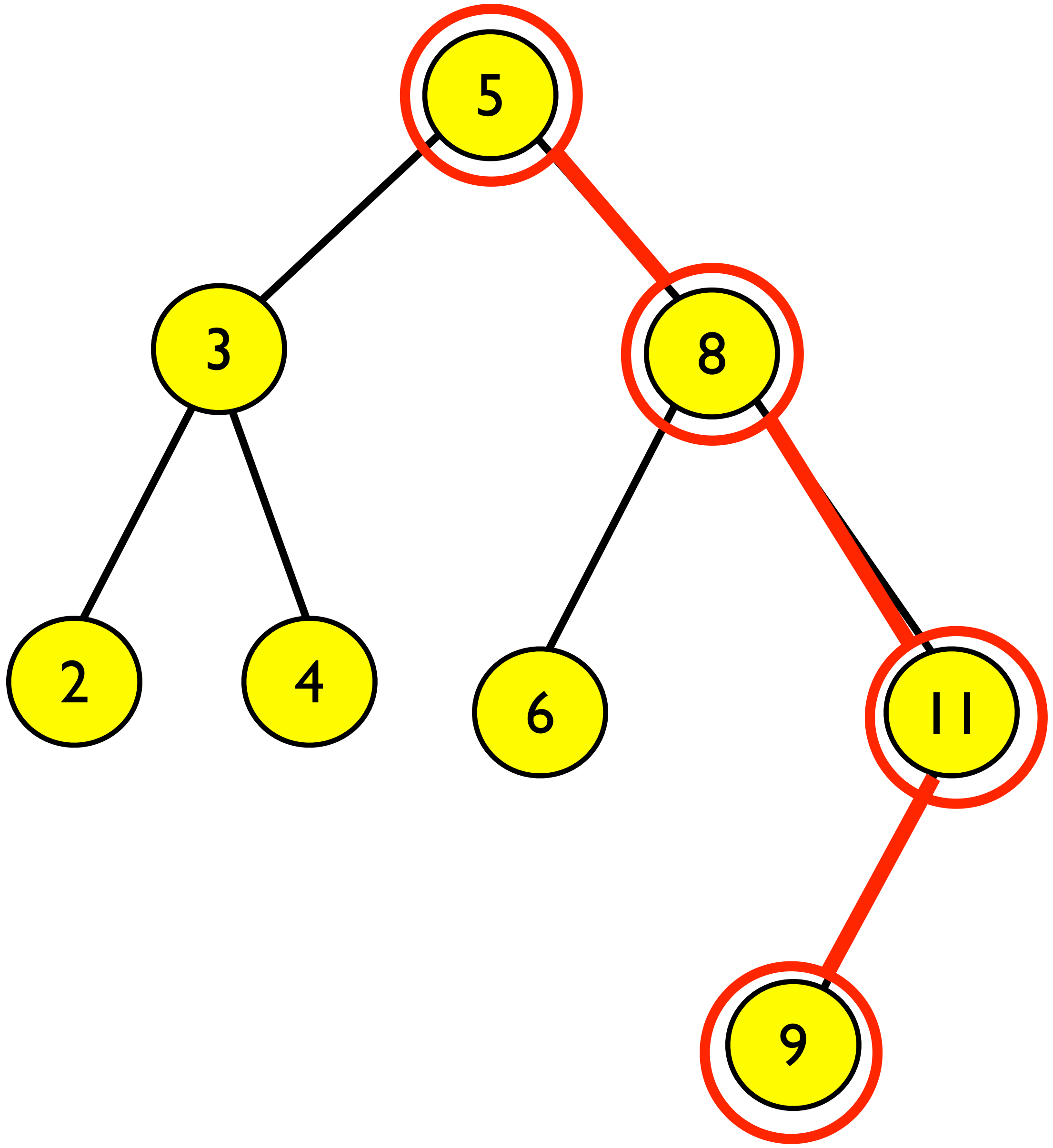
BST Find

Find $k = 9$:

Is $k < 5$? No, go right

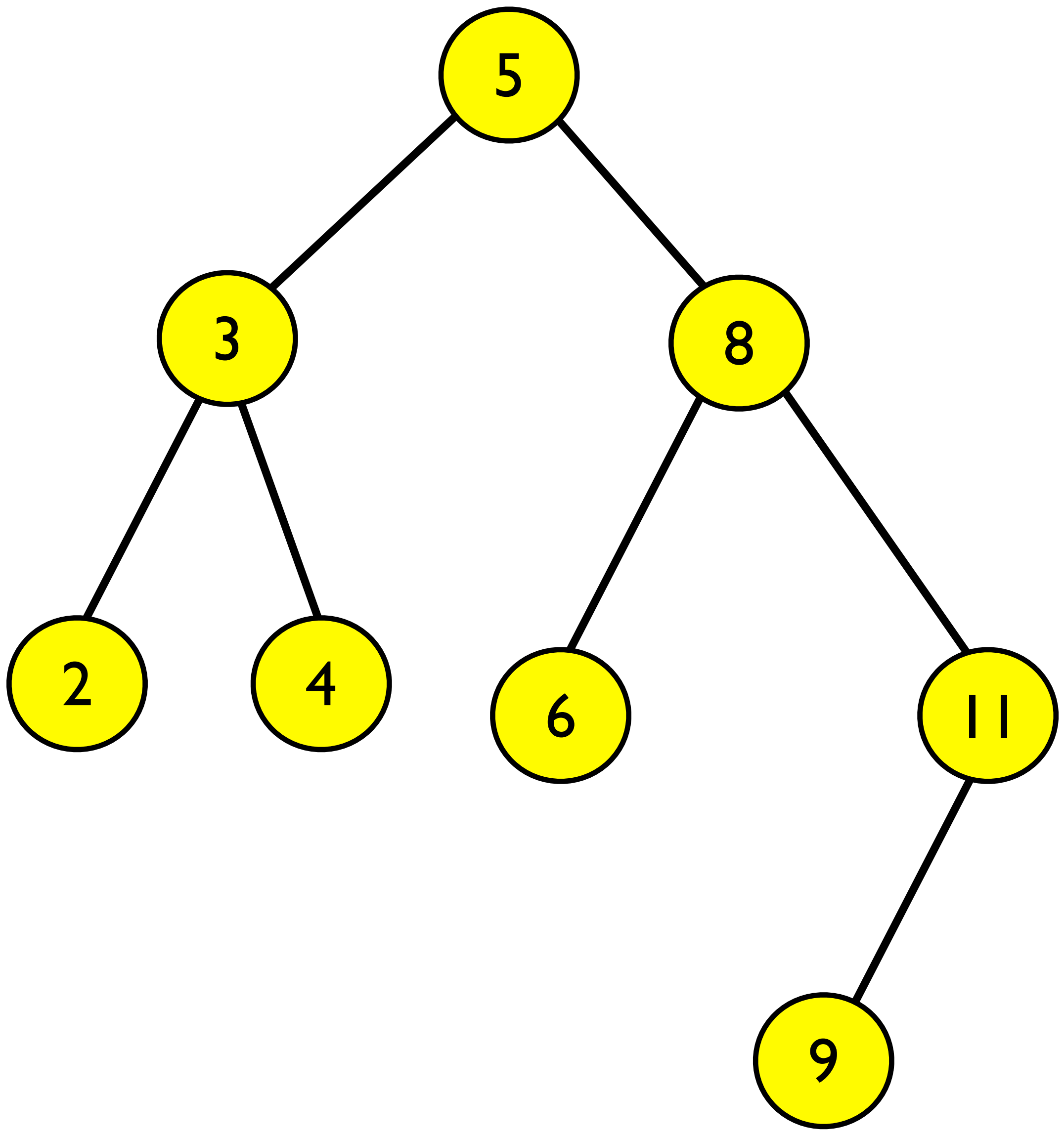
Is $k < 8$? No, go right

Is $k < 11$? Yes, go left



BST Find

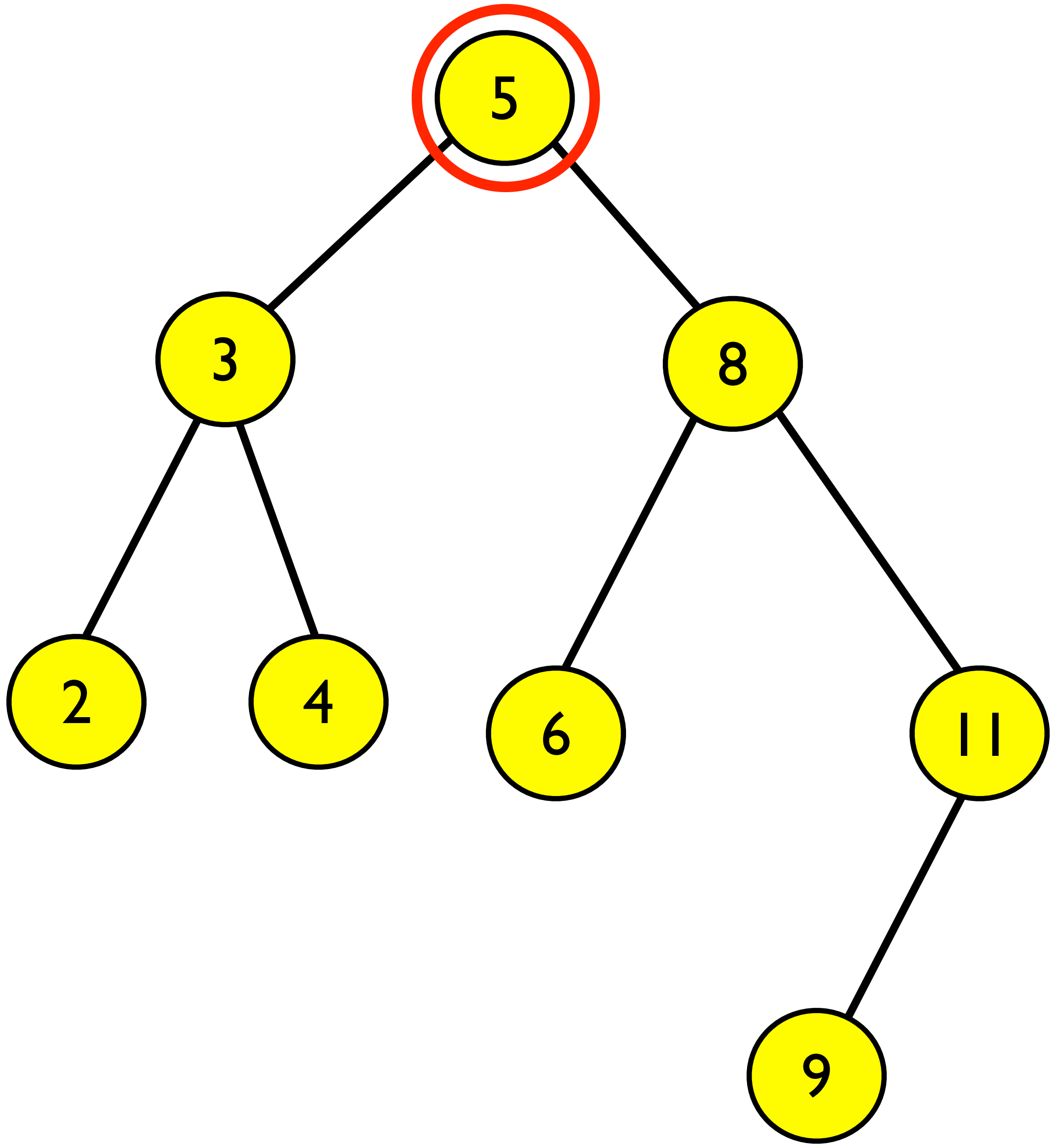
Find $k = 13$:



BST Find

Find $k = 13$:

Is $k < 5$?



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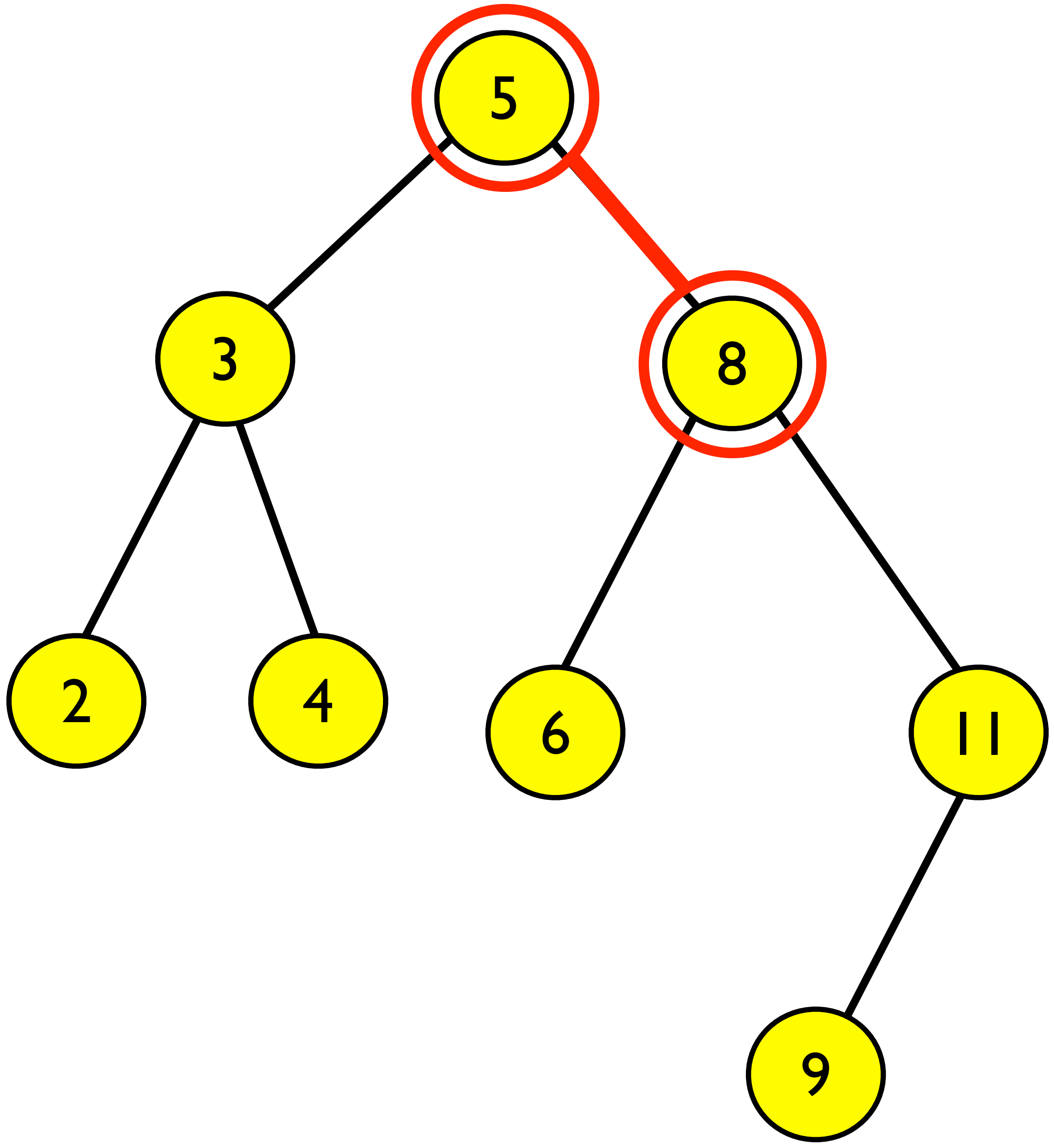

BST Find

Find $k = 13$:

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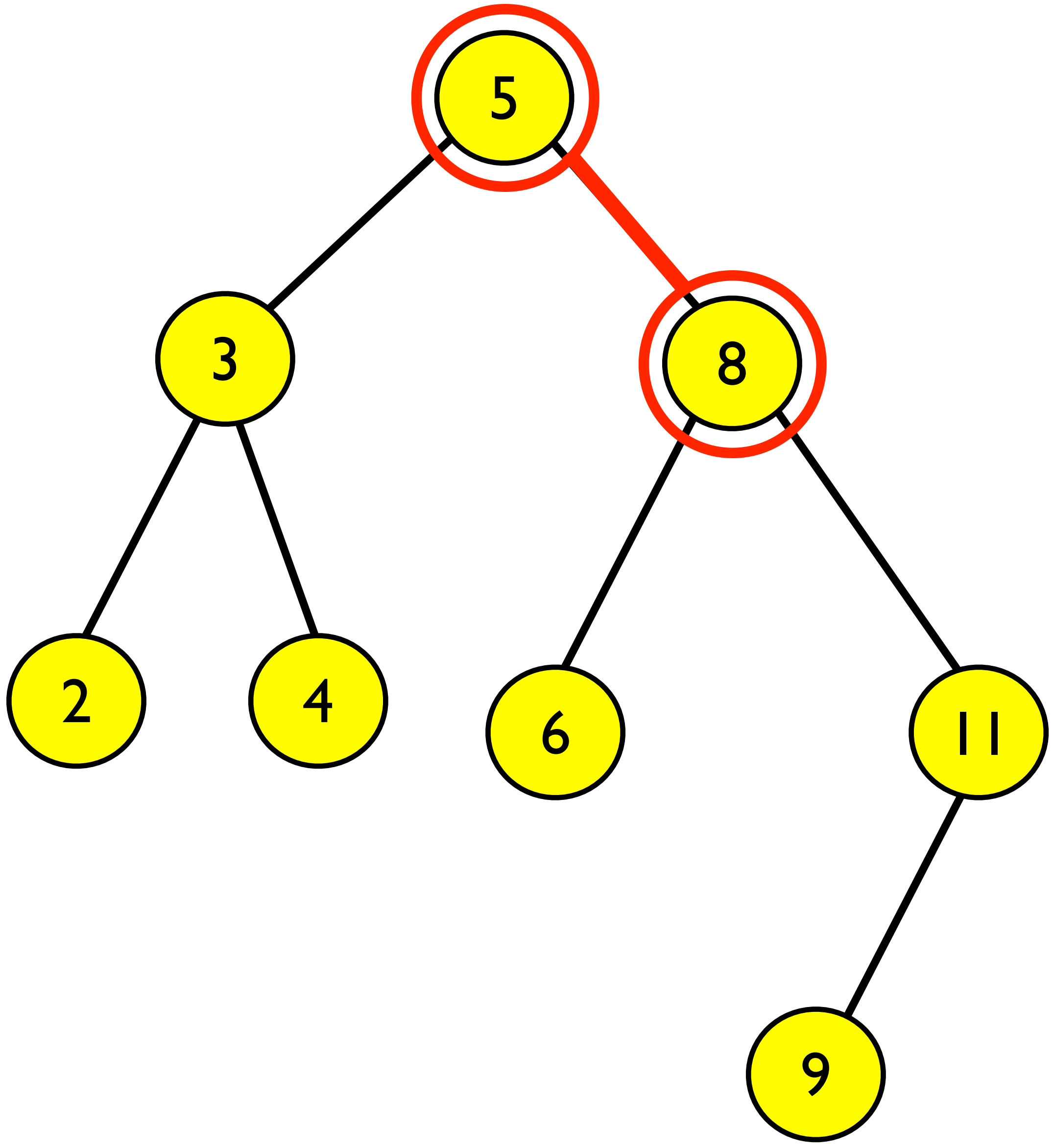
Is $k < 8$?



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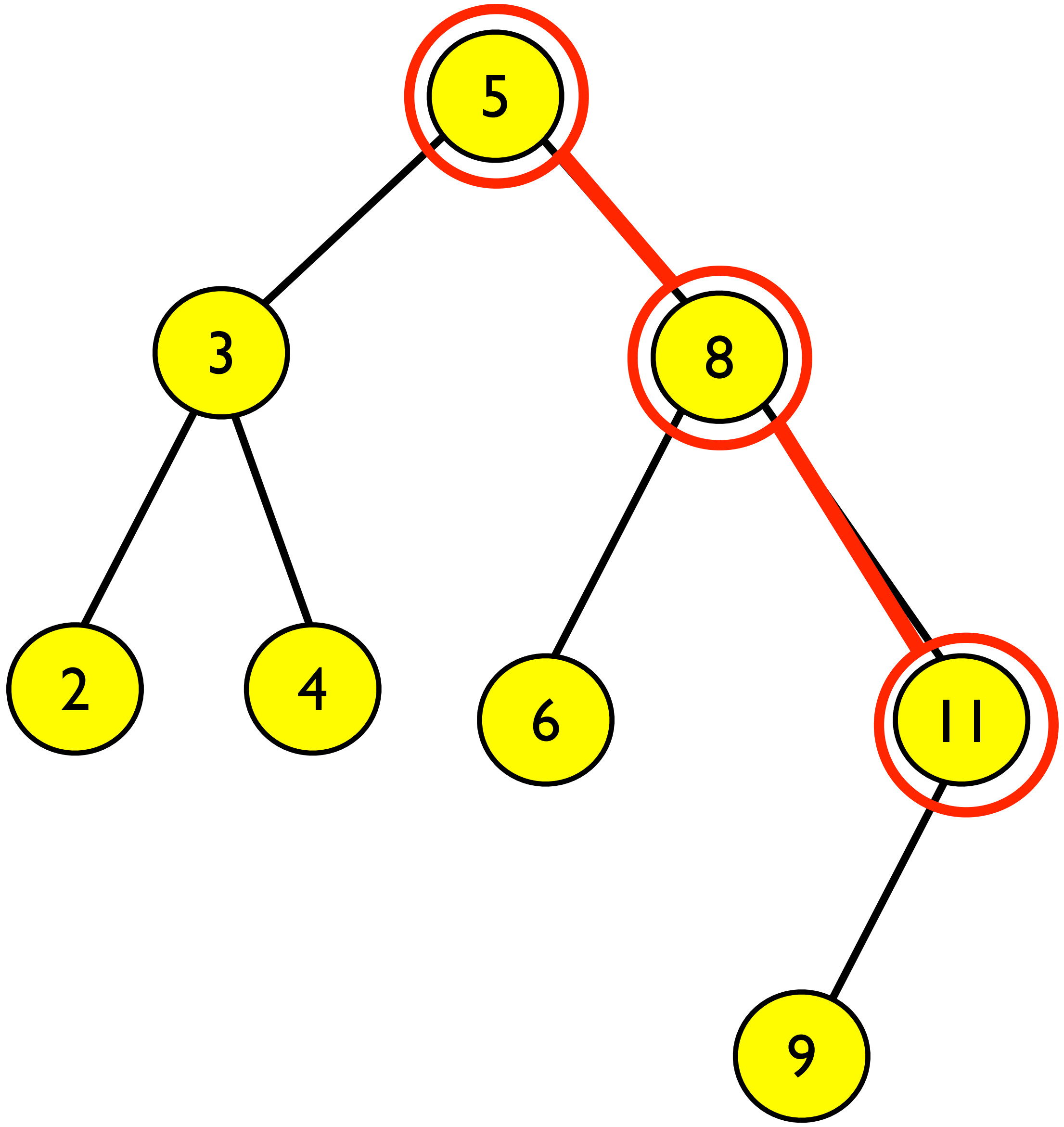

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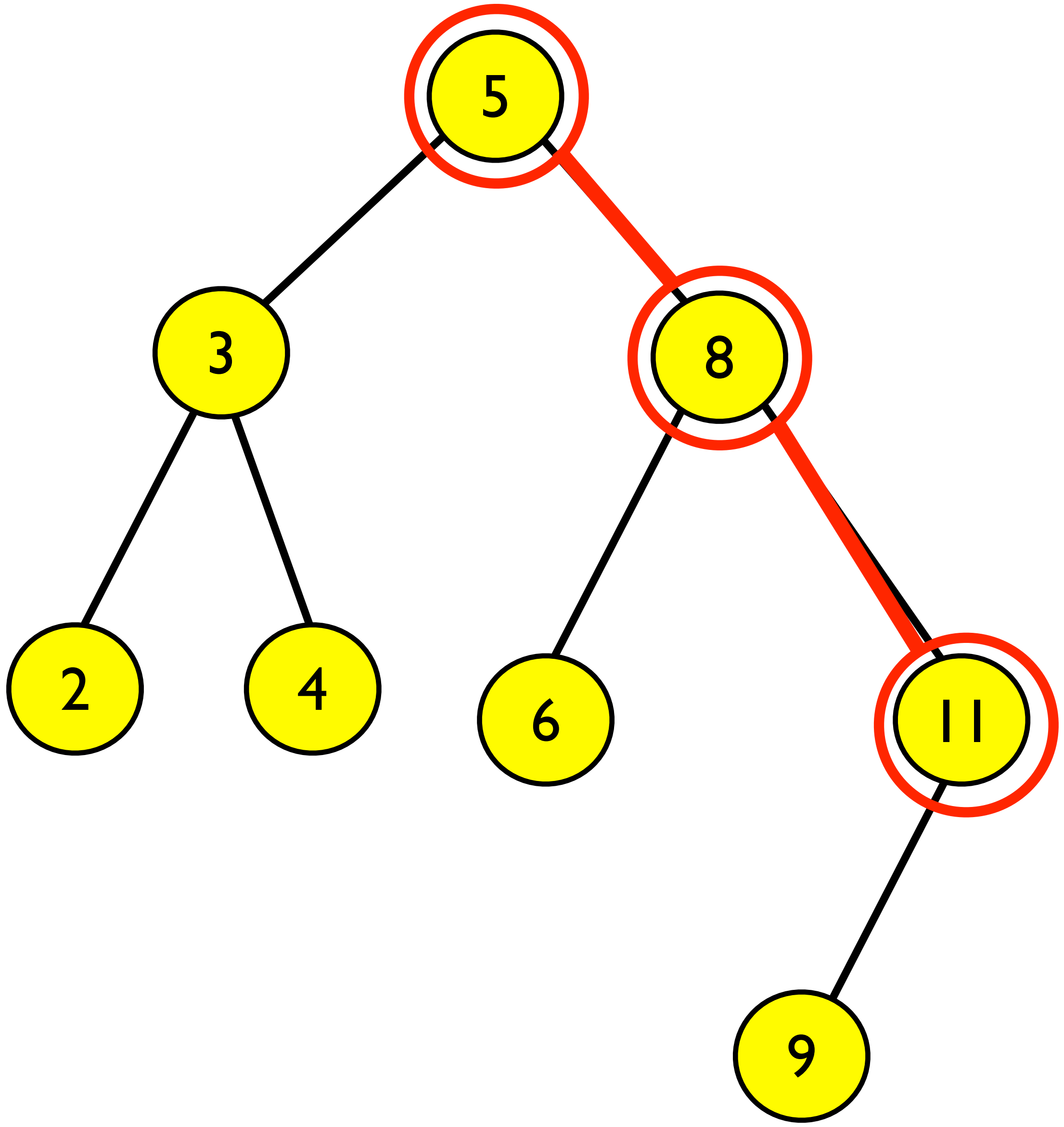

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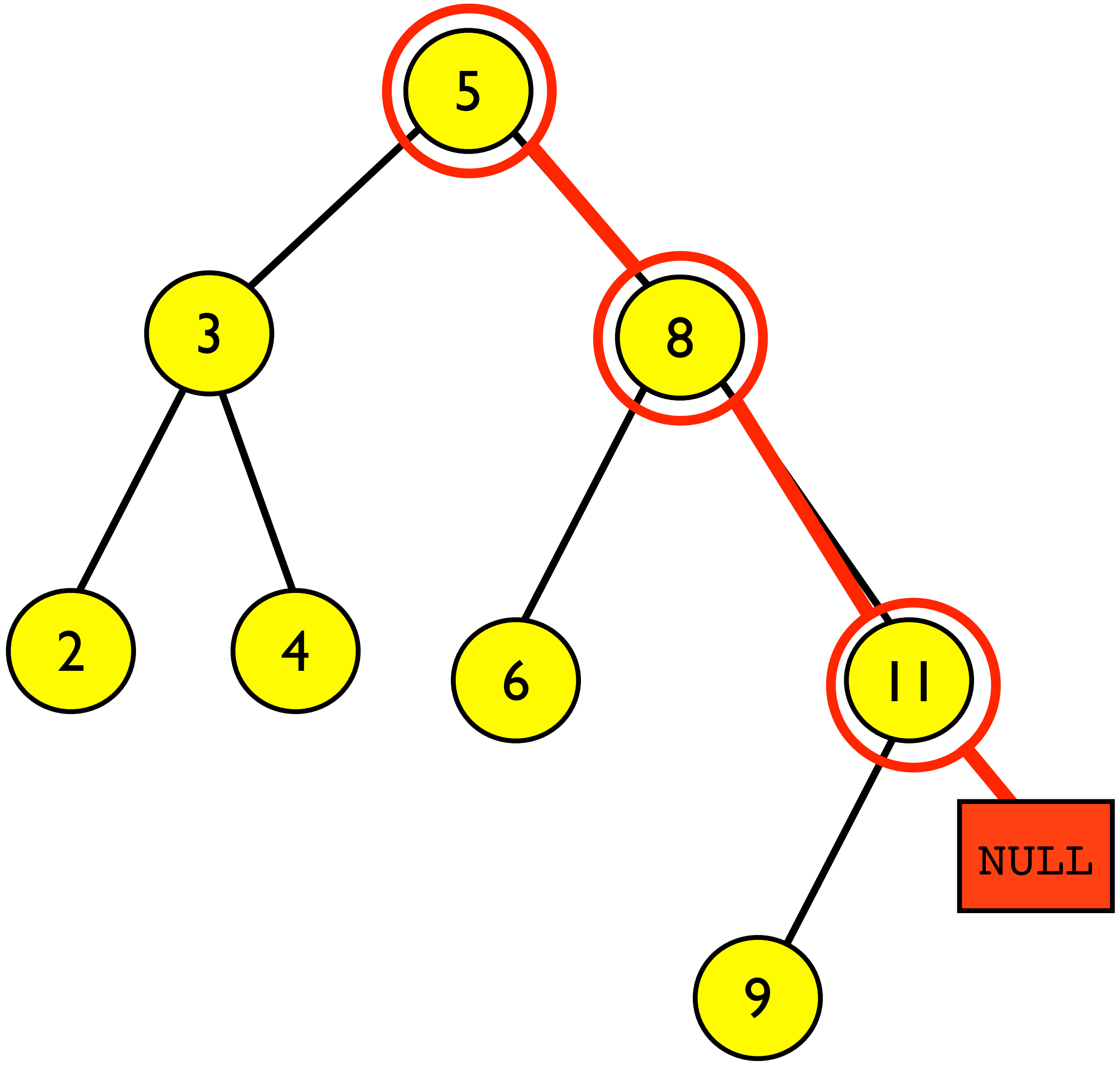


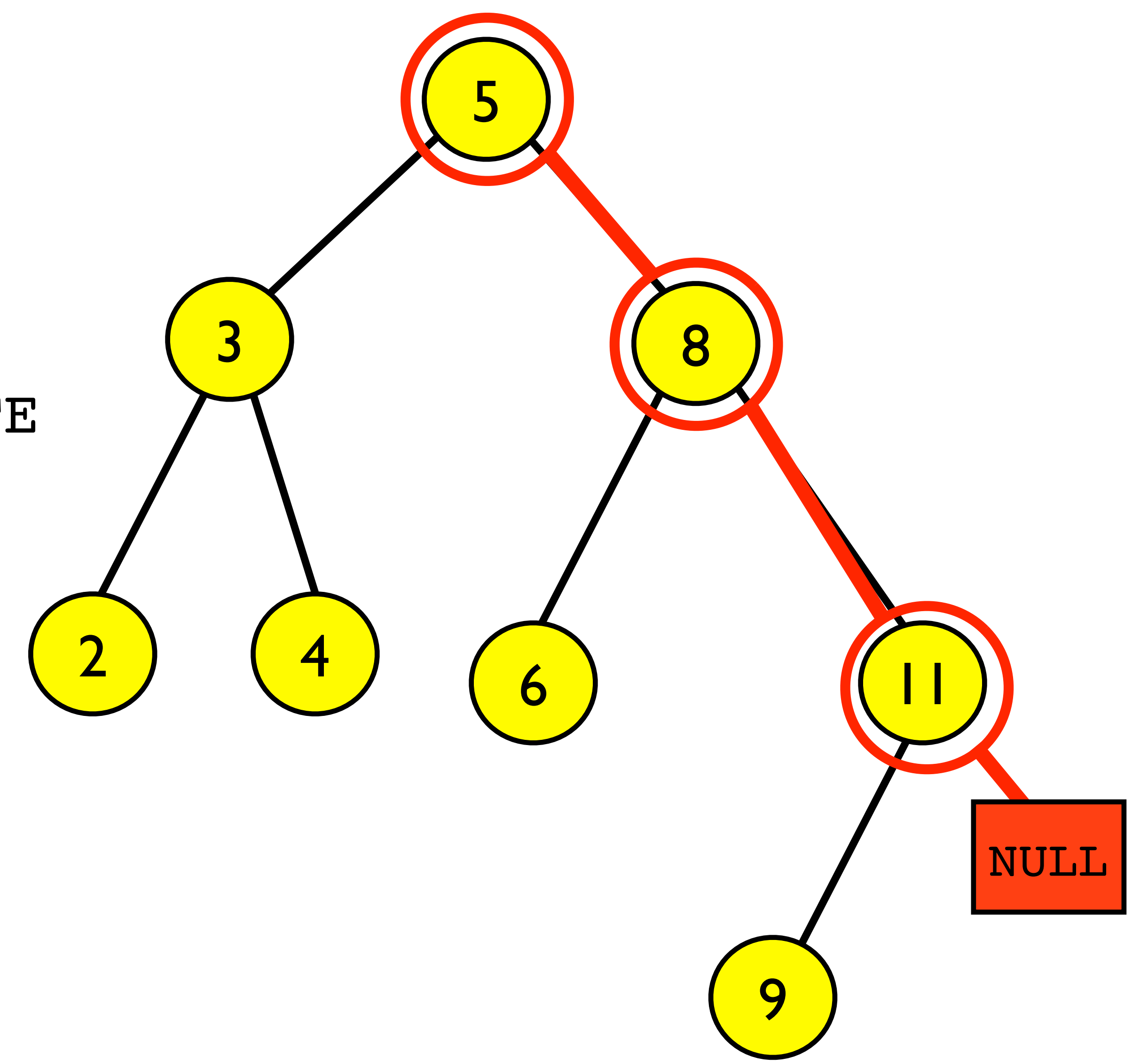
BST Insert

```
insert(T, K):
   q = NULL
   p = T
   while p != NULL and p.key != K:
      q = p
      if p.key < K:
        p = p.right
      else if p.key > K:
        p = p.left

   if p != NULL: error DUPLICATE

   N = new Node(K)
   if q.key > K:
     q.left = N
   else:
     q.right = N
```

Same idea as BST Find



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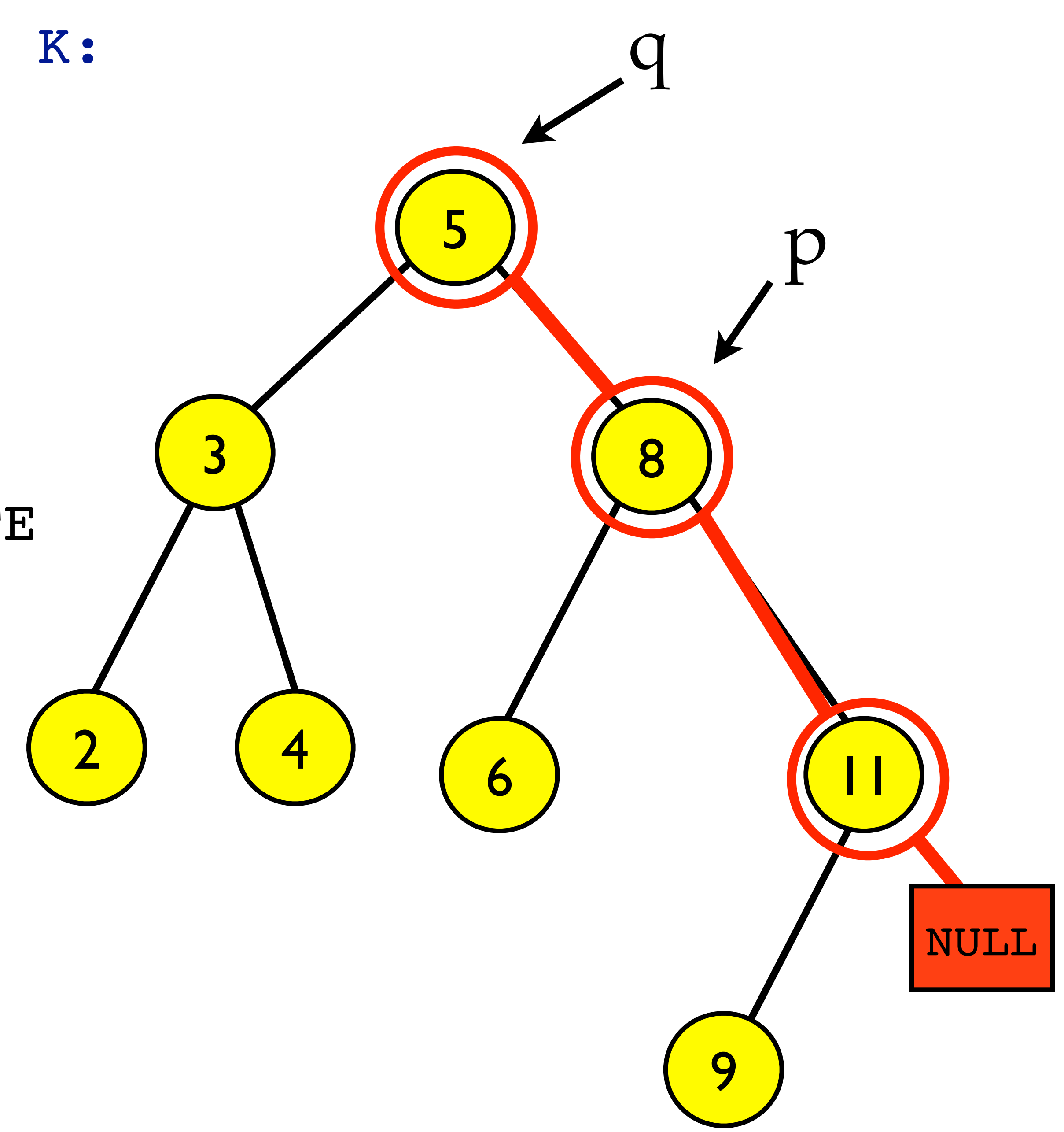

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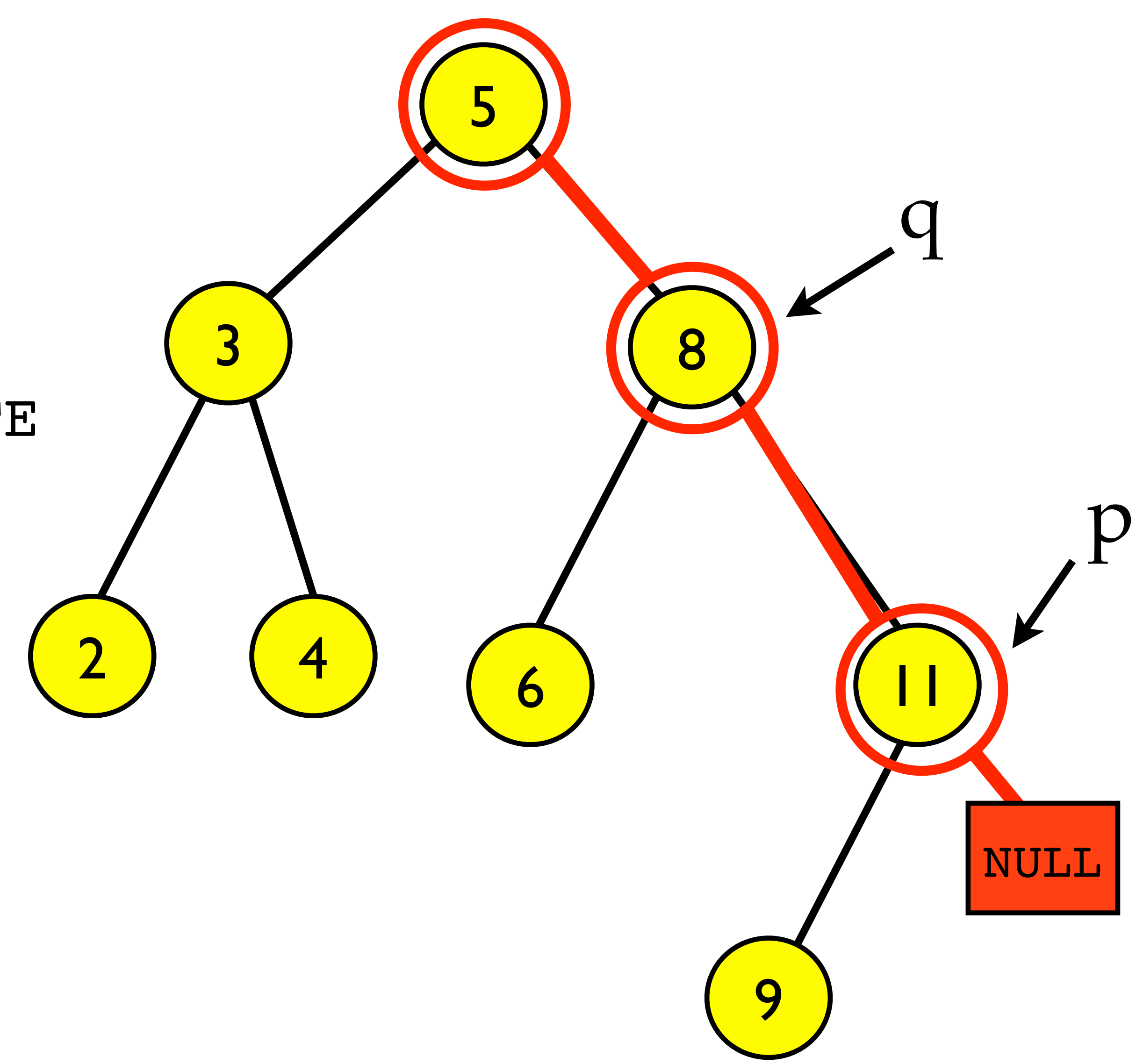

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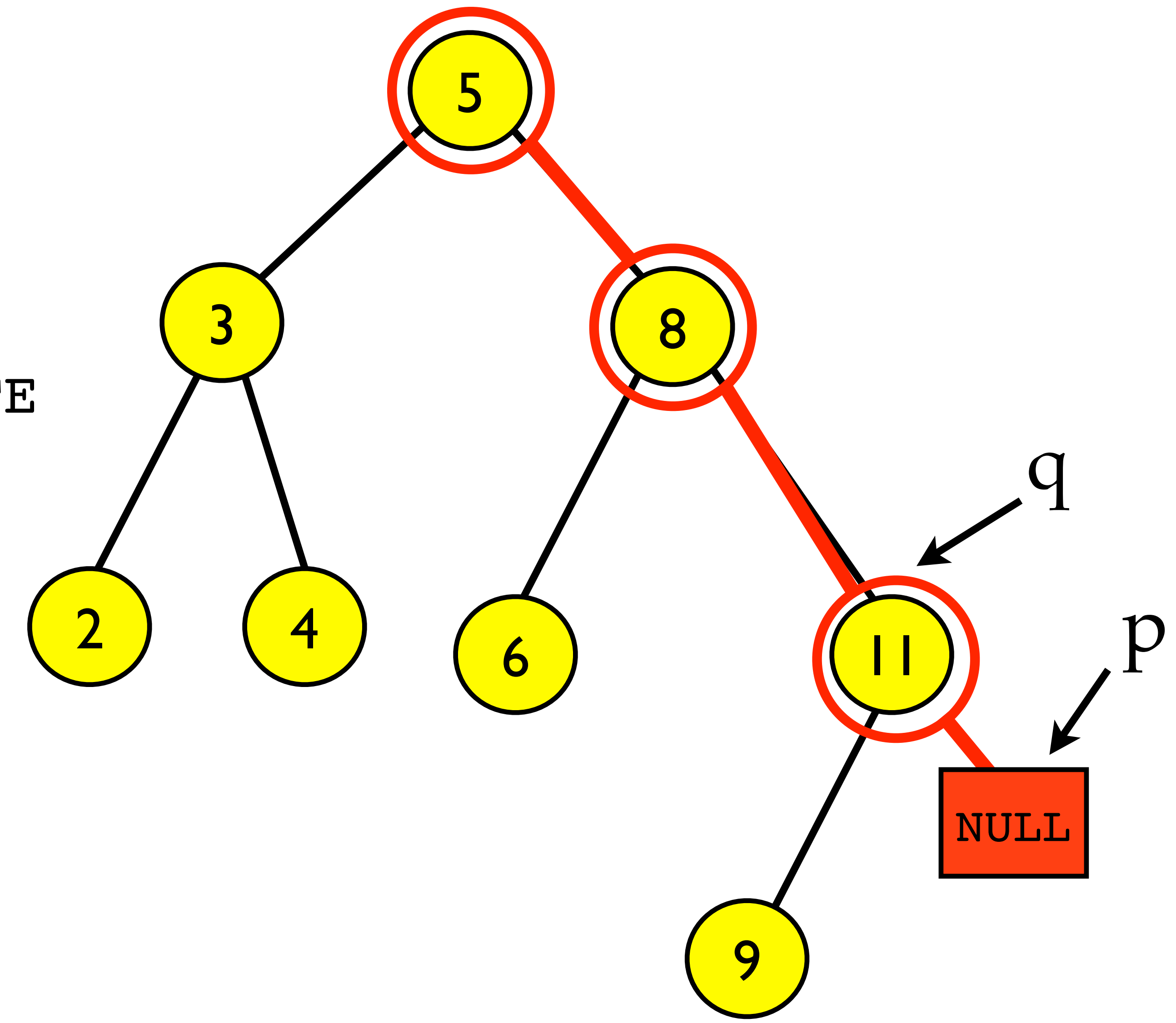
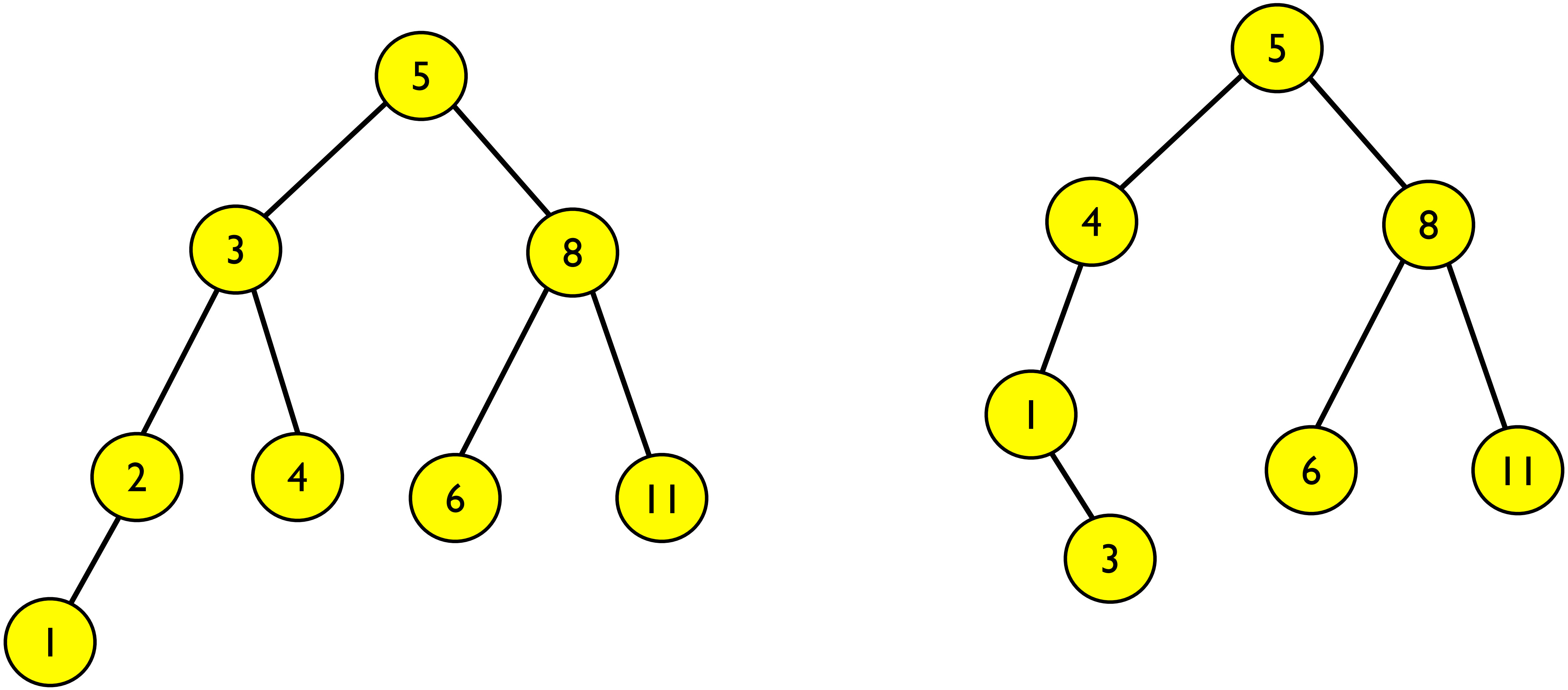
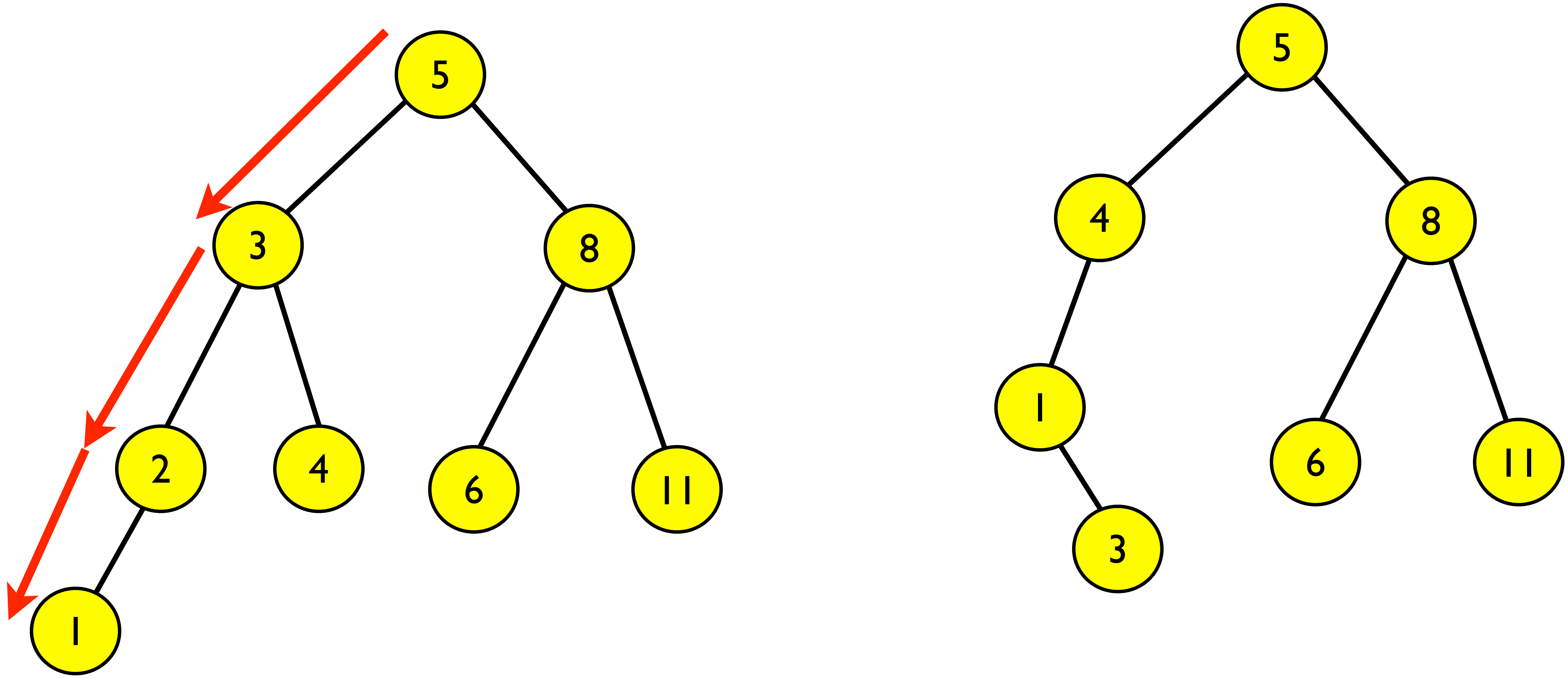


BST FindMin

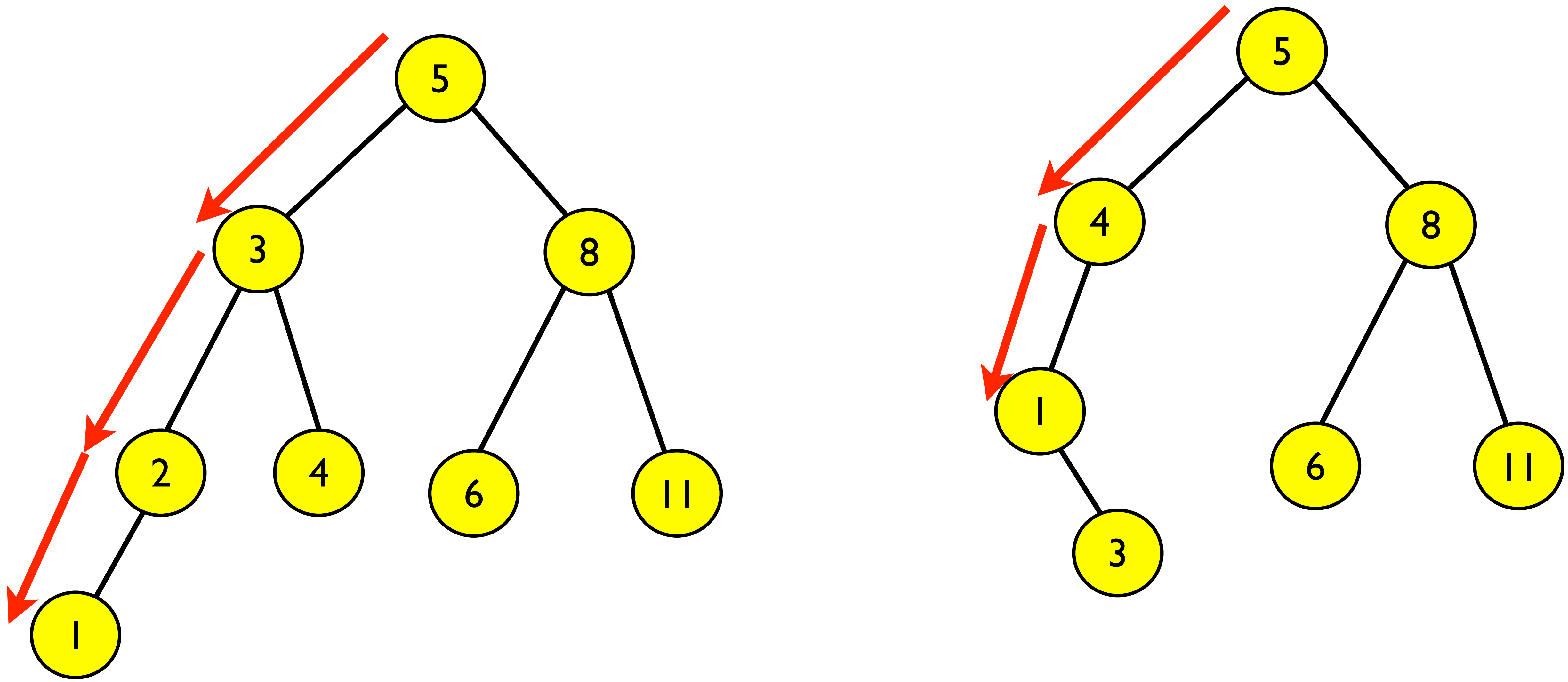


BST FindMin



Walk left until you can't go left any more

BST FindMin



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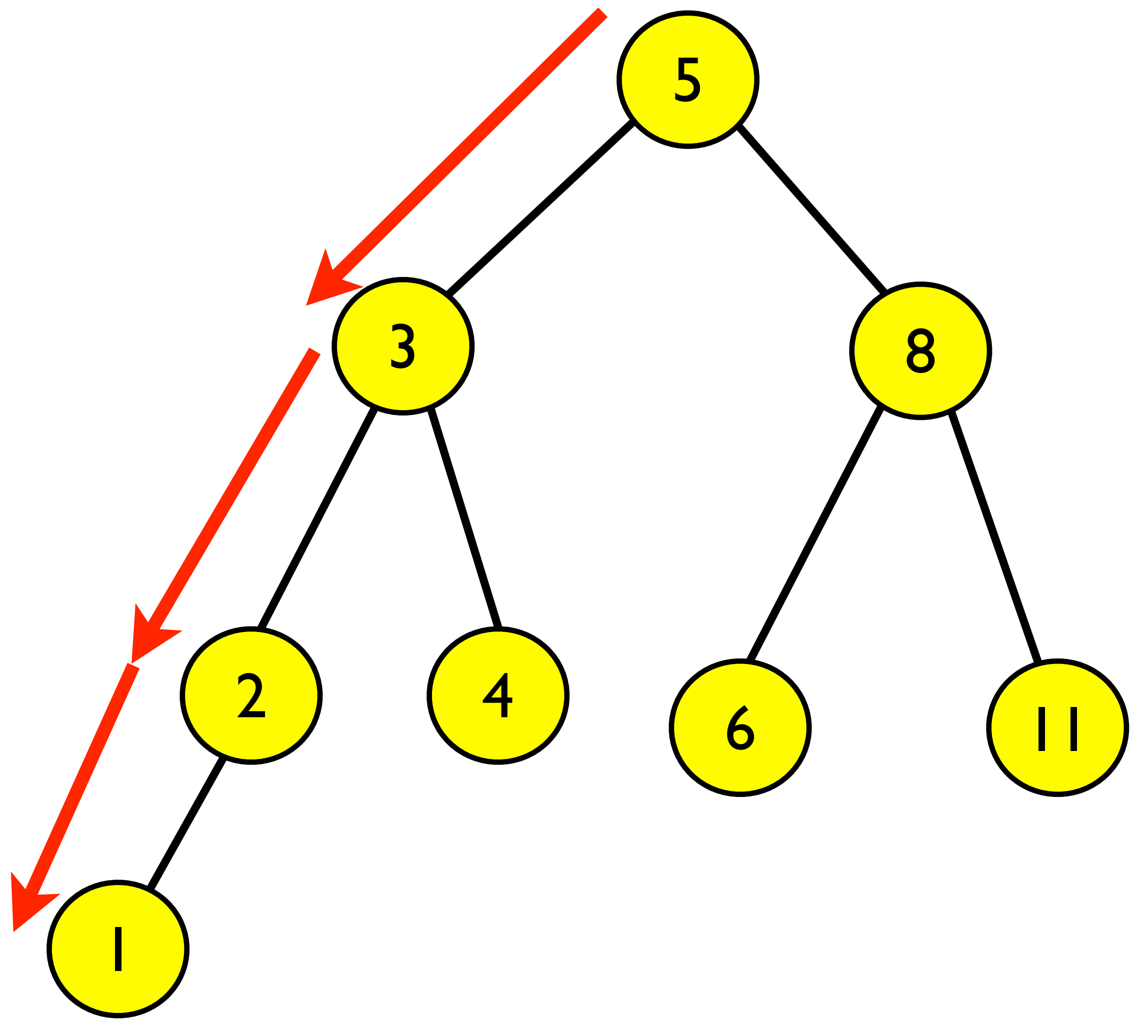
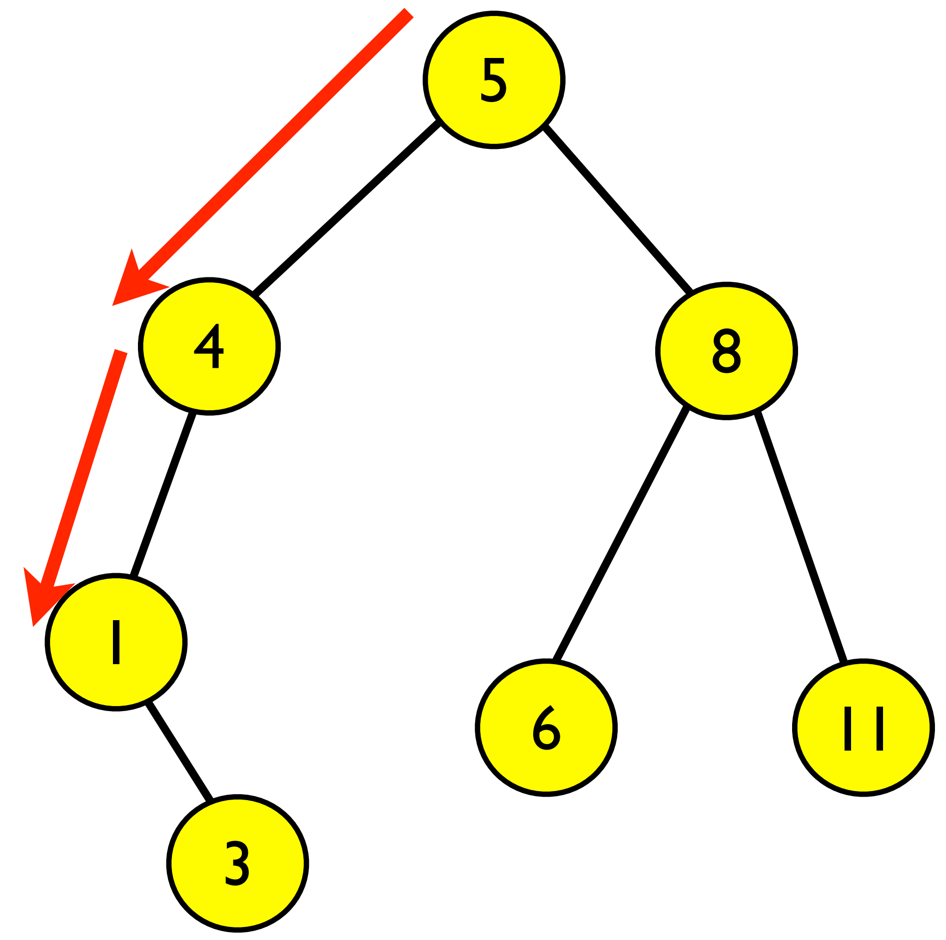
BST FindMin



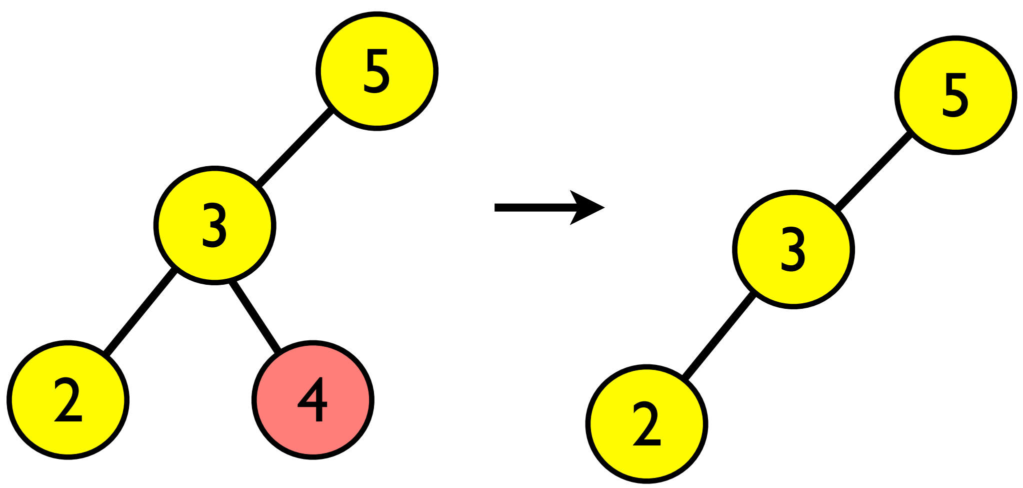


Walk left until you can't go left any more

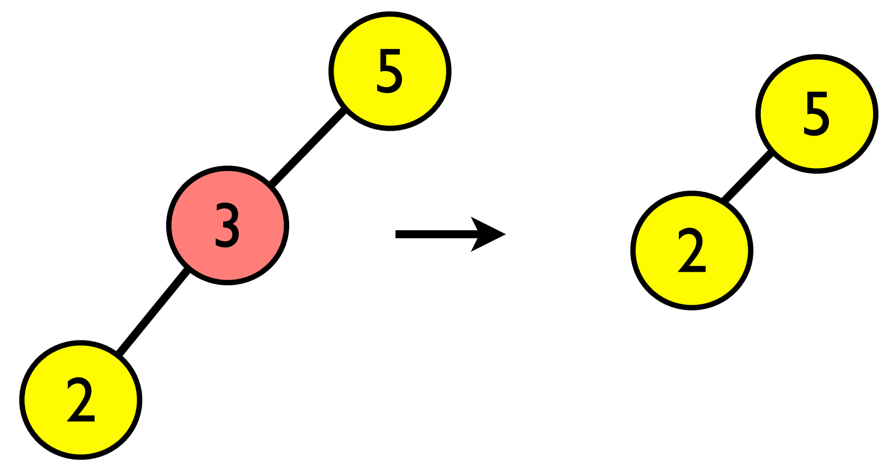
Can you express inorder_successor using find_min?

BST Delete

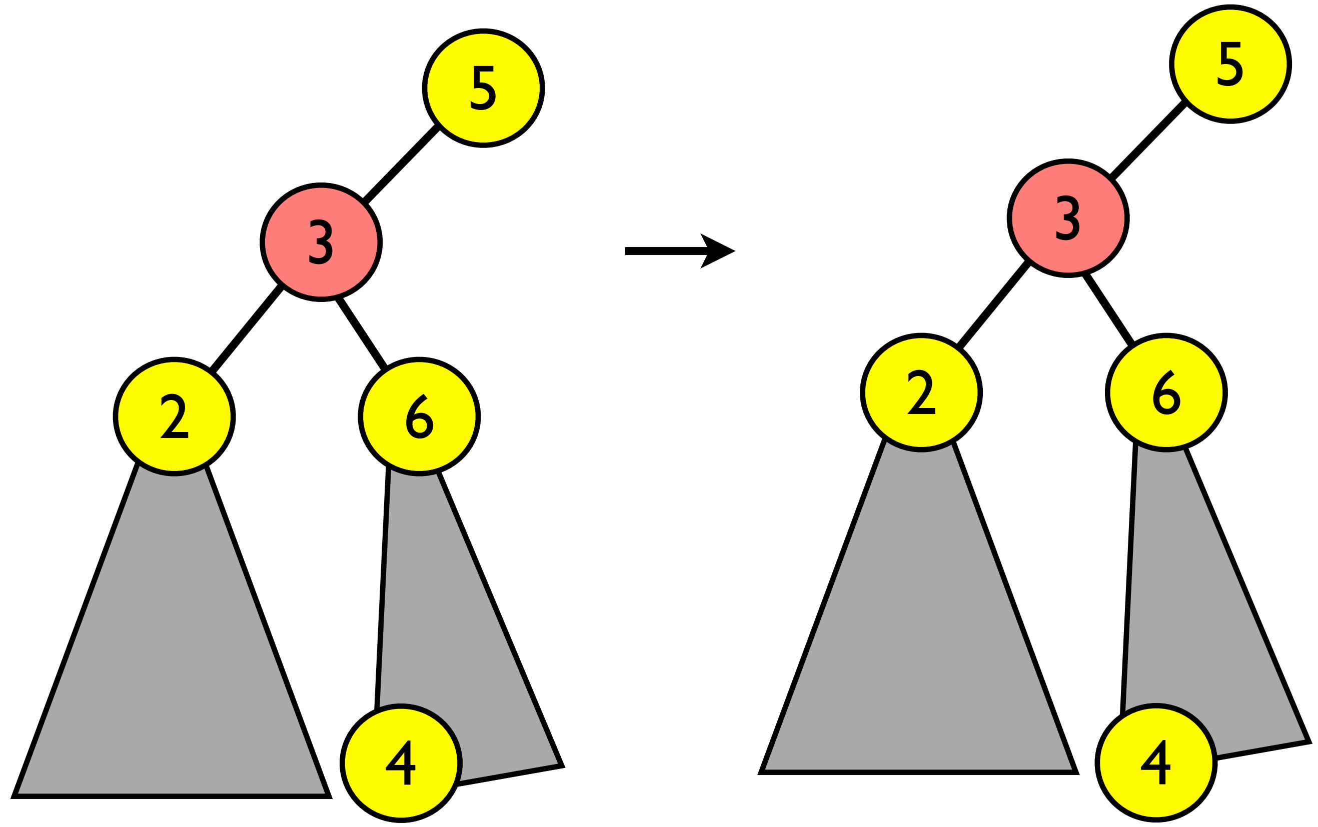
Node is leaf:



Node has 1 child:

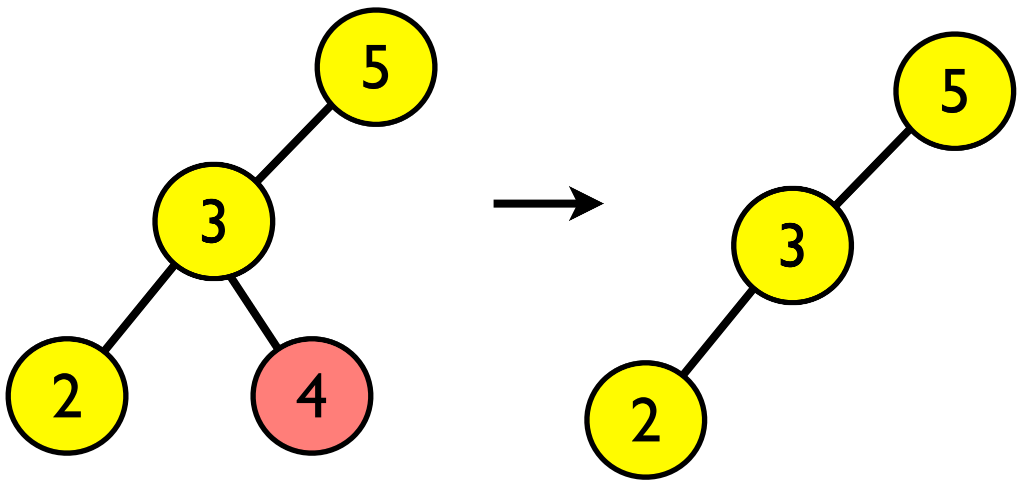


Node has 2 children:

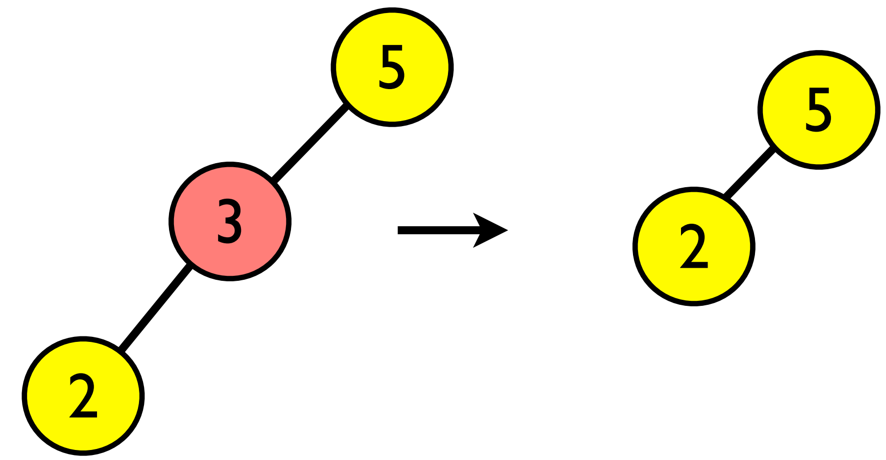


BST Delete

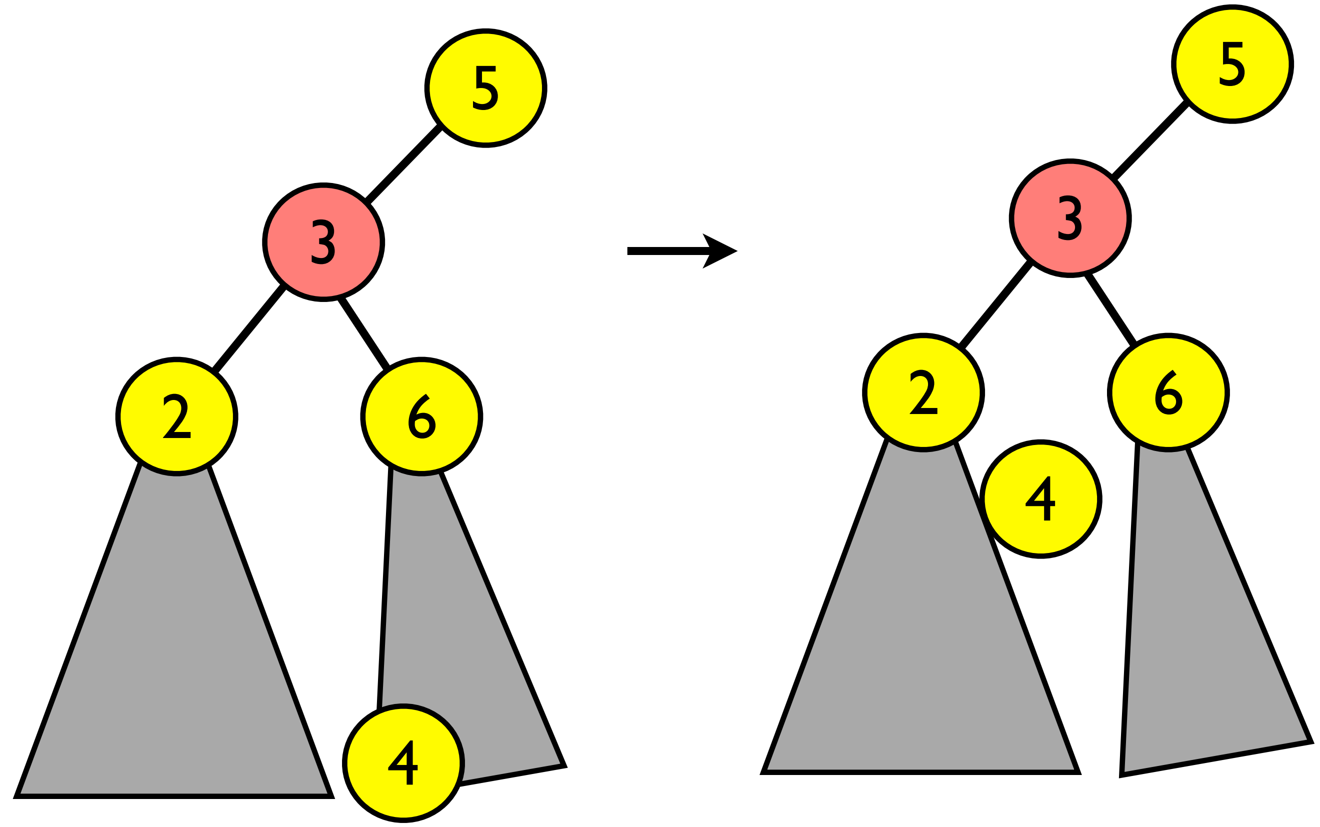
Node is leaf:



Node has 1 child:

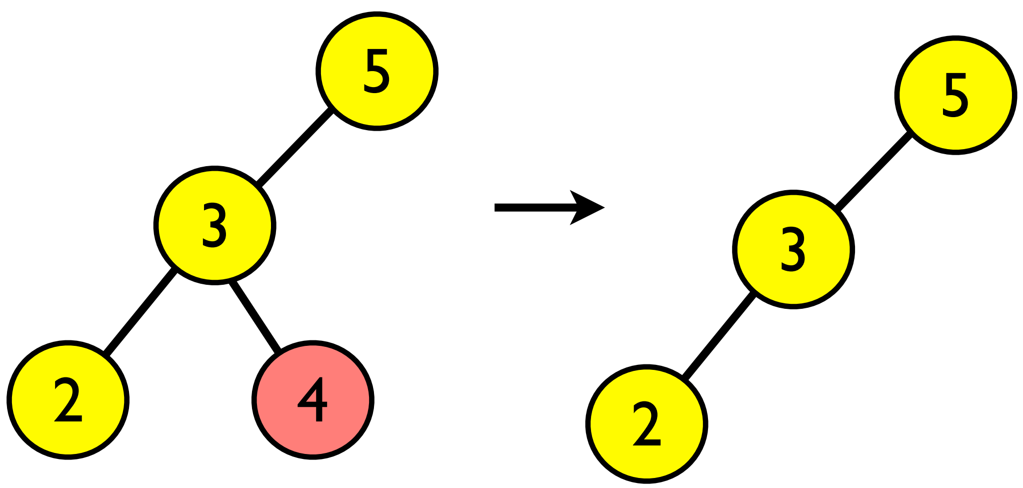


Node has 2 children:

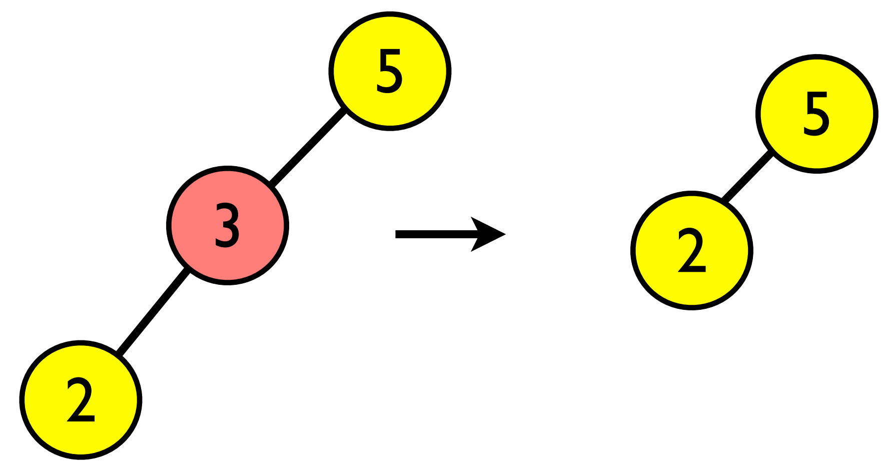


BST Delete

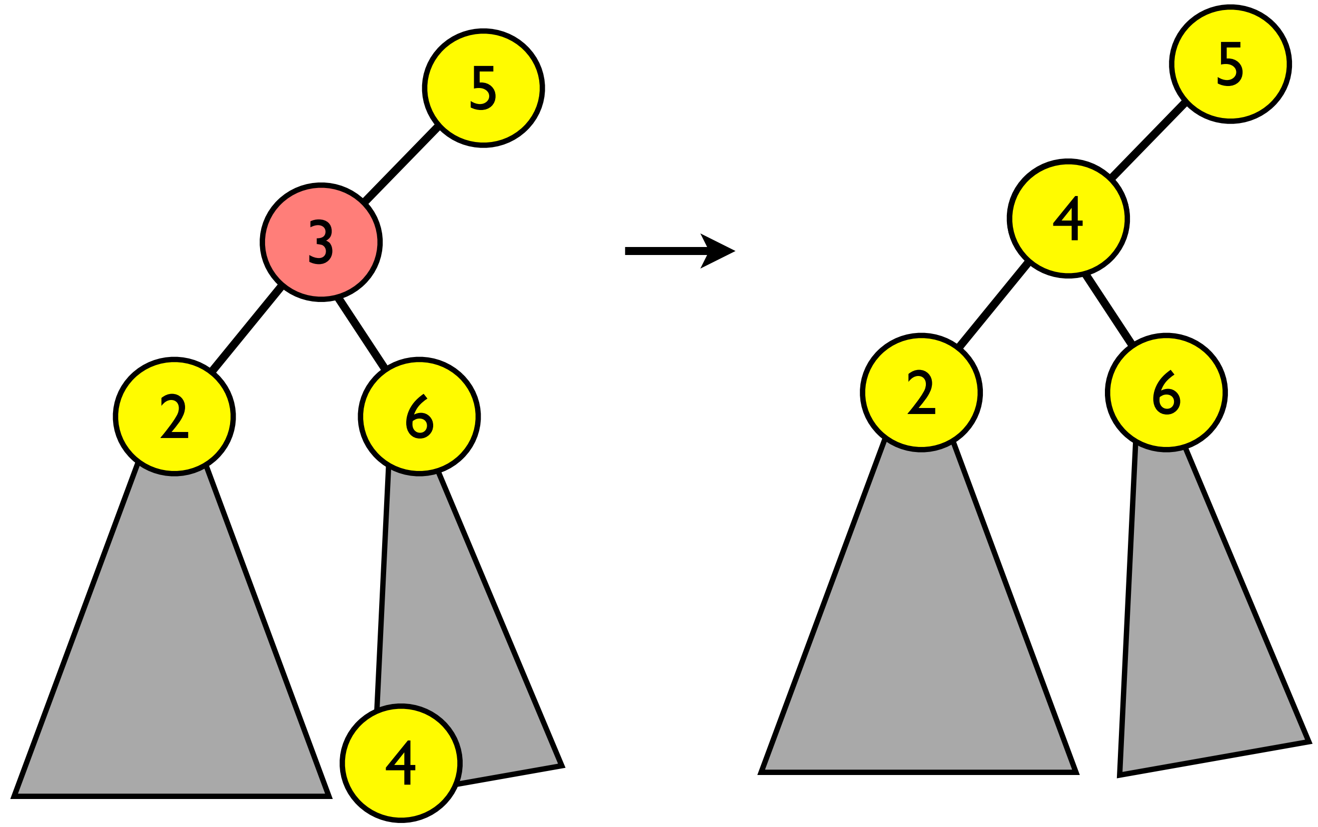
Node is leaf:



Node has 1 child:



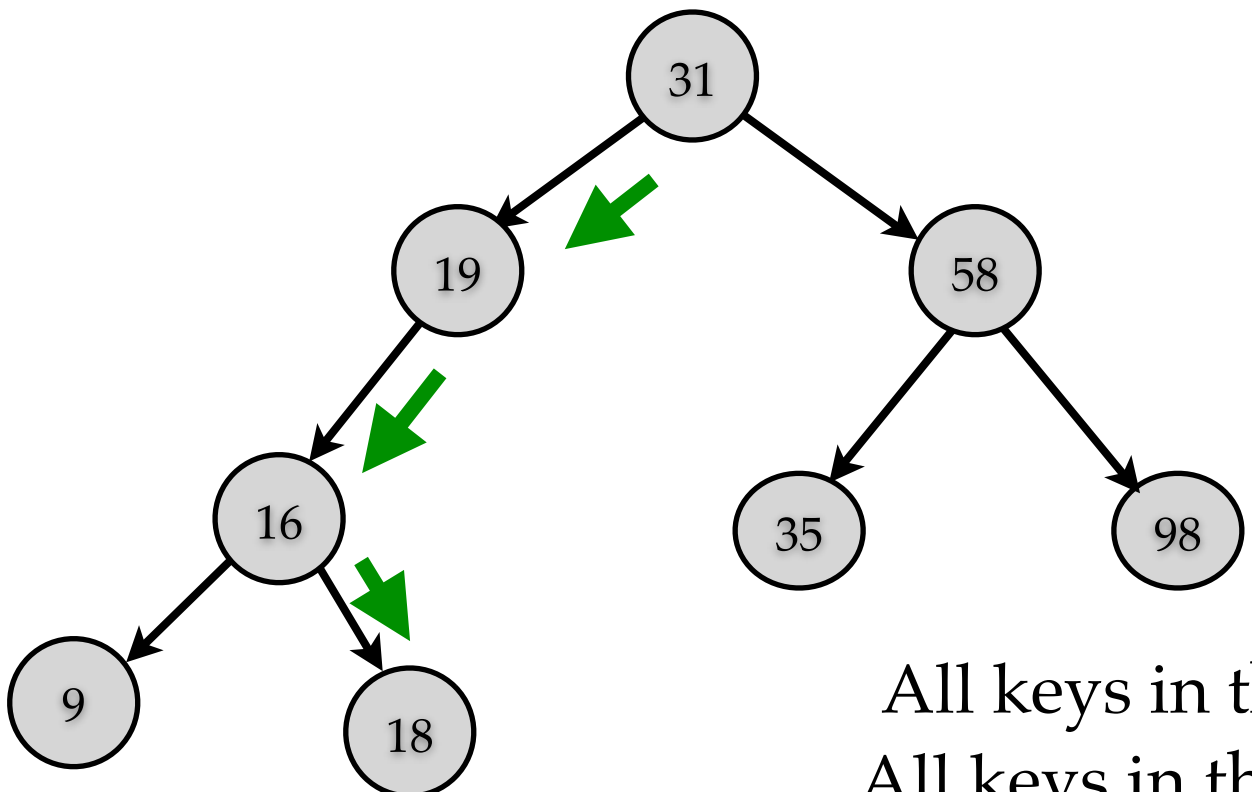
Node has 2 children:



Partitioning

- **Ordering implicitly gives a partitioning based on the "<" relation.**
- **Partitioning usually combined with linking to point to the two halves.**

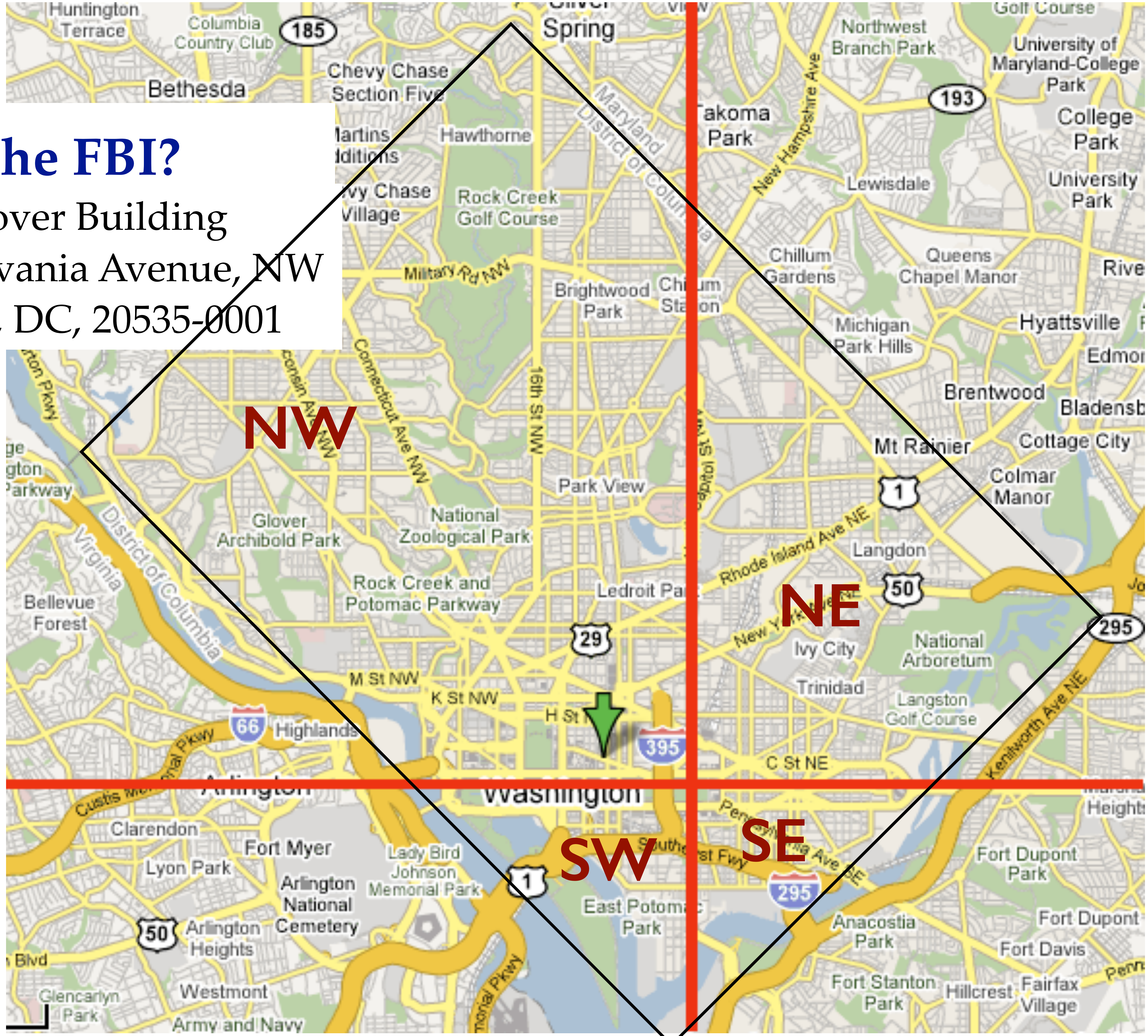
Find 18



All keys in the left subtree are < the root
All keys in the right subtree are ≥ the root

Where’s the FBI?

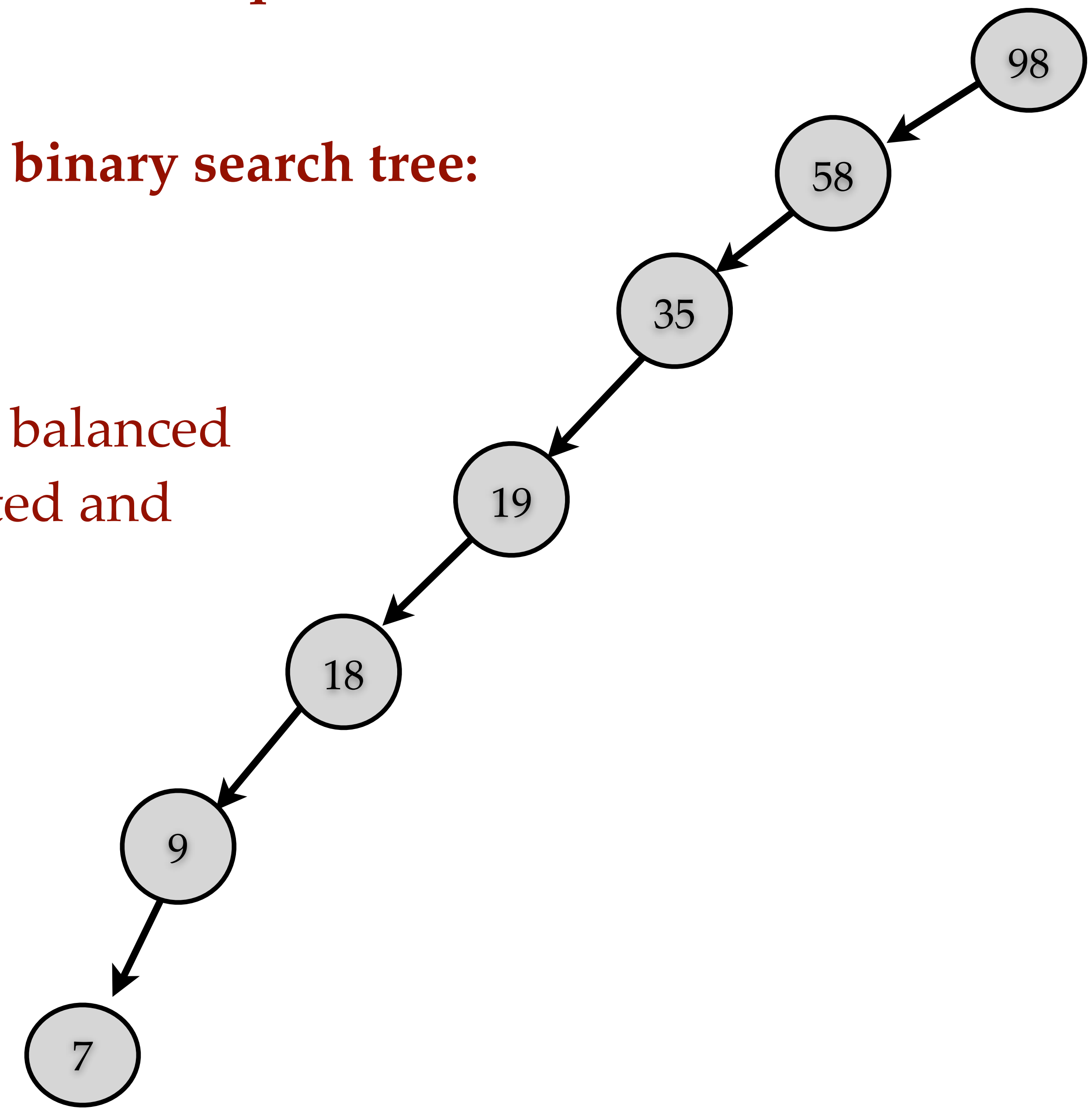
J. Edgar Hoover Building
935 Pennsylvania Avenue, NW
Washington, DC, 20535-0001



Why is the DC partitioning bad?

- **Everything interesting is in the northwest quadrant.**
- **Want a <u>balanced</u> partition!**
- **Another example: an unbalanced binary search tree: (becomes sequential search)**

- How can we force a BST tree to be balanced if items are constantly being inserted and deleted?



Splay Trees (Sleator & Tarjan, 1985)

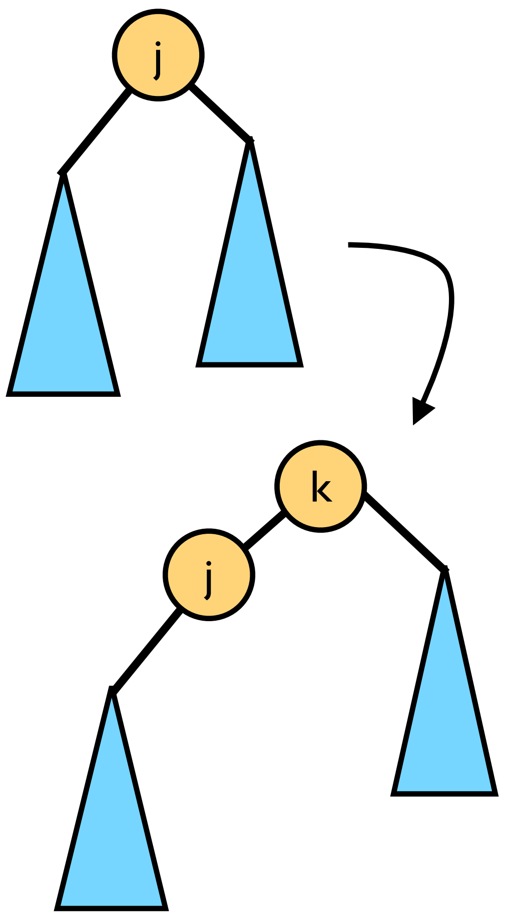
- no extra storage requirement
- simple to implement
- Main idea: move frequently accessed items up in tree
- amortized $O(\log n)$ performance
- worst case single operation is $\Omega(n)$

Splay Trees

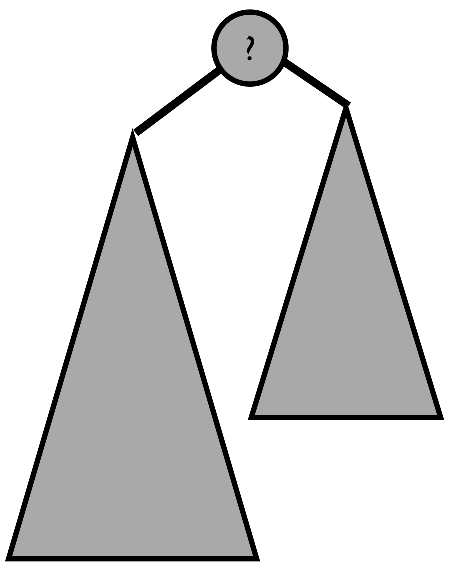
splay(T, *k*): if $k \in T$, then move *k* to the root using a particular set of transformations of the tree. Otherwise, move either the inorder successor or predecessor of *k* to the root.

Without knowing how *splay* is implemented, we can implement the dictionary ADT as follows:

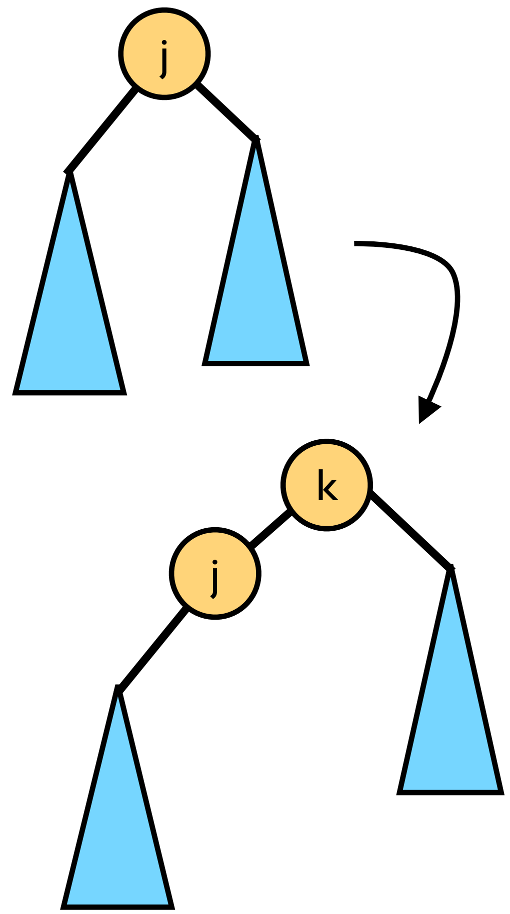
- *find*(T, k): *splay*(T, k). If *root*(T) = *k*, return *k*, otherwise return **not found**.
- *insert*(T, k): *splay*(T, k). If *root*(T) = *k*, return **duplicate!**; otherwise, make *k* the root and add children as in figure:
- *concat*(T_1, T_2): Assumes all keys in T_1 are < all keys in T_2. *Splay*(T_1, ∞). Now root T_1 contains the largest item, and has no right child. Make T_2 right child of T_1.
- *delete*(T, *k*): *splay*(T, *k*). If root *r* contains *k*, *concat*(LEFT(*r*), RIGHT(*r*)).



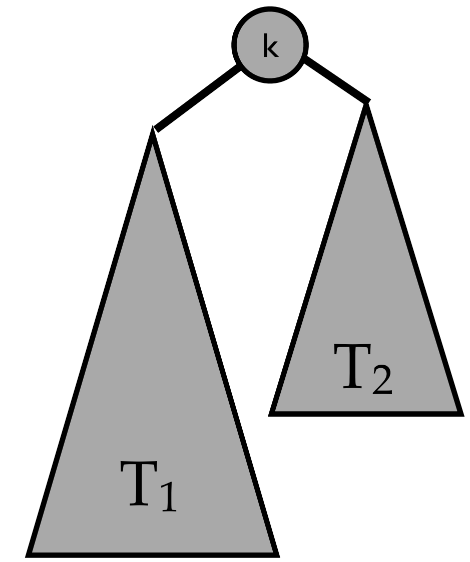
Dictionary Operations, in pictures



find(T, *k*): splay & check root



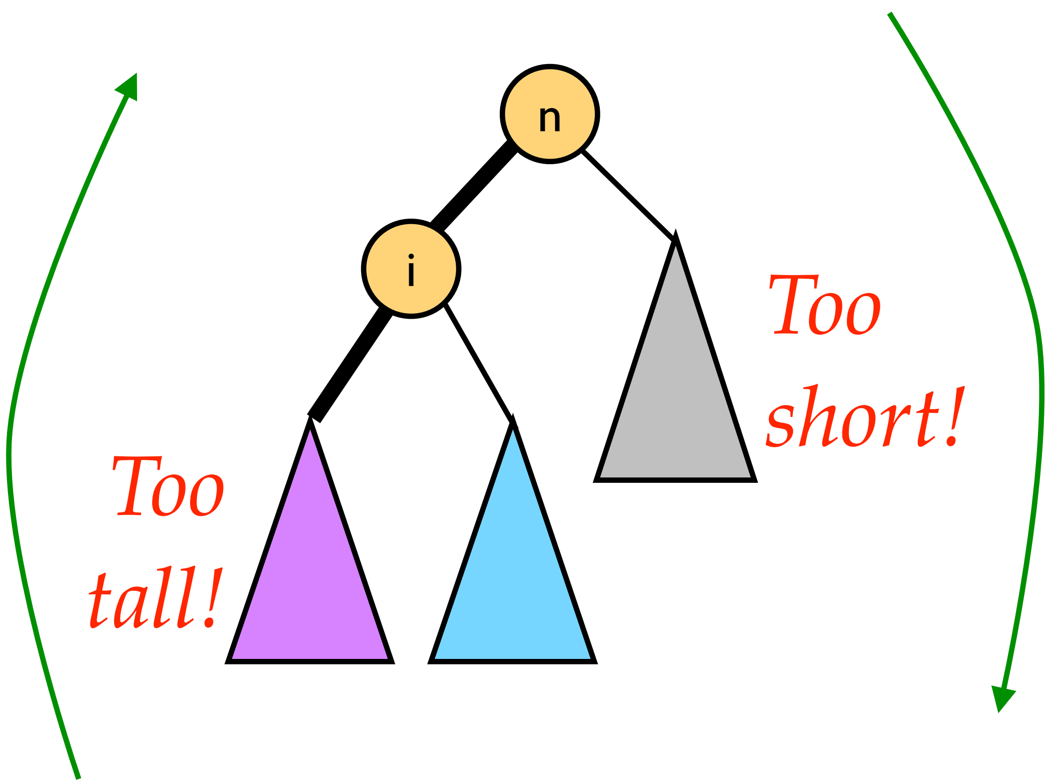
insert(T, k): splay and insert just below root



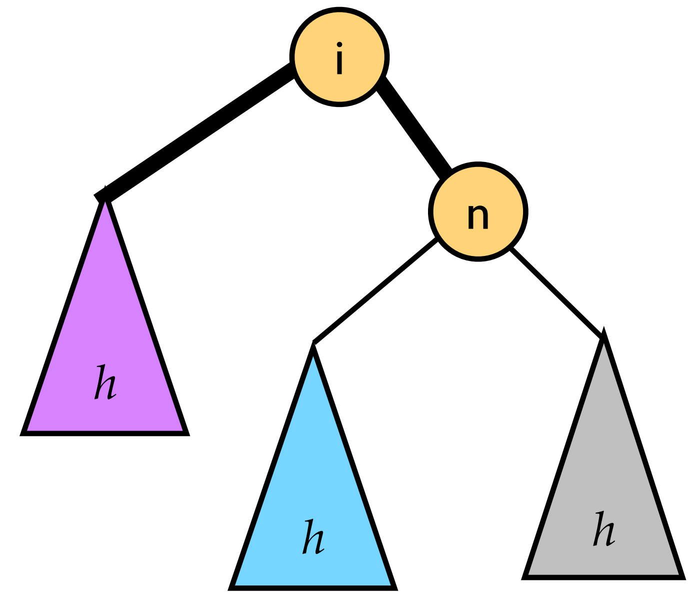
delete(T, *k*): splay & concat left & right subtrees

Implementing the Splay operation

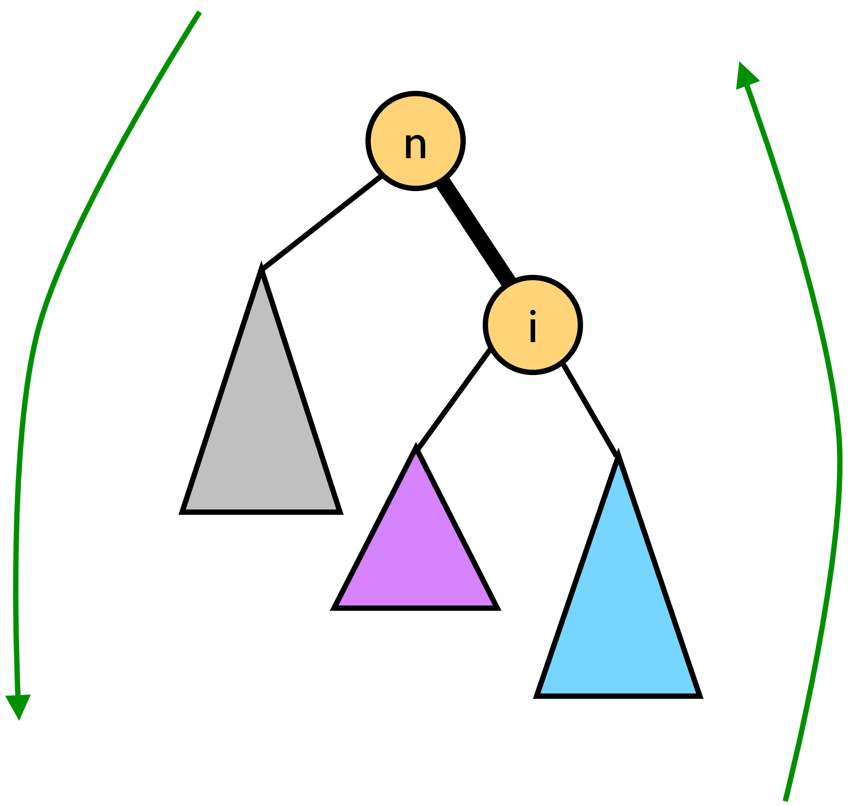
Right rotation (at n)

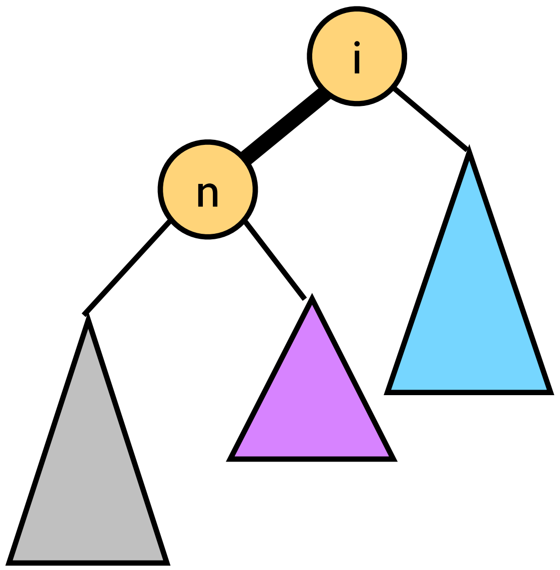


Right rotation
(aka clockwise rotation)



Left Rotation (at n)

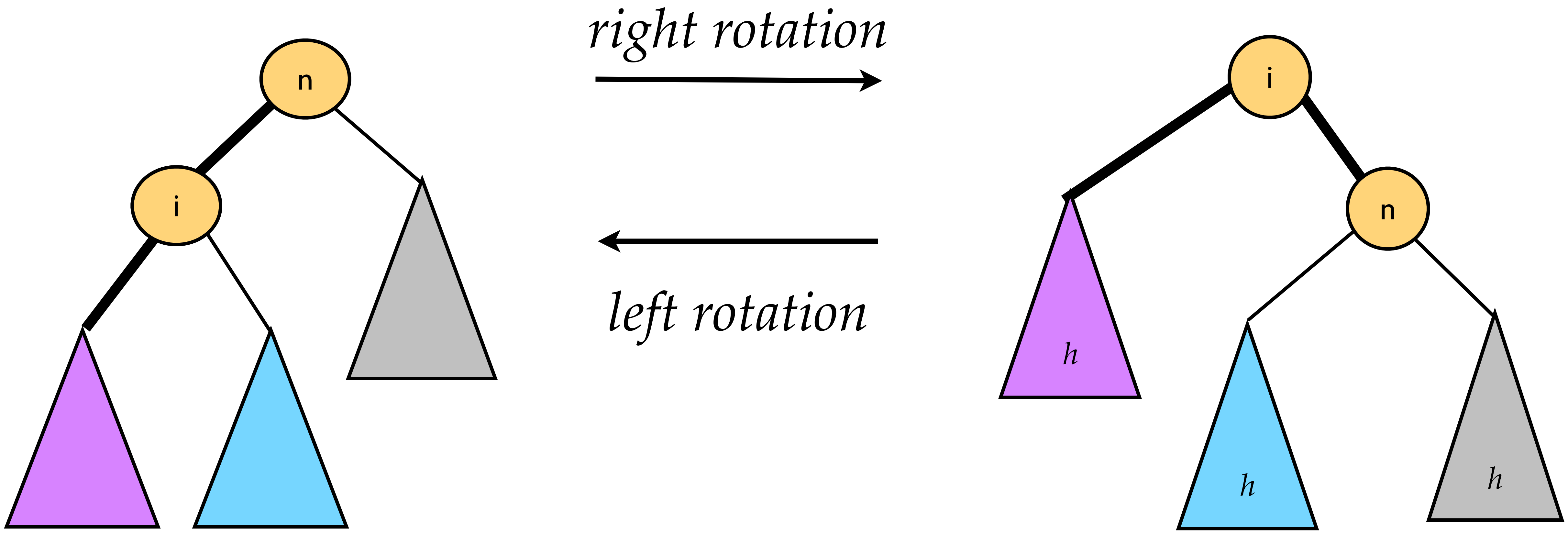




Left rotation
(aka counterclockwise rotation)

Only a constant # of pointers need to be updated for a rotation: O(1) time

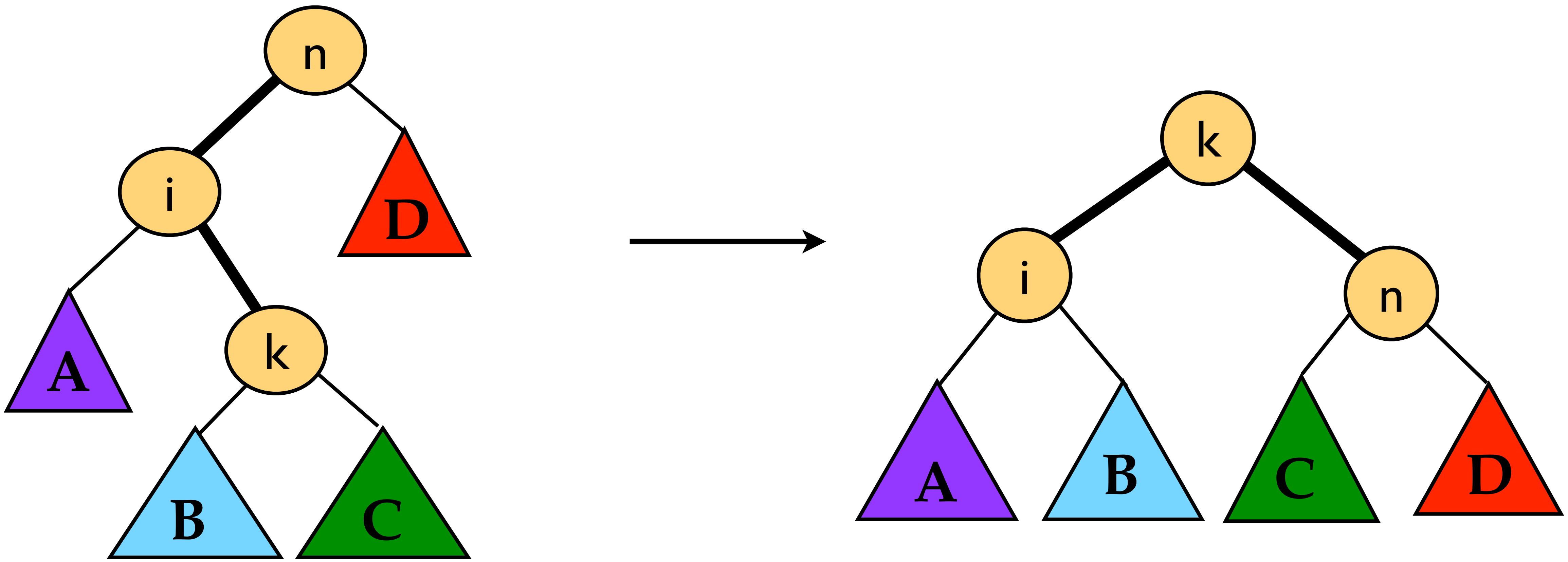
Right & Left Rotations are Inverses



i moves toward the root

n moves toward the root

Double Rotation



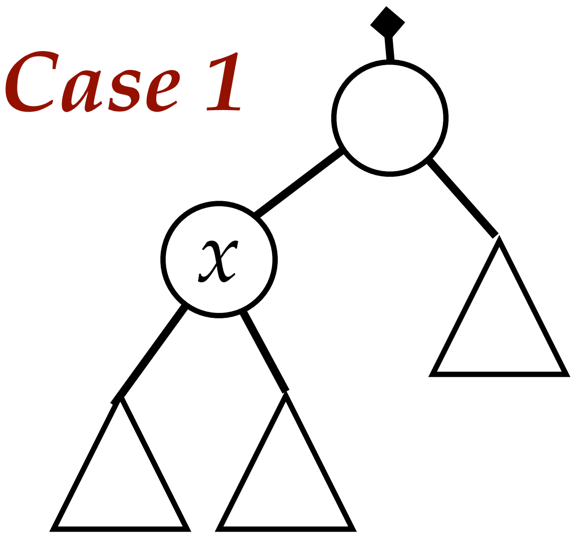
k moves toward the root

Splay Operation

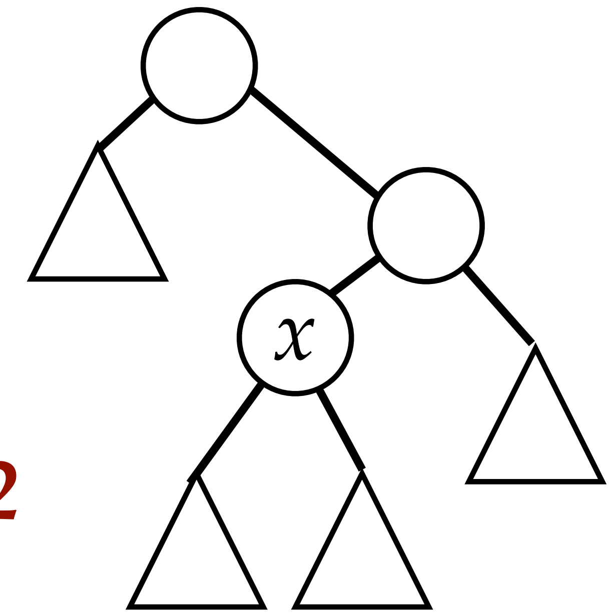
- *Splay*(T, *k*): find *k*, walk back up root. Let x be the current node.
- Cases:
  1. x has no grandparent
  2. x is left child of parent(x), which is the right child of parent^2(x).
  3. x is left child of parent(x), which is the left child of parent(parent(x)) = parent^2(x)

Rotations with goal: move x toward the root

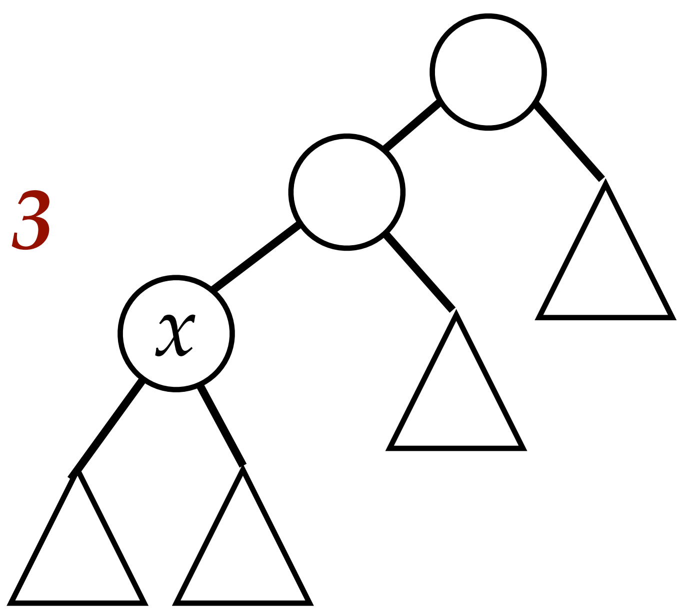
Case 1



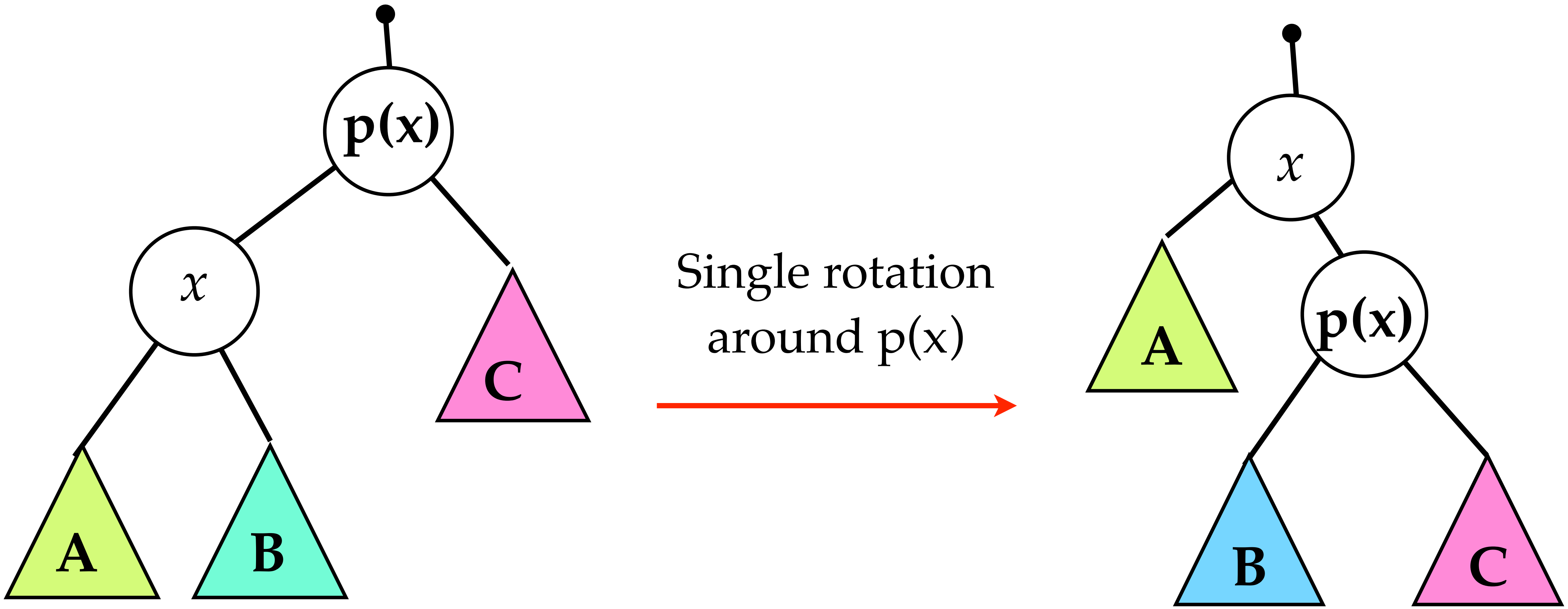
Case 2



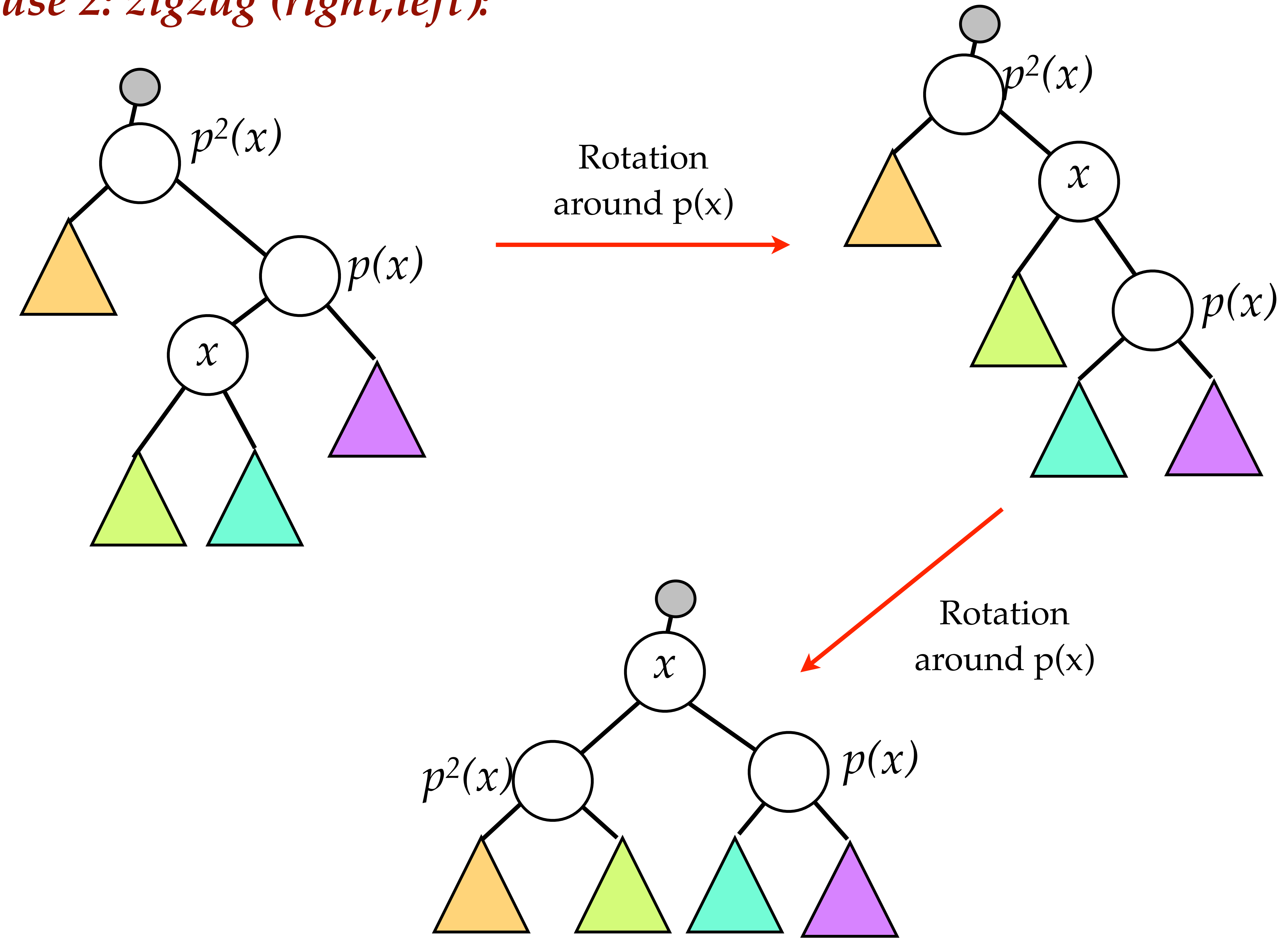
Case 3



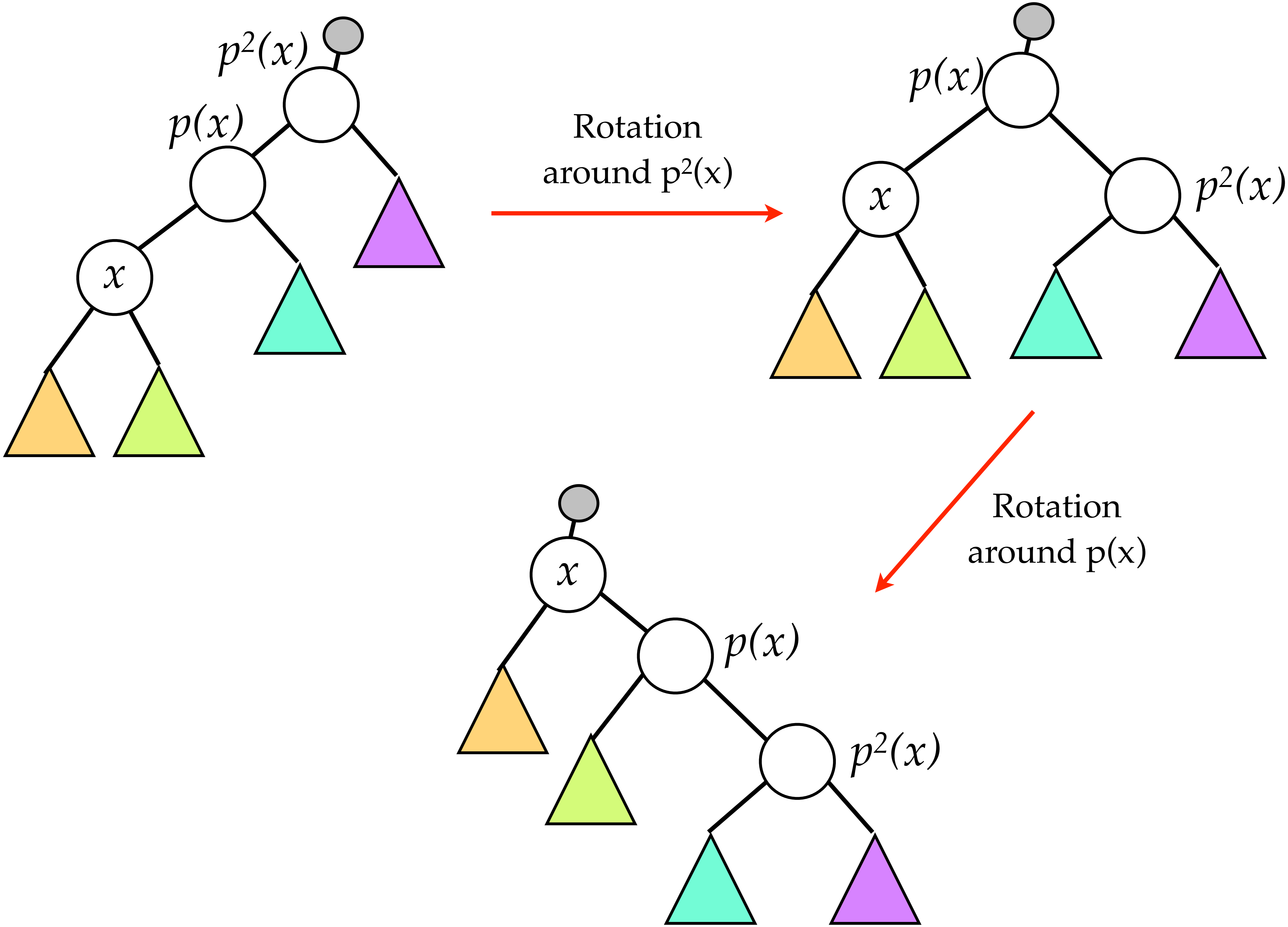
Case 1: no grandparent:



Case 2: zigzag (right,left):



Case 3: zigzig (left, left):



Splay Idea

- The find/insert/delete operations can be written in terms of the "splay" operation.

- Splay is implemented by doing a standard BST "find" and then applying particular rotations **walking back up toward the root.**

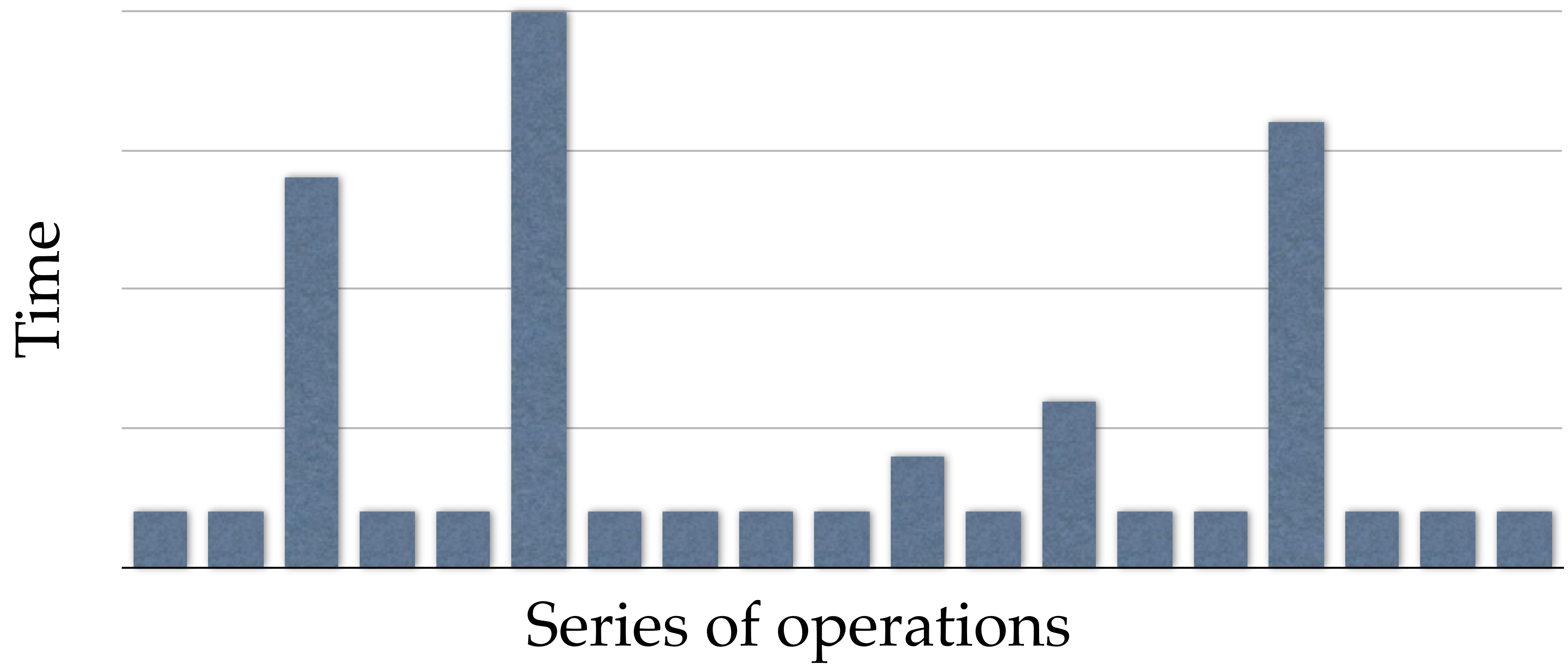
- This is somewhat like the idea of "path compression" for the tree-based union-find data structure: during a find, you flatten out the tree.

- Here: splay may actually make the tree worse, but over a series of operations the tree always gets better (e.g. a slow find results in a long splay, but this long splay tends to flatten the tree a lot).

Splay Notes

- Might make tree less balanced
- Might make tree taller
- So, how can they be good?

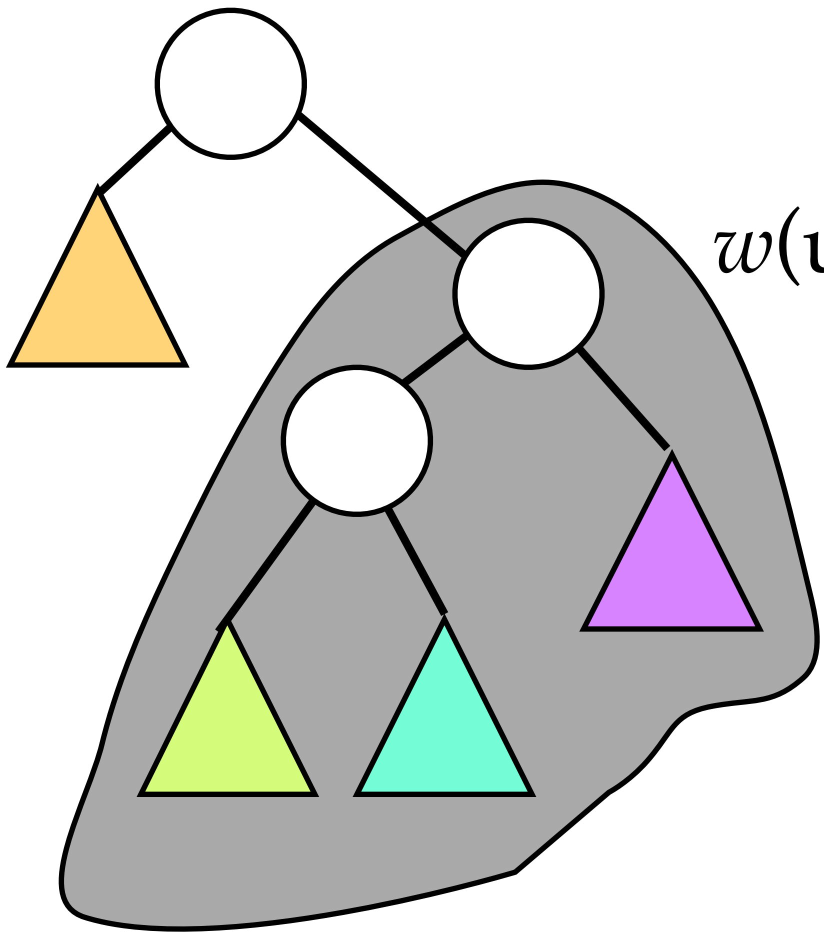
Amortized Analysis – Concept



- Some operations will be costly, some will be cheap
- Total area of *m* bars bounded by some function f(*m*,*n*).
  - m = number of operations, n = number of elements
- E.g. if area = O(m log n), each operation takes O(log n) amortized time

Analysis of the Splay operation

Node Ranks & Money Invariant



w(u) := weight of u

:= # nodes in the subtree rooted at u

rank(u) := [log $w(u)$]

[x] means floor(x)

Money Invariant: we will always keep *rank*(u) dollars stored at every node.

Each rotation/double rotation costs \$1.

O(1) amount of work

Also have to spend \$ to maintain invariant.

Idea:

> **Thm.** It costs at most 3[log n] + 1 new dollars to splay, keeping the money invariant

- So, for every splay, we're going to spend O(log n) new dollars; we might do more *work* than that if we use some of the $ already in the tree.
- If we start with an empty tree, after m splay operations, we'll have spent ≤ m(3[log n] +1) dollars.
- The dollars pay for both:
  - the money invariant
  - cost of all the rotations (time)
- So, total time for m splay operations is O(m log n).

Thm. It costs 3[log n] + 1 new dollars to splay, keeping the money invariant.

Suppose zig/zigzig/zigzag at x costs the following:

zig: $3(\mathit{rank}^1(x) - \mathit{rank}(x)) + 1$

zigzag: $3(\mathit{rank}^1(x) - \mathit{rank}(x))$

zigzig: $3(\mathit{rank}^1(x) - \mathit{rank}(x))$

Then cost of a whole splay =

$$
\begin{aligned}
&3(\mathit{rank}^1(x) - \mathit{rank}(x)) \\
&+ 3(\mathit{rank}^2(x) - \mathit{rank}^1(x)) \\
&+ 3(\mathit{rank}^3(x) - \mathit{rank}^2(x)) \\
&\\
&+ 3(\mathit{rank}^k(x) - \mathit{rank}^{(k-1)}(x)) + 1
\end{aligned}
$$

Then cost of a whole splay

$$
\begin{aligned}
&= 3(\mathit{rank}^k(x) - \mathit{rank}(x)) + 1 \\
&\leq 3(\mathit{rank}^k(x)) + 1 \\
&\leq 3[\log n] + 1
\end{aligned}
$$

Thm. It costs 3[log n] + 1 new dollars to splay, keeping the money invariant.

Suppose zig/zigzig/zigzag at x costs the following:

$$\text{zig:}\quad 3(\mathit{rank}^1(x) - \mathit{rank}(x)) + 1$$
$$\text{zigzag:}\quad 3(\mathit{rank}^1(x) - \mathit{rank}(x))$$
$$\text{zigzig:}\quad 3(\mathit{rank}^1(x) - \mathit{rank}(x))$$

Then cost of a whole splay =

$$\begin{aligned}
&3(\cancel{\mathit{rank}^1(x)} - \mathit{rank}(x)) \\
&+ 3(\mathit{rank}^2(x) - \cancel{\mathit{rank}^1(x)}) \\
&+ 3(\mathit{rank}^3(x) - \mathit{rank}^2(x)) \\
&+ 3(\mathit{rank}^k(x) - \mathit{rank}^{(k-1)}(x)) + 1
\end{aligned}$$

Then cost of a whole splay

$$= 3(\mathit{rank}^k(x) - \mathit{rank}(x)) + 1$$
$$\leq 3(\mathit{rank}^k(x)) + 1$$
$$\leq 3[\log n] + 1$$

Thm. It costs 3[log n] + 1 new dollars to splay, keeping the money invariant.

Suppose zig/zigzig/zigzag at x costs the following:

$$
\begin{aligned}
\text{zig:}\quad & 3(\mathit{rank}^1(x) - \mathit{rank}(x)) + 1 \\
\text{zigzag:}\quad & 3(\mathit{rank}^1(x) - \mathit{rank}(x)) \\
\text{zigzig:}\quad & 3(\mathit{rank}^1(x) - \mathit{rank}(x))
\end{aligned}
$$

Then cost of a whole splay =

$$
\begin{aligned}
& 3(\cancel{\mathit{rank}^1(x)} - \mathit{rank}(x)) \\
+\ & 3(\cancel{\mathit{rank}^2(x)} - \cancel{\mathit{rank}^1(x)}) \\
+\ & 3(\mathit{rank}^3(x) - \cancel{\mathit{rank}^2(x)}) \\
\\
+\ & 3(\mathit{rank}^{k}(x) - \mathit{rank}^{(k-1)}(x)) + 1
\end{aligned}
$$

Then cost of a whole splay

$$
\begin{aligned}
&= 3(\mathit{rank}^k(x) - \mathit{rank}(x)) + 1 \\
&\leq 3(\mathit{rank}^k(x)) + 1 \\
&\leq 3[\log n] + 1
\end{aligned}
$$

Thm. It costs 3[log n] + 1 new dollars to splay, keeping the money invariant.

Suppose zig/zigzig/zigzag at x costs the following:

zig: $3(rank^1(x) - rank(x)) + 1$

zigzag: $3(rank^1(x) - rank(x))$

zigzig: $3(rank^1(x) - rank(x))$

Then cost of a whole splay =

$$
\begin{aligned}
& 3(\cancel{rank^1(x)} - rank(x)) \\
&+ 3(\cancel{rank^2(x)} - \cancel{rank^1(x)}) \\
&+ 3(\cancel{rank^3(x)} - \cancel{rank^2(x)}) \\
&\\
&+ 3(rank^k(x) - \cancel{rank^{(k-1)}(x)}) + 1
\end{aligned}
$$

Then cost of a whole splay

$= 3(rank^k(x) - rank(x)) + 1$

$\leq 3(rank^k(x)) + 1$

$\leq$ 3[log n] + 1

Thm. It costs 3[log n] + 1 new dollars to splay, keeping the money invariant.

Suppose zig/zigzig/zigzag at x costs the following:

zig: $3(rank^1(x) - rank(x)) + 1$
zigzag: $3(rank^1(x) - rank(x))$
zigzig: $3(rank^1(x) - rank(x))$

Then cost of a whole splay =

$$
\begin{aligned}
& 3(\cancel{rank^1(x)} - rank(x)) \\
&+ 3(\cancel{rank^2(x)} - \cancel{rank^1(x)}) \\
&+ 3(\cancel{rank^3(x)} - \cancel{rank^2(x)}) \\
&+ 3(rank^k(x) - \cancel{rank^{(k-1)}(x)}) + 1
\end{aligned}
$$

Telescoping sum

Then cost of a whole splay

$= 3(rank^k(x) - rank(x)) + 1$

$\leq 3(rank^k(x)) + 1$

$\leq 3[\log n] + 1$

Thm. It costs 3[log n] + 1 new dollars to splay, keeping the money invariant.

Suppose zig/zigzig/zigzag at x costs the following:

$$\text{zig:}\quad 3(\mathit{rank}^1(x) - \mathit{rank}(x)) + 1$$
$$\text{zigzag:}\quad 3(\mathit{rank}^1(x) - \mathit{rank}(x))$$
$$\text{zigzig:}\quad 3(\mathit{rank}^1(x) - \mathit{rank}(x))$$

Then cost of a whole splay =

$$
\begin{aligned}
&3(\cancel{\mathit{rank}^1(x)} - \mathit{rank}(x)) \\
&+ 3(\cancel{\mathit{rank}^2(x)} - \cancel{\mathit{rank}^1(x)}) \\
&+ 3(\cancel{\mathit{rank}^3(x)} - \cancel{\mathit{rank}^2(x)}) \\
&\\
&+ 3(\mathit{rank}^k(x) - \cancel{\mathit{rank}^{(k-1)}(x)}) + 1
\end{aligned}
$$

Telescoping sum

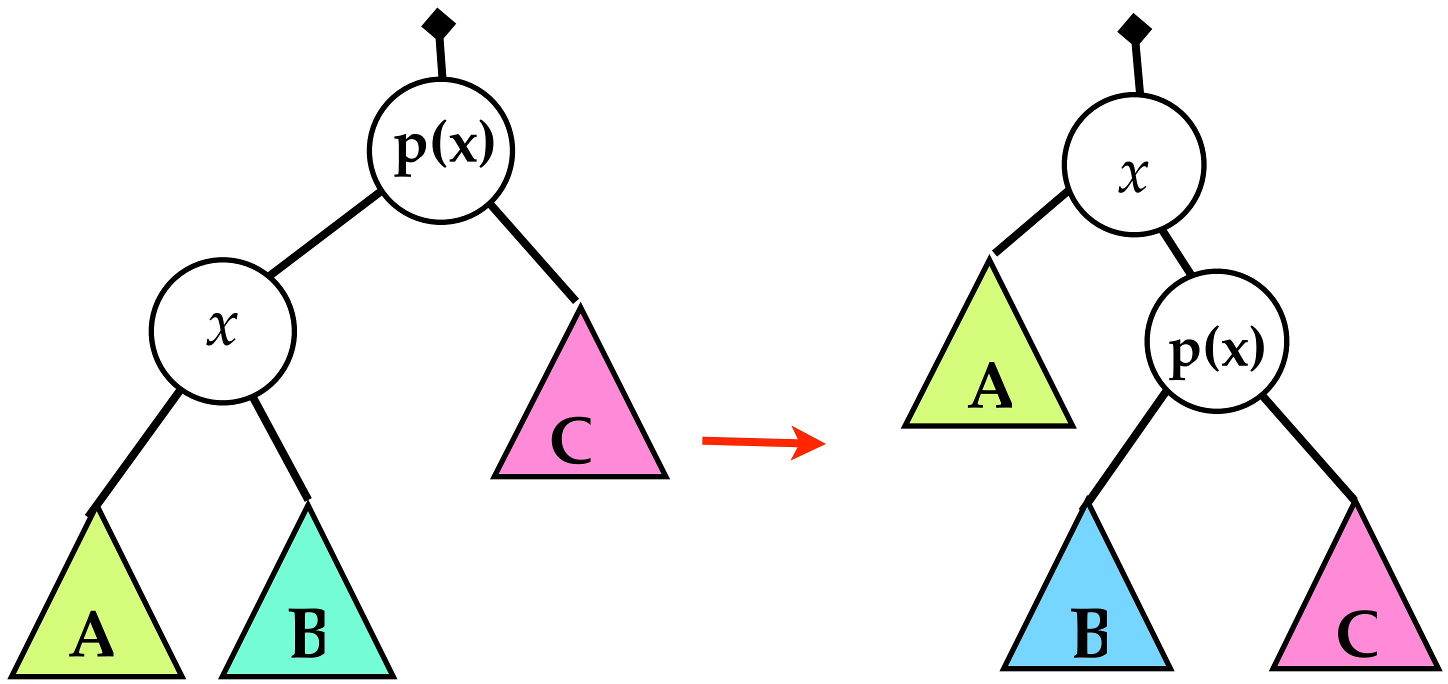
Then cost of a whole splay

$= 3(\mathit{rank}^k(x) - \mathit{rank}(x)) + 1$

$\leq 3(\mathit{rank}^k(x)) + 1$ $\quad$ *$\mathit{rank}^k(x)$ = after k steps, x is at the root*

$\leq 3[\log n] + 1$

case 1: $3(rank^1(x) - rank(x)) + 1$



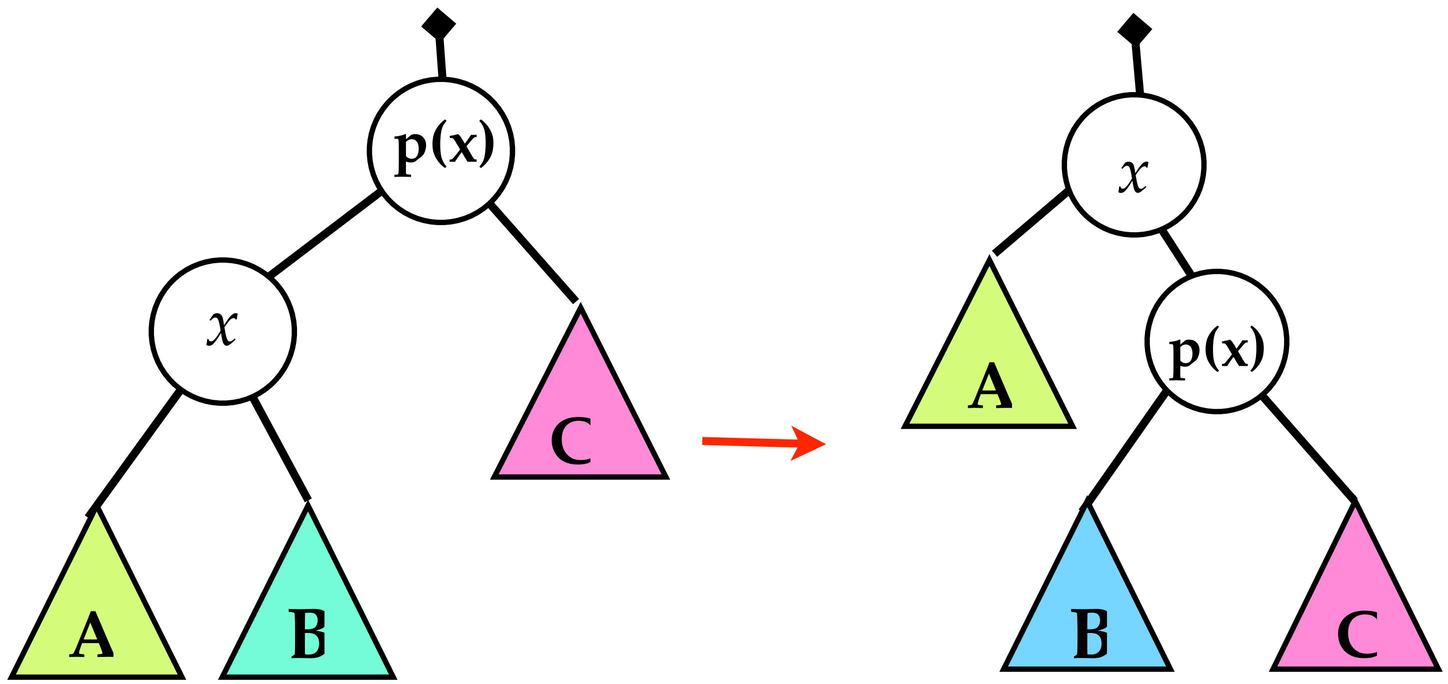
+1 pays for the rotation

$rank^1(x) = rank(p(x))$

Extra $ to keep the invariant is:

$rank^1(x) + rank^1(p(x)) - (rank(x) + rank(p(x))$

case 1: $3(rank^1(x) - rank(x)) + 1$



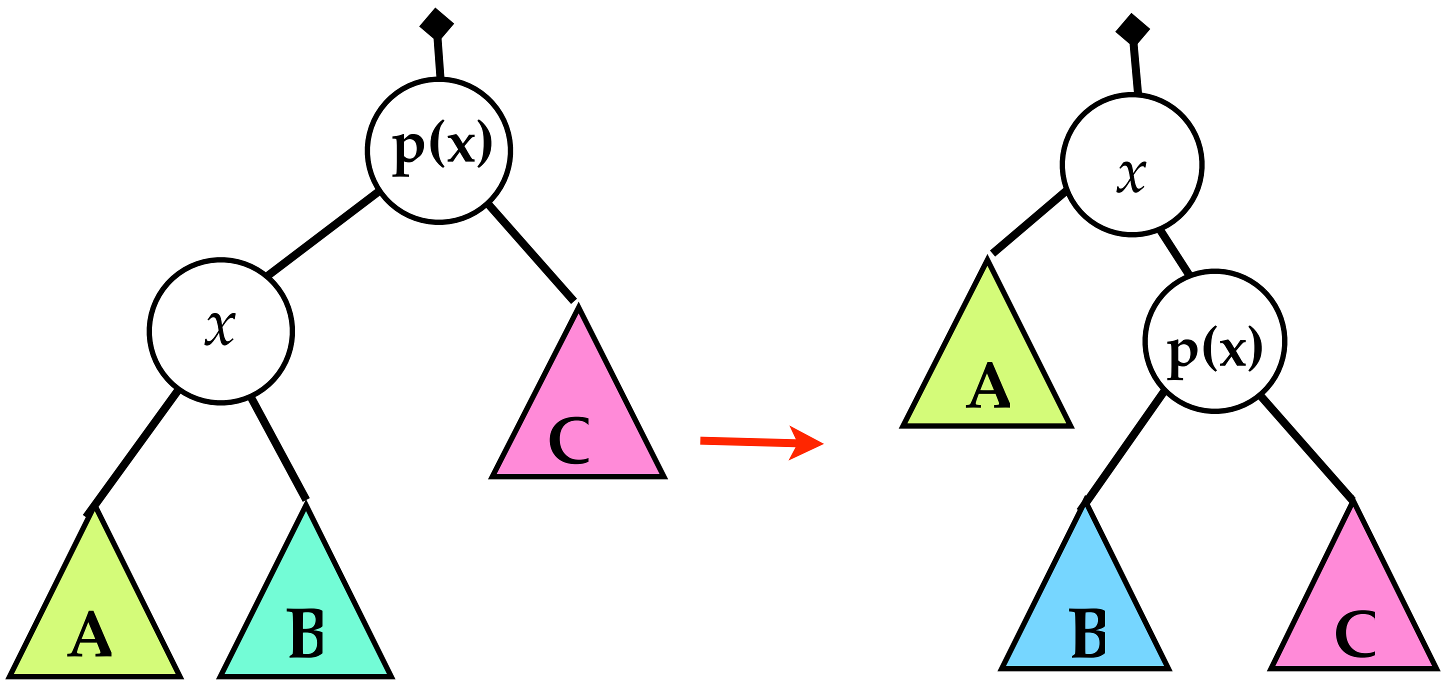
+1 pays for the rotation

$rank^1(x) = rank(p(x))$

Extra $ to keep the invariant is:

$\underbrace{\boxed{rank^1(x) + rank^1(p(x))}}_{\text{\$ needed for x and p(x)}} - (rank(x) + rank(p(x)))$

case 1: $3(rank^1(x) - rank(x)) + 1$



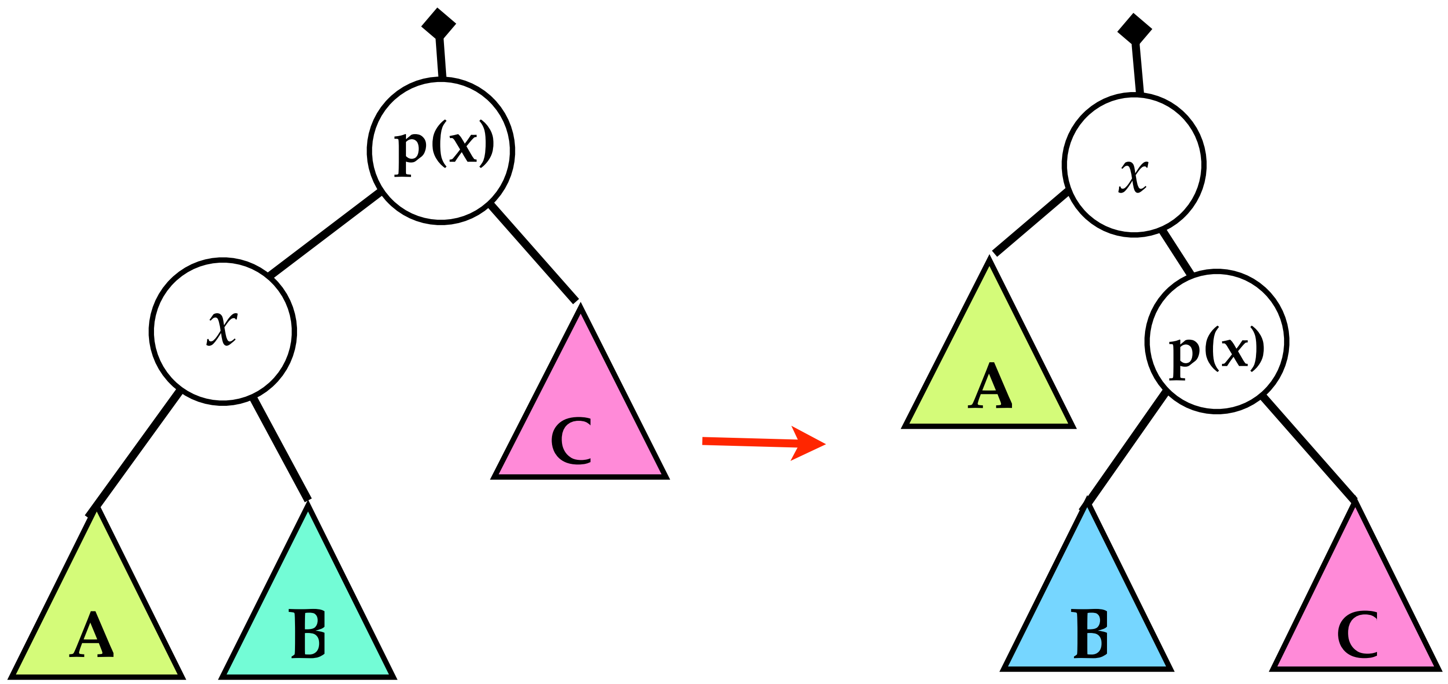
+1 pays for the rotation

$rank^1(x) = rank(p(x))$

Extra $ to keep the invariant is:

$\underbrace{\boxed{rank^1(x) + rank^1(p(x))}}_{\text{\$ needed for x and p(x)}} - \underbrace{\boxed{(rank(x) + rank(p(x))}}_{\text{\$ already on x and p(x)}}$

case 1: $3(\mathit{rank}^1(\text{x}) - \mathit{rank}(\text{x})) + 1$



+1 pays for the rotation

$\text{rank}^1(\text{x}) = \text{rank}(\text{p}(\text{x}))$

Extra \$ to keep the invariant is:

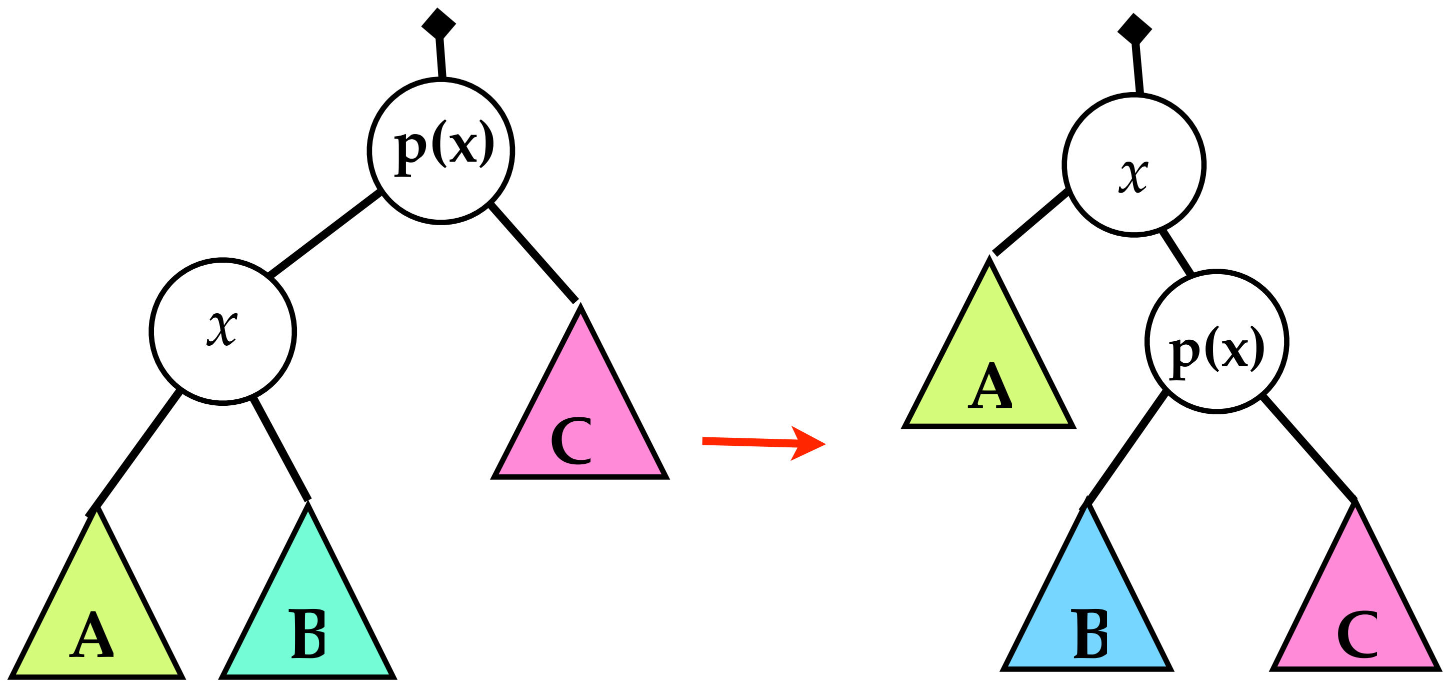
$$\boxed{\cancel{\text{rank}^1(\text{x})} + \text{rank}^1(\text{p}(\text{x}))} - \boxed{(\text{rank}(\text{x}) + \cancel{\text{rank}(\text{p}(\text{x}))})}$$

\$ needed for x and p(x)

\$ already on x and p(x)

r(x) after = r(p(x)) before.

case 1: $3(\mathit{rank}^1(x) - \mathit{rank}(x)) + 1$



+1 pays for the rotation

$\mathrm{rank}^1(x) = \mathrm{rank}(p(x))$

Extra $ to keep the invariant is:

$$\boxed{\cancel{\mathrm{rank}^1(x)} + \mathrm{rank}^1(p(x))} - \boxed{(\mathrm{rank}(x) + \cancel{\mathrm{rank}(p(x))})}$$

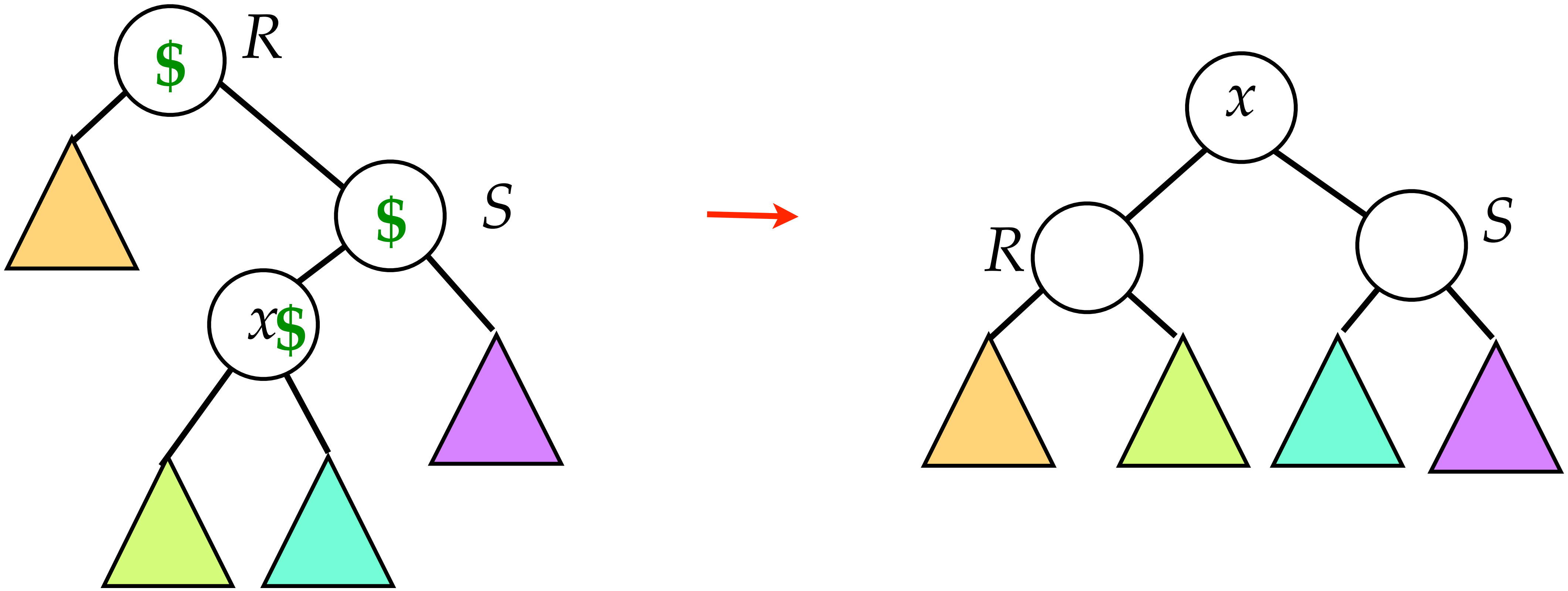
$ needed for x and p(x) $\qquad$ $ already on x and p(x)

r(x) after = r(p(x)) before.

$$= \mathrm{rank}^1(p(x)) - \mathrm{rank}(x)$$

$$\leq \mathrm{rank}^1(x) - \mathrm{rank}(x) \leq 3(\mathrm{rank}^1(x) - \mathrm{rank}(x))$$

case 2: $3(rank^1(x) - rank(x))$



$ *needed to add* $\leq$ $\text{rank}^1(R) - \text{rank}(x)$

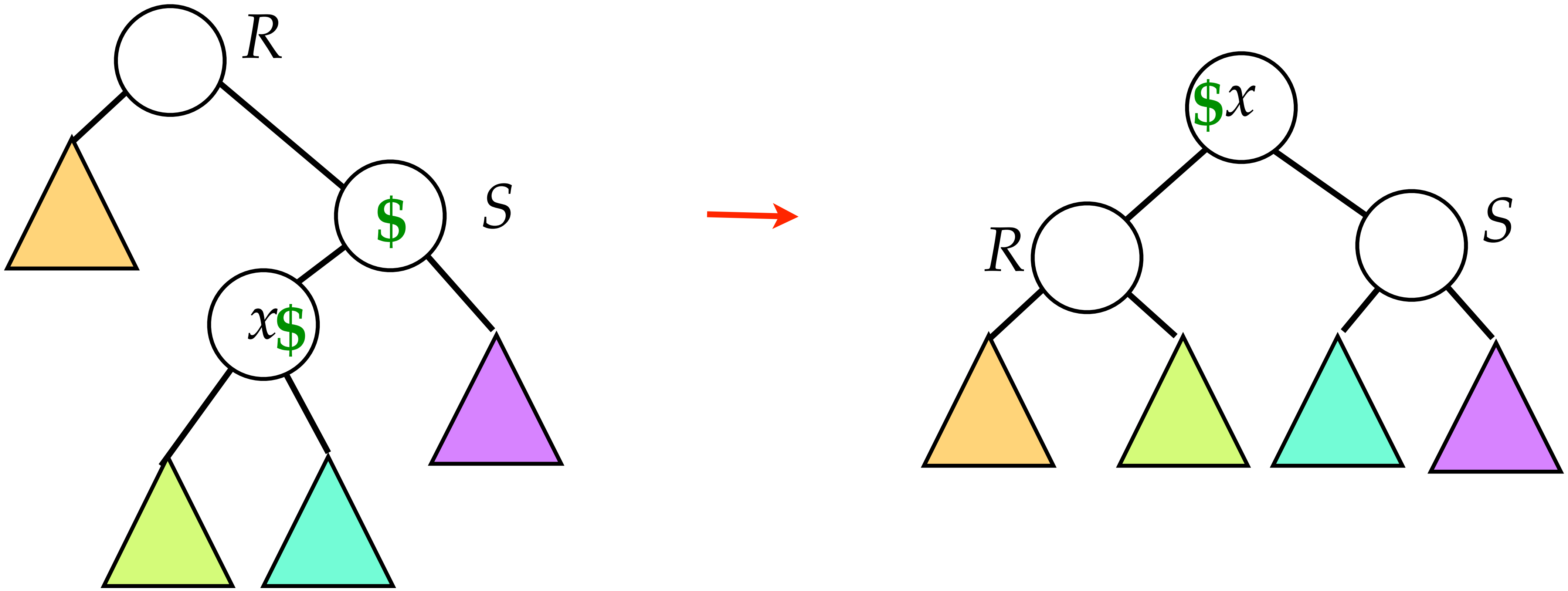
$\leq$ $\text{rank}^1(x) - \text{rank}(x)$

But how do we pay for the rotations?

If $\text{rank}^1(x) - \text{rank}(x) > 0$, then
we need to add $\leq r^1(x) - r(x)$ but we have $3(r^1(x) - r(x)) > 1$
budgeted, so we have at least $1 to pay for the rotations.

Otherwise see next slide.

case 2: $3(\mathit{rank}^1(x) - \mathit{rank}(x))$



$\$$ *needed to add* $\leq \; \mathrm{rank}^1(R) - \mathrm{rank}(x)$

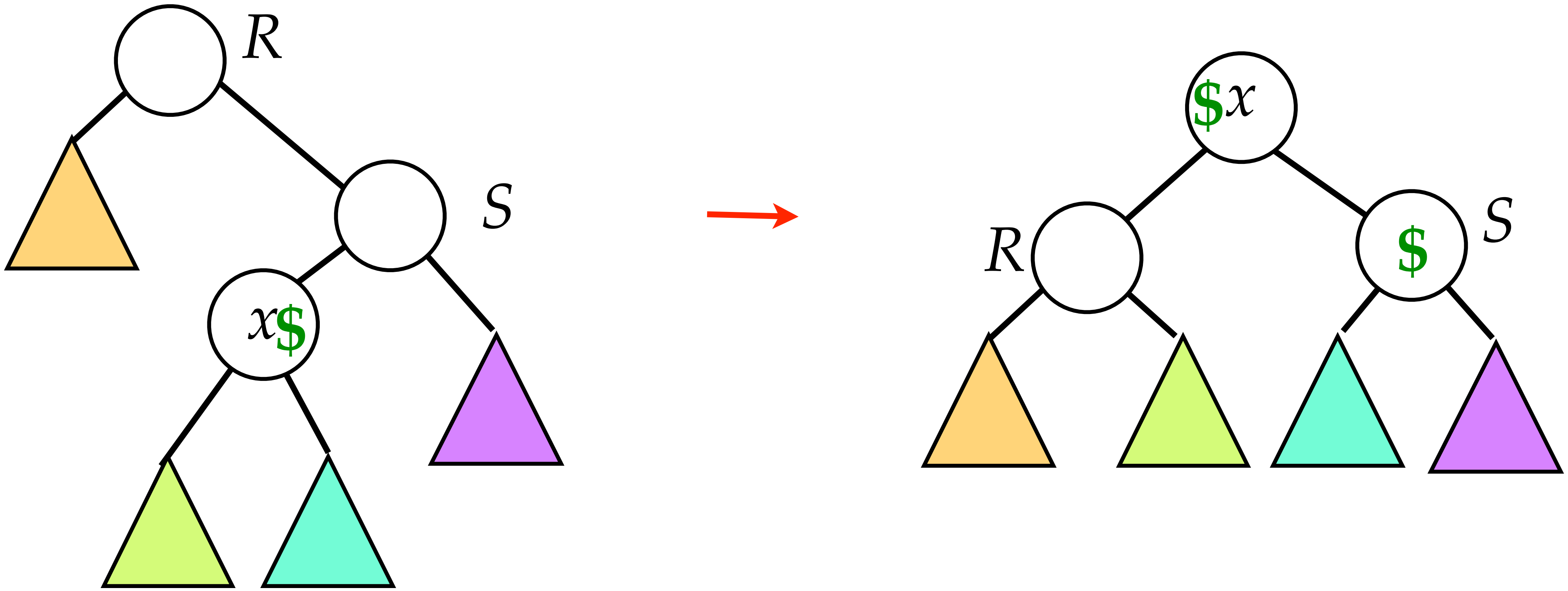
$\leq \; \mathrm{rank}^1(x) - \mathrm{rank}(x)$

But how do we pay for the rotations?

If $\mathrm{rank}^1(x) - \mathrm{rank}(x) > 0$, then
we need to add $\leq r^1(x) - r(x)$ but we have $3(r^1(x) - r(x)) > 1$
budgeted, so we have at least \$1 to pay for the rotations.

Otherwise see next slide.

case 2: $3(\mathit{rank}^1(x) - \mathit{rank}(x))$



$ needed to add $\leq$ $\text{rank}^1(R) - \text{rank}(x)$

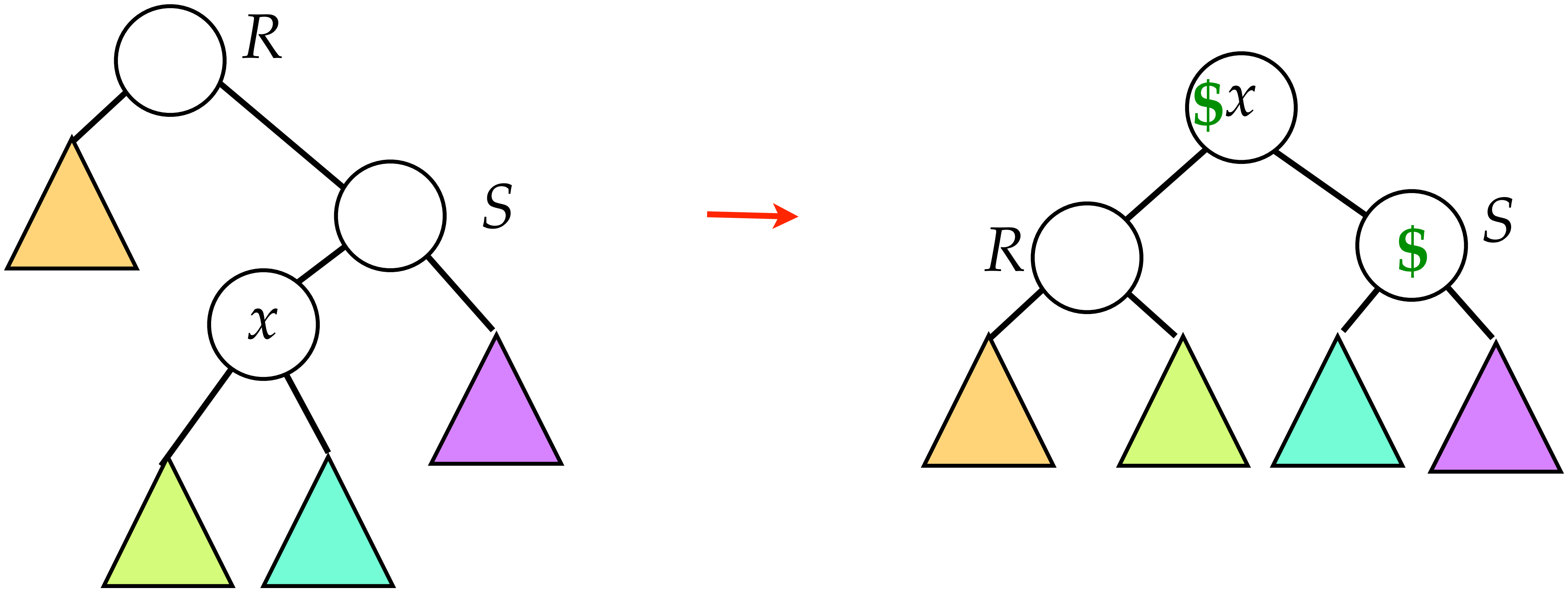
$\leq$ $\text{rank}^1(x) - \text{rank}(x)$

But how do we pay for the rotations?

If $\text{rank}^1(x) - \text{rank}(x) > 0$, then
we need to add $\leq r^1(x) - r(x)$ but we have $3(r^1(x) - r(x)) > 1$ budgeted, so we have at least $1 to pay for the rotations.

Otherwise see next slide.

case 2: $3(\mathit{rank}^1(x) - \mathit{rank}(x))$



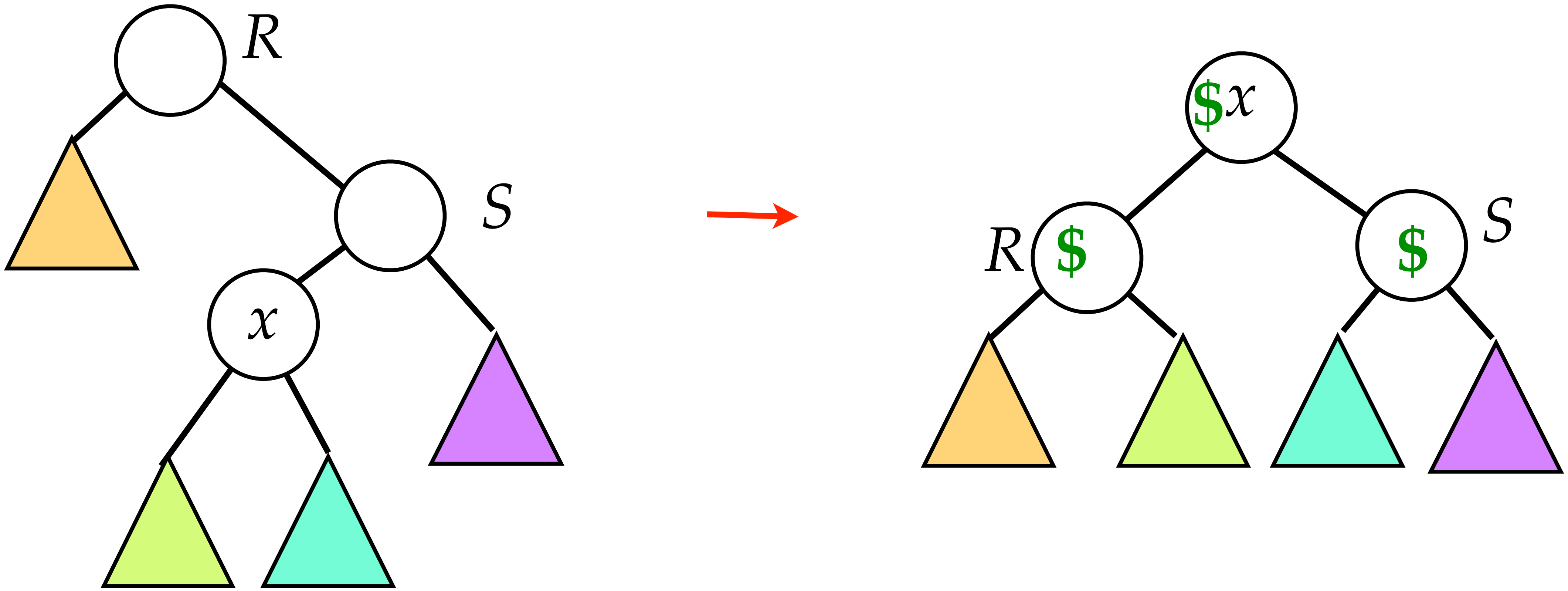
$$\textit{\$ needed to add} \leq \mathrm{rank}^1(R) - \mathrm{rank}(x)$$
$$\leq \mathrm{rank}^1(x) - \mathrm{rank}(x)$$

But how do we pay for the rotations?

If $\mathrm{rank}^1(x) - \mathrm{rank}(x) > 0$, then
we need to add $\leq r^1(x) - r(x)$ but we have $3(r^1(x) - r(x)) > 1$
budgeted, so we have at least \$1 to pay for the rotations.

Otherwise see next slide.

case 2: $3(rank^1(x) - rank(x))$



$\$$ *needed to add* $\leq \text{rank}^1(R) - \text{rank}(x)$

$\leq \text{rank}^1(x) - \text{rank}(x)$

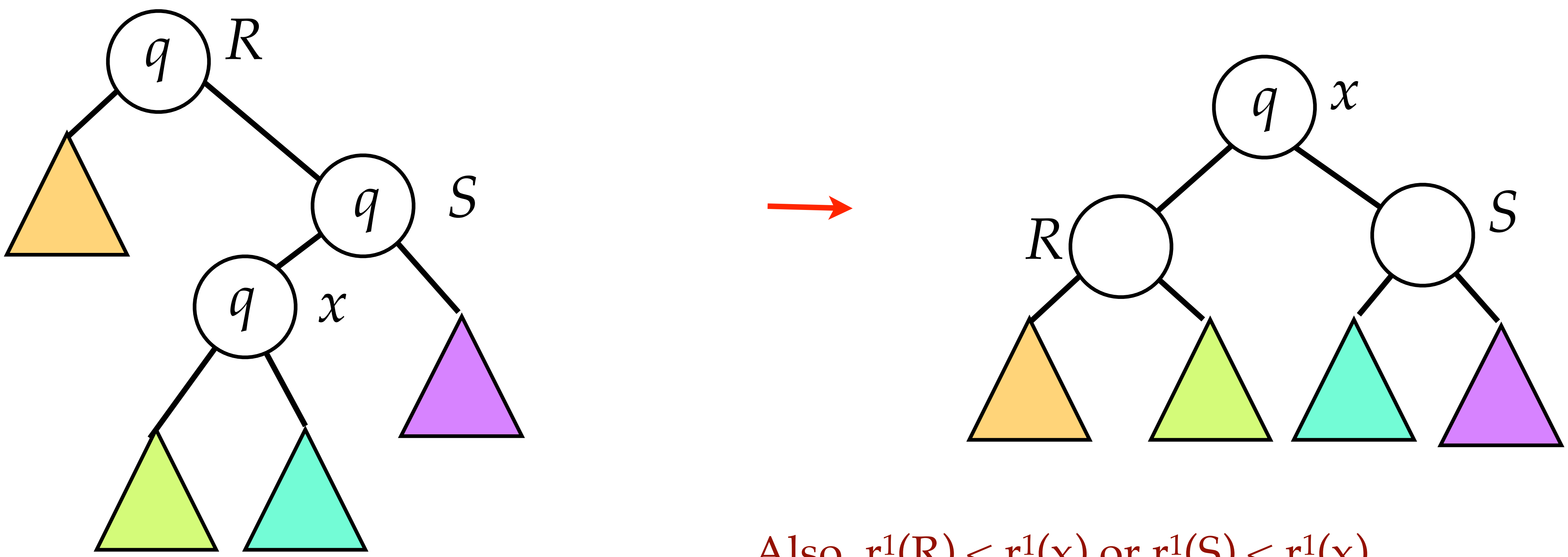
But how do we pay for the rotations?

If $\text{rank}^1(x) - \text{rank}(x) > 0$, then
we need to add $\leq r^1(x) - r(x)$ but we have $3(r^1(x) - r(x)) > 1$ budgeted, so we have at least \$1 to pay for the rotations.

Otherwise see next slide.

case 2: What if ($rank^1(x) - rank(x)$) is 0?

Since $r^1(x) = r(x)$, and $r^1(x) = r(R)$, we must have $r(x) = r(R) = r(S)$:



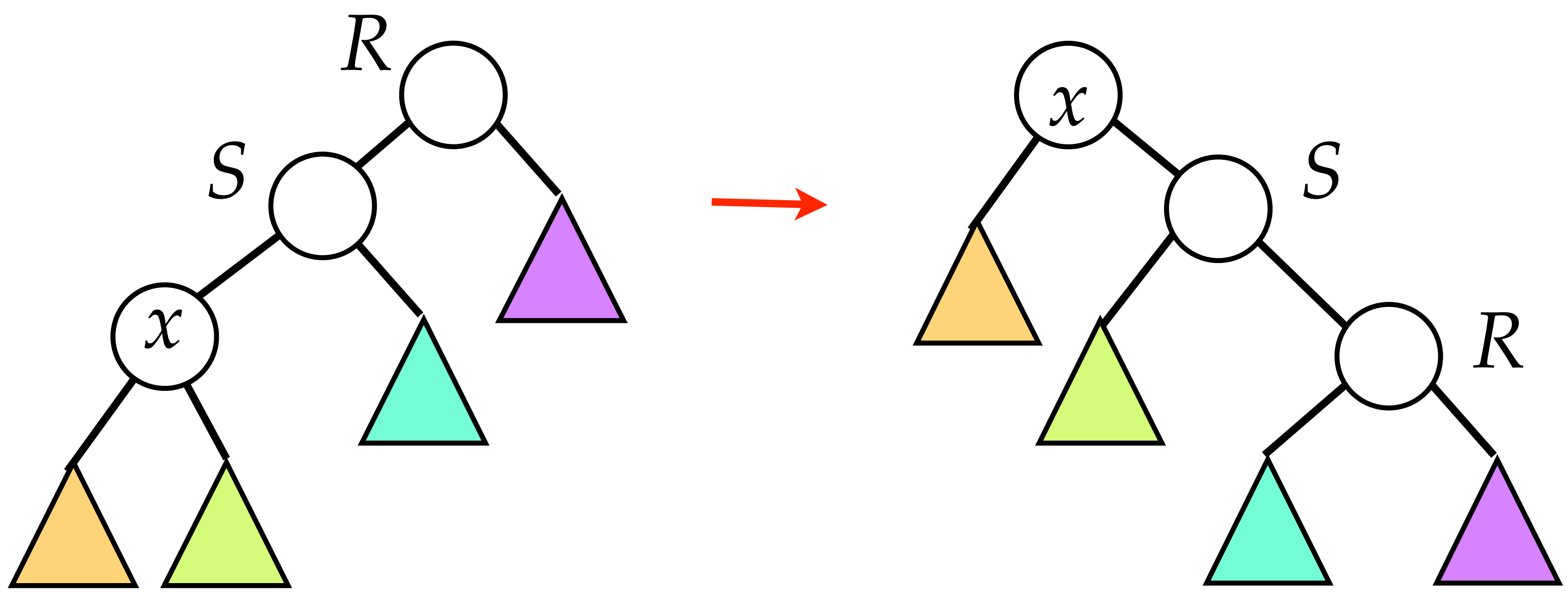
Also, $r^1(R) < r^1(x)$ or $r^1(S) < r^1(x)$

Suppose both $r(R) = r(S) = q$.

Then R would have at least 2^q nodes under it and S would have at least 2^q nodes under it. So x would have at least $2(2^q)$ nodes under it and x would then have rank $q+1$ instead of q.

Had $3q$ \$ on tree before, now need $< 3q$, so have 1 \$ left to pay for rotations.

case 3: $3(rank^1(x) - rank(x))$

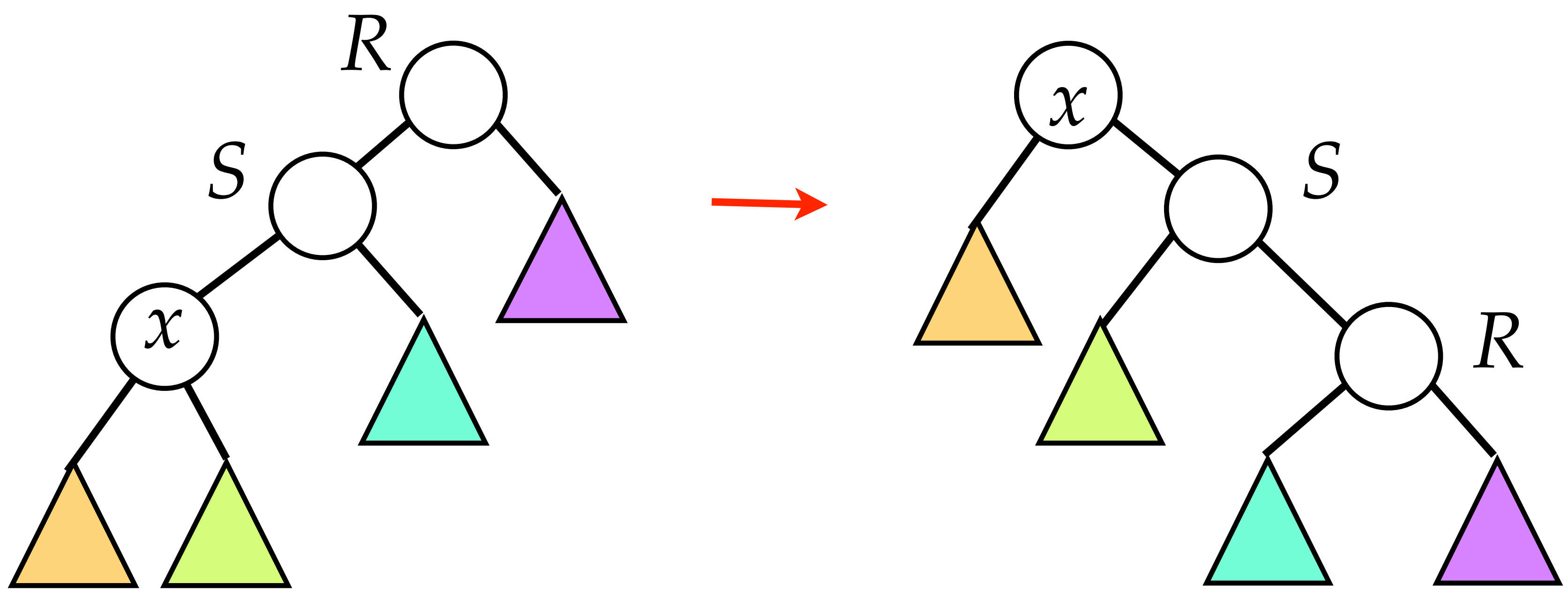


$ needed to add:

$$\boxed{r^1(x) + r^1(S) + r^1(R)} - \boxed{(r(x) + r(S) + r(R))}$$

$ needed for moved nodes $ already on moved nodes

case 3: $3(\mathit{rank}^1(x) - \mathit{rank}(x))$



$ needed to add:

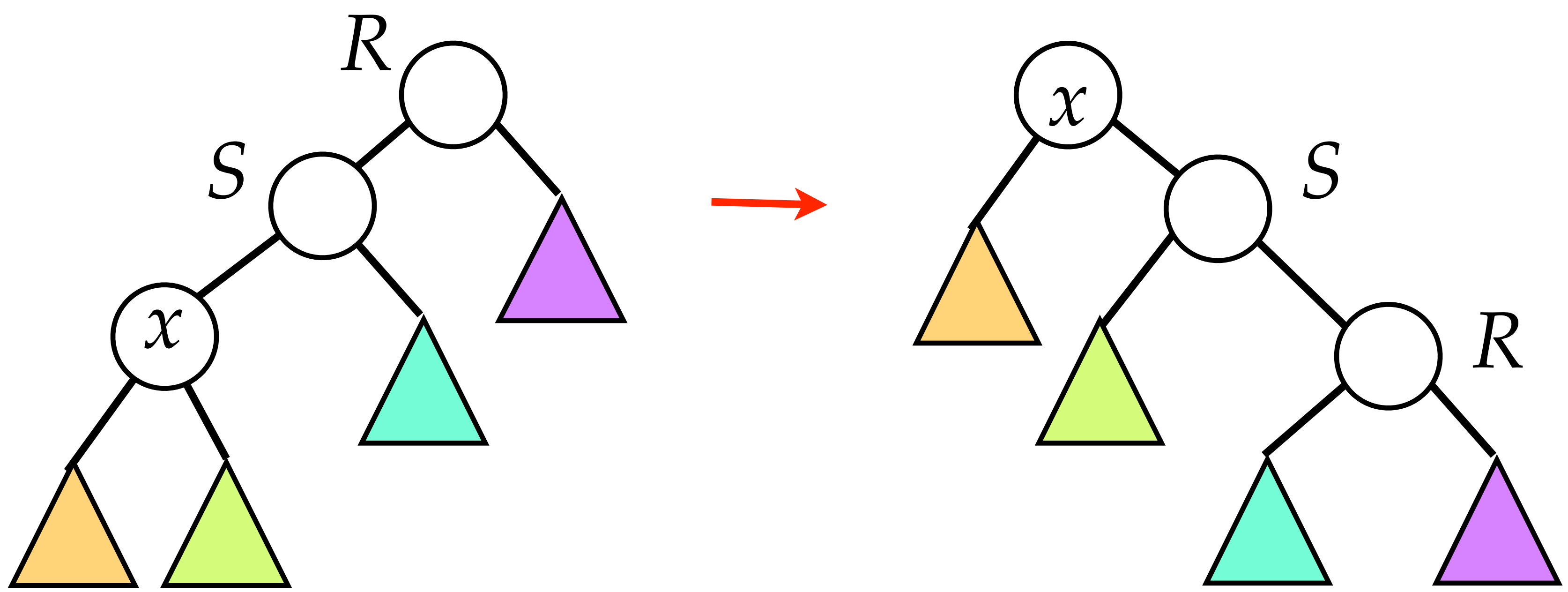
$$\boxed{r^1\cancel{(x)} + r^1(S) + r^1(R)} - \boxed{(r(x) + r(S) + r\cancel{(R)})}$$

$ needed for moved nodes $ already on moved nodes

$$= r^1(S) + r^1(R) - r(x) - r(S)$$

$r^1(x) = r(R)$

case 3: $3(rank^1(x) - rank(x))$



$ needed to add:

$\boxed{r^1\cancel{(x)} + r^1(S) + r^1(R)} - \boxed{(r(x) + r(S) + r\cancel{(R)})}$

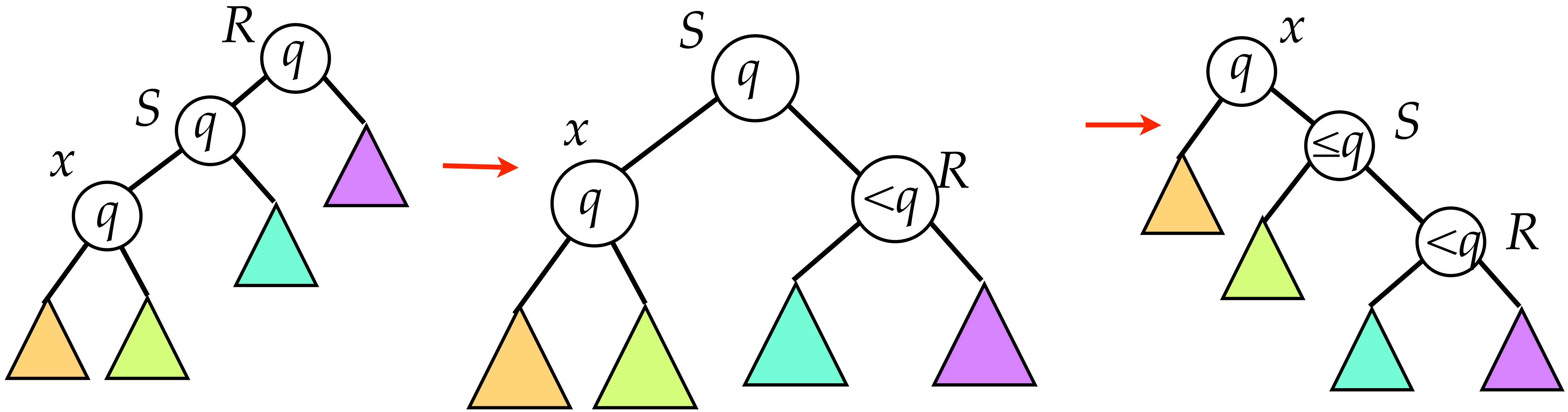
$ needed for moved nodes $ already on moved nodes

$= r^1(S) + r^1(R) - r(x) - r(S)$ $r^1(x) = r(R)$

$\leq r^1(x) + r^1(x) - r(x) - r(x)$ $r^1(R) \leq r^1(S) \leq r^1(x)$

$\leq 2(r^1(x) - r(x))$ $r(x) \leq r(S)$

case 3: What if $rank^1(x) - rank(x) = 0$?



Why must R have rank $< q$?

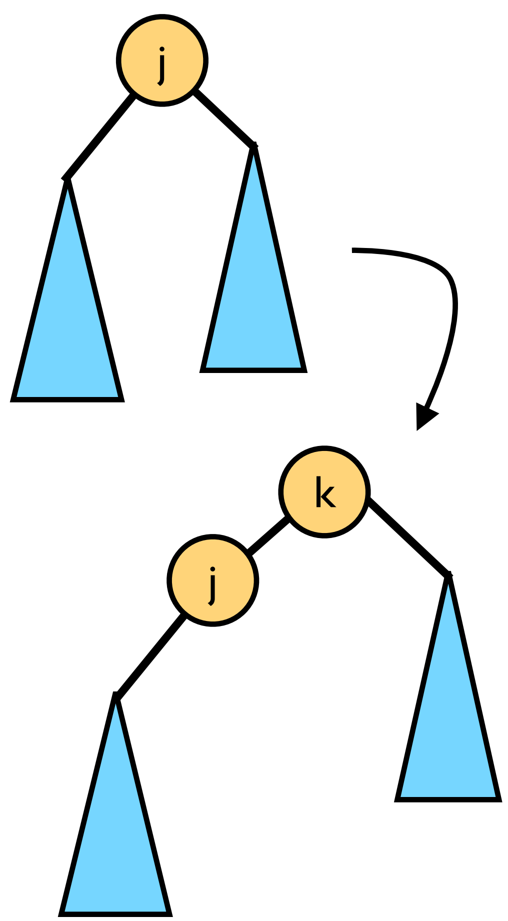
Suppose not. Then x has at least 2^q nodes under it, and R has at least 2^q nodes under it.

So S has at least $2(2^q) = 2^{q+1}$ nodes under it,
so it should have rank Q+1, but it has rank Q, which is a contradiction.

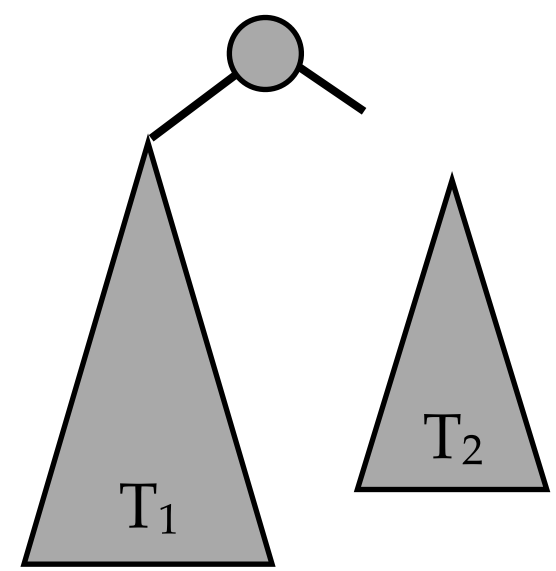
So before we had $3q$ \$ on the nodes, now we need only $\leq 2q + q - 1$

Additional Cost of Insert & Concat

- Cost of *insert* & *concat* more than the cost of a splay because may have to add $s to root to maintain invariant:



insert(T, k): k has *n* descendants, so need to put [log n] $ on k



concat(T_1, T_2): root gets at most *n* new descendants from T_2, so need to put [log n] dollars on root.