Closest Pair of Points

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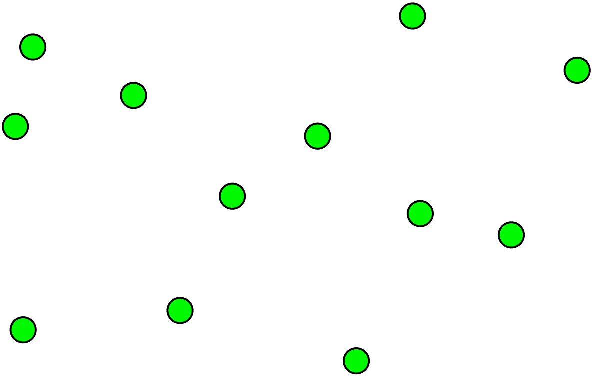
Based on AD Section 5.4

Finding closest pair of points

Problem. Given a set of points $\{p_1, \ldots, p_n\}$ in the plane find the pair of points $\{p_i, p_j\}$ that are closest together.



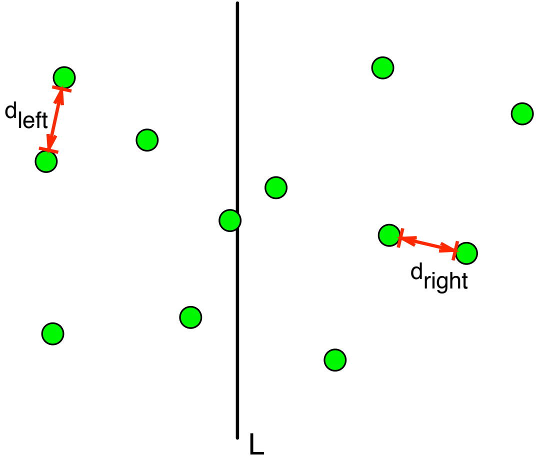
Goal

- Brute force gives an $O(n^2)$ algorithm: just check every pair of points.
- Can we do it faster? Seems like no: don't we have to check every pair?
- In fact, we can find the closest pair in $O(n \log n)$ time.
- What's a reasonable first step?

Divide

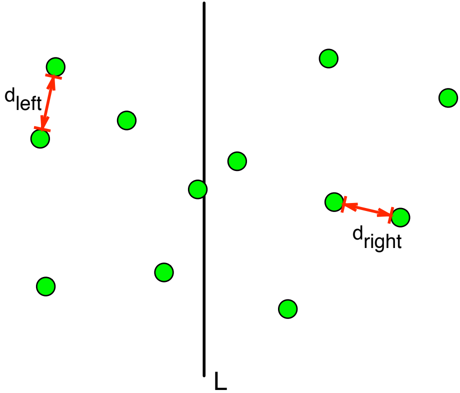
Split the points with line L so that half the points are on each side.

Recursively find the pair of points closest in each half.



Merge: the hard case

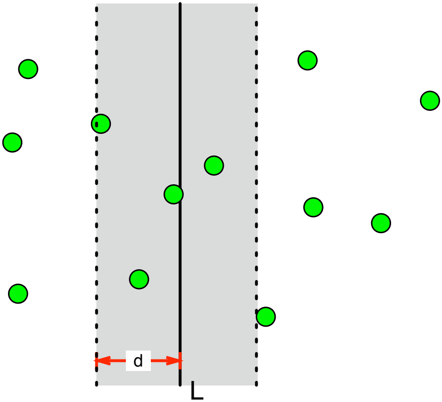
Let $d = \min\{d_{\text{left}}, d_{\text{right}}\}$.



- d would be the answer, except maybe L split a close pair!

Region Near L

If there is a pair $\{p_i, p_j\}$ with $\mathrm{dist}(p_i, p_j) < d$ that is split by the line, then both p_i and p_j must be within distance d of L.



Let S_y be an array of the points in that region, sorted by decreasing y-coordinate value.

Slab Might Contain All Points

- Let S_y be an array of the points in that region, sorted by decreasing y-coordinate value.
- S_y might contain all the points, so we can't just check every pair inside it.

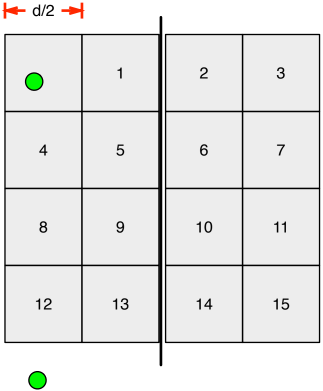
Theorem. *Suppose* $S_y = [p_1, \ldots, p_m]$. *If* $dist(p_i, p_j) < d$ *then* $j - i \leq 15$.

In other words, if two points in S_y are close enough in the plane, they are close in the array S_y.

Proof, 1

Divide the region up into squares with sides of length $d/2$:

How many points in each box?



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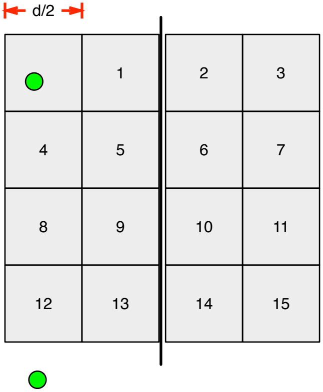
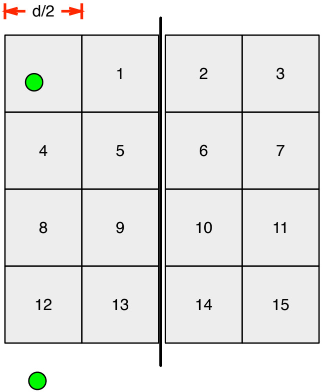


At most 1 because each box is completely contained in one half and no two points in a half are closer than d.

Proof, 2

Suppose the theorem is false and two points x, y with $\mathrm{dist}(x, y) < d$ are separated by > 15 indices.



- Then, at least 3 full rows separate them (the packing shown is the smallest possible).
- But the height of 3 rows is $> 3d/2$, which is $> d$.
- So the two points are father than d apart. Contradiction!

Linear Time Merge

Therefore, we can scan S_y for pairs of points separated by $< d$ in linear time.

```
ClosestPair(Px, Py):
   if |Px| == 2: return dist(Px[1],Px[2])    // base

   d1 = ClosestPair(FirstHalf(Px,Py))     // divide
   d2 = ClosestPair(SecondHalf(Px,Py))
   d = min(d1,d2)

   Sy = points in Py within d of L     // merge
   For i = 1,...,|Sy|:
       For j = 1,...,15:
           d = min( dist(Sy[i], Sy[i+j]), d )
   Return d
```

Total Running Time

- Divide set of points in half each time:
  $O(\log n)$ depth recursion
- Merge takes $O(n)$ time.
- Recurrence: $T(n) \leq 2T(n/2) + cn$
- Same as MergeSort $\implies$ $O(n \log n)$ time.