Divide and Conquer

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Based on AD Sections 5.1–5.3

Divide and Conquer

Divide and Conquer is general algorithmic design framework.

Related to *induction*:

- Suppose you have a "box" that can solve problems of size $\leq k < n$
- You use this box on some subset of the input items to get partial answers
- You combine these partial answers to get the full answer.

But: you construct the "box" by recursively applying the same idea until the problem is small enough to be solved by brute force.

Merge Sort

```
MergeSort(L):
    if |L| = 2:
        return [min(L), max(L)]
    else:
        L1 = MergeSort(L[0, |L|/2])
        L2 = MergeSort(L[|L|/2+1, |L|-1])
        return Combine(L1, L2)
```

- In practice, you sort in-place rather than making new lists.
- `Combine(L1,L2)` walks down the sorted lists putting the smaller number onto a new list. Takes $O(n)$ time
- Total time: $T(n) \leq 2T(n/2) + cn$.

Runtime via a Recurrence

Given a recurrence such as $T(n) \leq 2T(n/2) + cn$, we want a simple upper bound on the total running time.

Two common ways to "solve" such a recurrence:

1. Unroll the recurrence and see what the pattern is.
   Typically, you'll draw the recursion tree.

2. Guess an answer and prove that it's right.

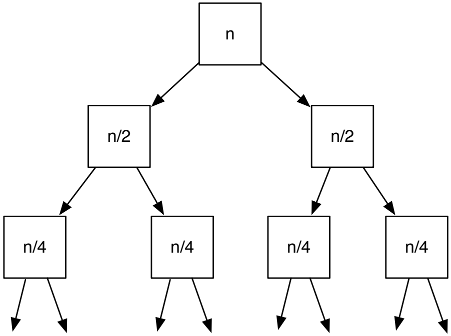
Solving Recurrences

Draw the first few levels of the tree.

Write the amount of work done at each level in terms of the level.

Figure out the height of the tree.

Sum over all levels of the tree.



$T(n) \leq 2T(n/2) + cn$. Each level is cn. There are $\log n$ levels, so $T(n)$ is $O(n \log n)$.

Substitution Method

Substitution method is based on induction. We:

1. Show $T(k) \leq f(k)$ for some small k.
2. Assume $T(k) \leq f(k)$ for all $k < n$.
3. Show $T(n) \leq f(n)$.

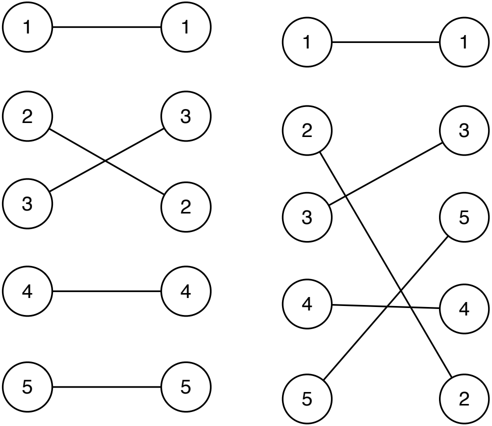
$T(n) \leq 2T(n/2) + cn$ **Base Case:** $2c \log 2 = 2c \geq T(2)$
Induction Step:

$$
\begin{aligned}
T(n) &\leq 2T(n/2) + cn \\
&\leq 2c(n/2)\log(n/2) + cn \\
&= cn[(\log n) - 1] + cn \\
&= cn \log n
\end{aligned}
$$

Counting Inversions

Comparing Rankings

Suppose two customers rank a list of movies.



similar more different

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We can count the number of inversions:

- Assume one of the rankings is $1, 2, 3, \ldots, n$.
- Denote the other ranking by $a_1, a_2, \ldots, a_n$.
- An inversion is a pair (i, j) such that $i < j$ but $a_j < a_i$.

Two identical rankings have no inversions.

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How many inversions do opposite rankings have? $\binom{n}{2}$

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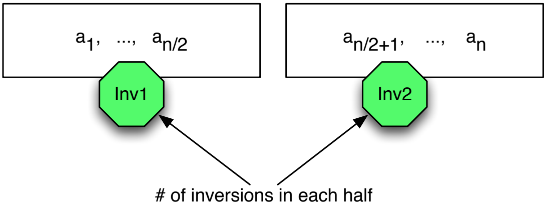
- **Brute Force:** check every pair: $O(n^2)$.
- Some sequences might have $O(n^2)$ inversions, so you might think that it might take as much as $O(n^2)$ time to count them.
- In fact, with divide and conquer, you can count them in $O(n \log n)$ time.

Basic Divide and Conquer

Count the number of inversions in the sequence $a_1, \ldots, a_n$.

Suppose I told you the number of inversions in the first half of the list and in the second half of the list:



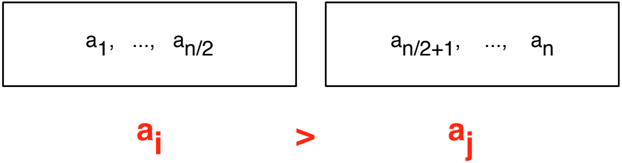
What kinds of inversions are not accounted for in `Inv1 + Inv2`?

SortAndCount

```
SortAndCount(List L):
   If |L| == 1: Return 0

   A, B = first & second halves of L

   invA, SortedA = SortAndCount(A)
   invB, SortedB = SortAndCount(B)

   crossInv, SortedL = MergeAndSort(SortedA, SortedB)
   Return invA + invB + crossInv  and SortedL
```

Half-Crossing Inversions

The inversions we have to count during the merge step:

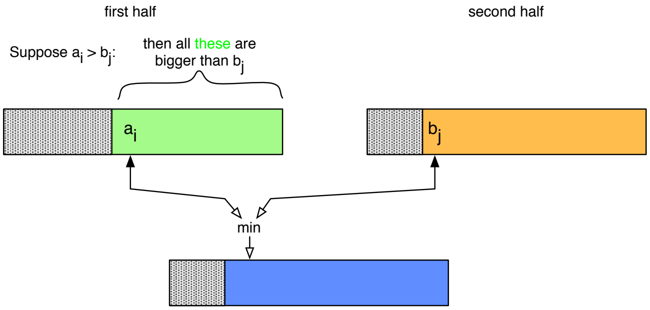


The crux is that we have to count these kinds of inversion in $O(n)$ time.

What if each of the half lists were sorted?

Suppose each of the half lists were sorted.

If we find a pair $a_i > b_j$, then we can infer many other inversions:



Each of the green items is an inversion with b_j.

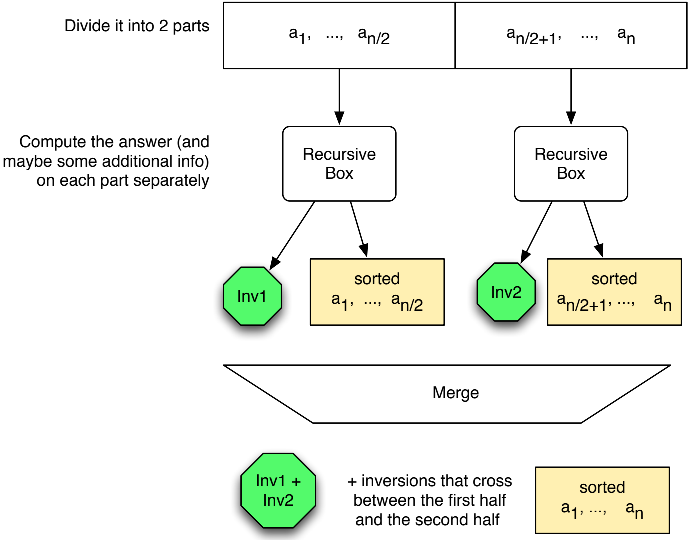
Merge-and-Count

```
MergeAndCount(SortedList A, SortedList B):
  a = b = CrossInvCount = 0
  OutList = empty list
  While a < |A| and b < |B|:    // not at end of a list
     next = min(A[a], B[b])
     OutList.append(next)

     If B[b] == next:
        b = b + 1
        CrossInvCount += |A| - a //inc by # left in A
     Else
        a = a + 1
   EndWhile
   Append the non-empty list to OutList
   Return CrossInvCount and OutList
```

Sorted!

Note that MergeAndCount will produce a sorted list as well as the number of cross inversions.

Algorithm Schematic



Running time?

What's the running time of `SortAndCount`?

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Break the problem into two halves.

Merge takes $O(n)$ time.

$$T(n) \leq 2T(n/2) + cn$$

$\Longrightarrow$ Total running time is $O(n \log n)$.