Shortest Paths in a Graph

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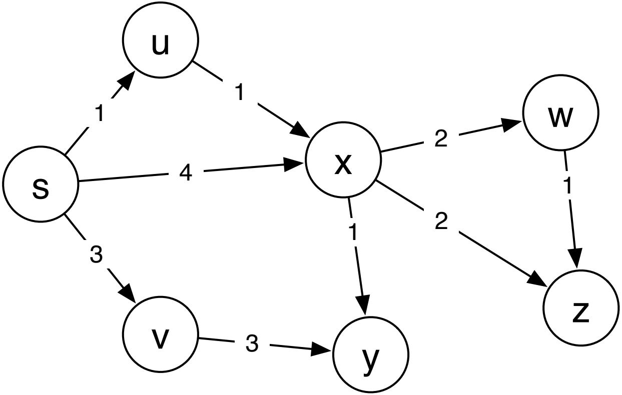
Based on/Reading: Chapter 4.5 of Kleinberg & Tardos

Shortest Paths in a Weighted, Directed Graph

Given a directed graph G with lengths $\ell_e > 0$ on each edge e:

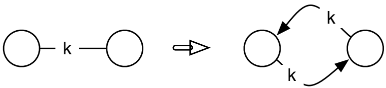


Goal: Find the shortest path from a given node s to every other node in the graph.

Shortest Paths

Shortest Paths. Given directed graph G with n nodes, and non-negative lengths on each edge, find the n shortest paths from a given node s to each v_i.

- Dijkstra's algorithm (1959) solves this problem.
- If we have an undirected graph, we can replace each undirected edge by 2 directed edges:



- If all the edge lengths are $= 1$, how can we solve this?

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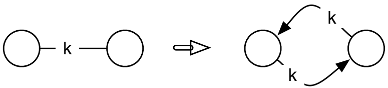
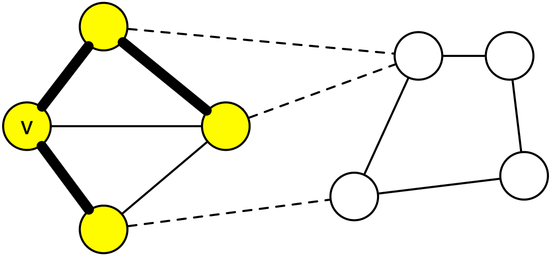


- If all the edge lengths are $= 1$, how can we solve this? BFS

General Tree Growing

Dijkstra's algorithm is just a special case of tree growing:

- Let T be the current tree T, and
- Maintain a list of frontier edges: the set of edges of G that have one endpoint in T and one endpoint not in T:



- Repeatedly choose a frontier edge (somehow) and add it to T.

Tree Growing

```
TreeGrowing(graph G, vertex v, func nextEdge):
    T = (v,∅)
    S = set of edges incident to v
    While S is not empty:
        e = nextEdge(G, S)
        T = T + e                  // add edge e to T
        S = updateFrontier(G, S, e)
    return T
```

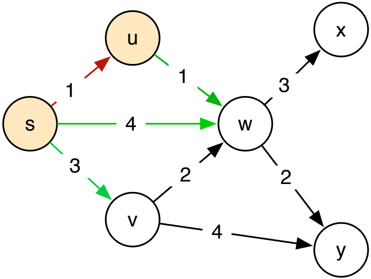
- The function `nextEdge(G, S)` returns a frontier edge from S.
- `updateFrontier(G, S, e)` returns the new frontier after we add edge e to T.

`nextEdge` for Shortest Path

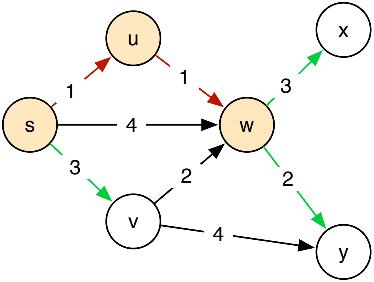
- Let u be some node that we've already visited (it will be in S).
- Let $d(u)$ be the length of the $s - u$ path found for node $u \in S$.
- `nextEdge`: return the frontier edge (u, v) for which $d(u) + \text{length}(u, v)$ is minimized.
- The "$d(u)$" term is the difference from Prim's algorithm.

Example

d[s] = 0; d[u] = 1
(green gives frontier)



d[w] = 2



Proof of Correctness

Theorem. *Let T be the set of nodes explored at some point during the algorithm. For each $u \in T$, the path to u found by Dijkstra's algorithm is the shortest.*

Proof. By induction on the size of T. **Base case**: When $|T| = 1$, the only node in T is s, for which we've obviously found the shortest path.

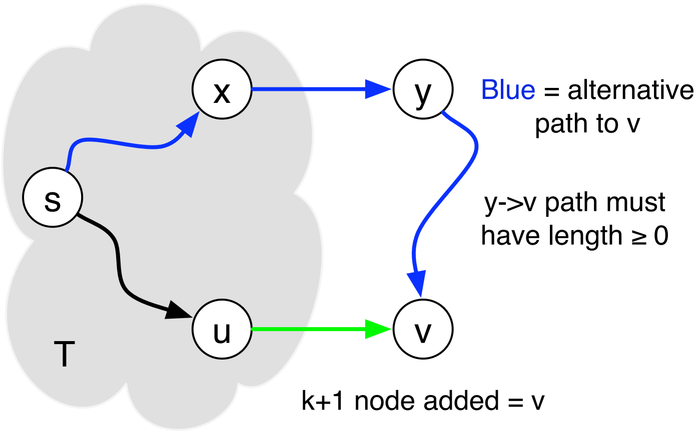
Induction Hypothesis: Assume theorem is true when $|T| \leq k$.

Let v be the $(k+1)^{\text{st}}$ node added using edge (u, v).

Let P_v be the path chosen by Dijkstra's to v and let P be any other path from s to v.

Then we have the situation on the next slide. □

Proof, cont.



The path to v chosen by Dijkstra's is of length $\leq$ the alternative blue path.

Shortest Paths $\implies$ Tree

Theorem. *There is some optimal set of shortest paths from source* s *such that their union forms a tree.*

Proof. Dijkstra's algorithm is correct and produces a tree.

□

Implementation of Dijkstra

```
for u ∈ V do dist[u] ← ∞
H ← MakeHeap( )
u ← s # (s is an arbitrary start vertex)
while u ≠ null do
    for v ∈ Neighbors(u) do
        # If the distance is smaller than before, we have to update
        if dist[u] + d(u,v) < dist[v] then
            dist[v] ← dist[u] + d(u,v)
            if v ∉ H then
                Insert(H, v, dist[v])
            else
                ReduceKey(H, v, dist[v]) # Sift up for new key
            parent[v] ← u
    u ← DeleteMin(H)
return parent
```

Running time of Dijkstra's Algorithm

Same as Prim's MST algorithm:

- Every edge is processed in the **for** loop at most once.
- In response to that processing, we may either
  1. do nothing; $O(1)$,
  2. insert a item into the heap of at most $|V|$ items; $O(\log|V|)$, or
  3. reduce the key of an item in a heap of at most $|V|$ items; $O(\log|V|)$
- Total time is therefore: $O(|E|\log|V|)$.