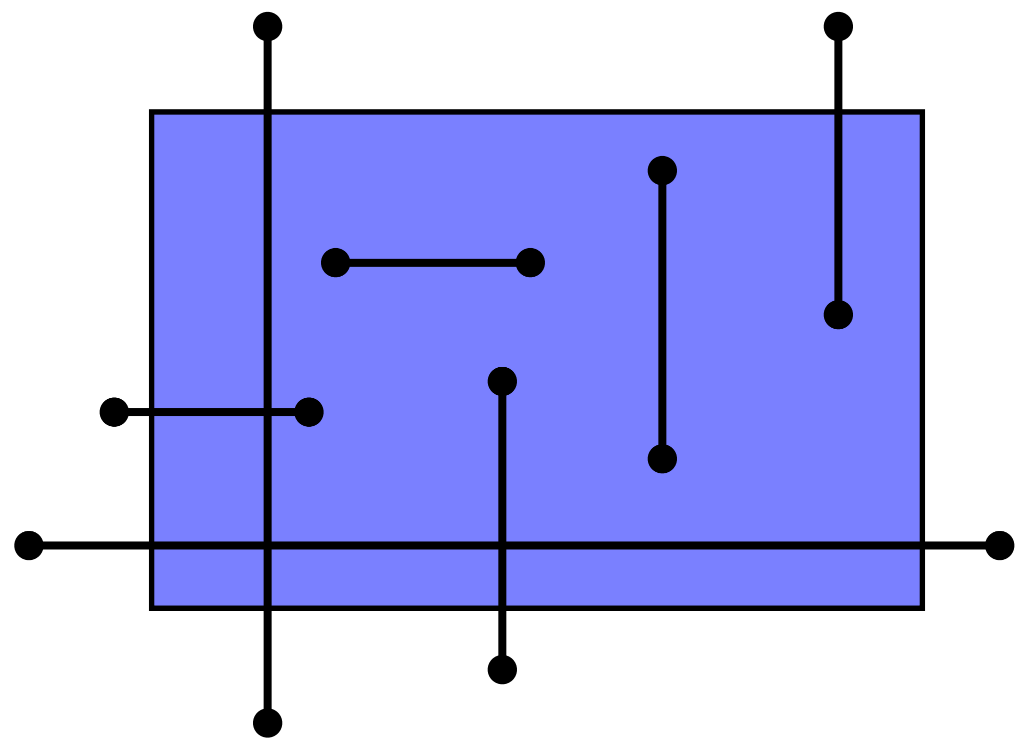
Interval Trees

Storing and Searching Intervals

- Instead of points, suppose you want to keep track of axis-aligned *segments:*

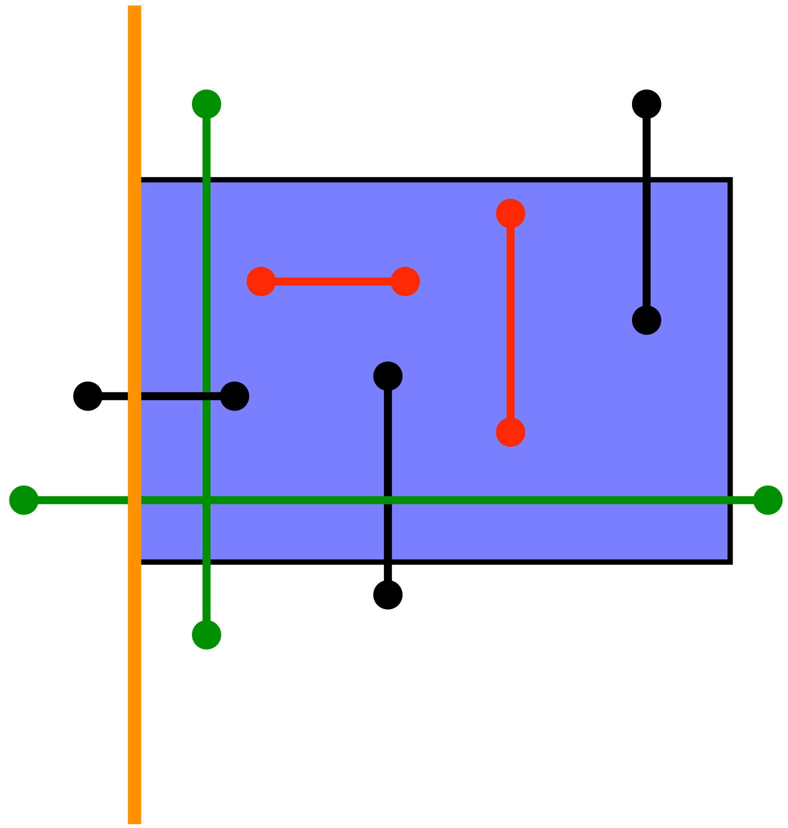


- Range queries: return all segments that have any part of them inside the rectangle.
- *Motivation:* wiring diagrams, genes on genomes

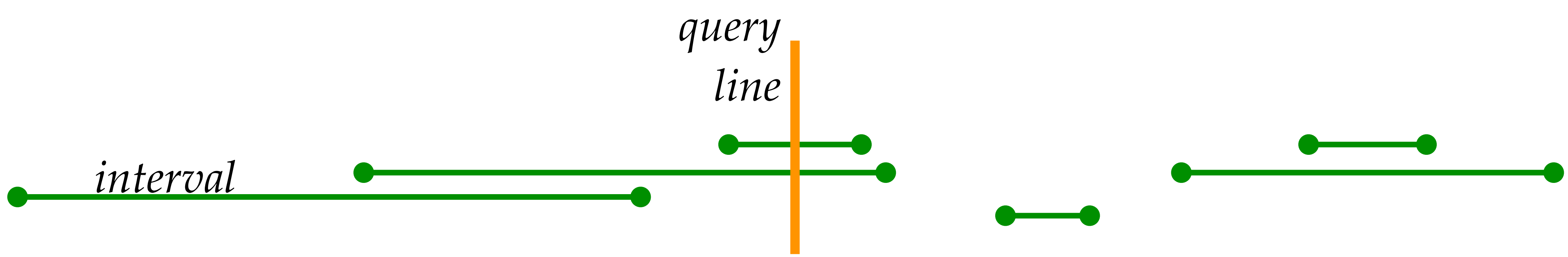
Simpler Problem: 1-d intervals

- Segments *with at least one* endpoint in the rectangle can be found by building a 2d range tree on the 2n endpoints.
  - Keep pointer from each endpoint stored in tree to the segments
  - Mark segments as you output them, so that you don't output contained segments twice.

- Segments with *no* endpoints in range are the harder part.
  - Consider just horizontal segments
  - They must cross a vertical side of the region
  - Leads to subproblem: Given a vertical line, find segments that it crosses.
  - (y-coords become irrelevant for this subproblem)



Interval Trees



Recursively build tree on interval set S as follows:

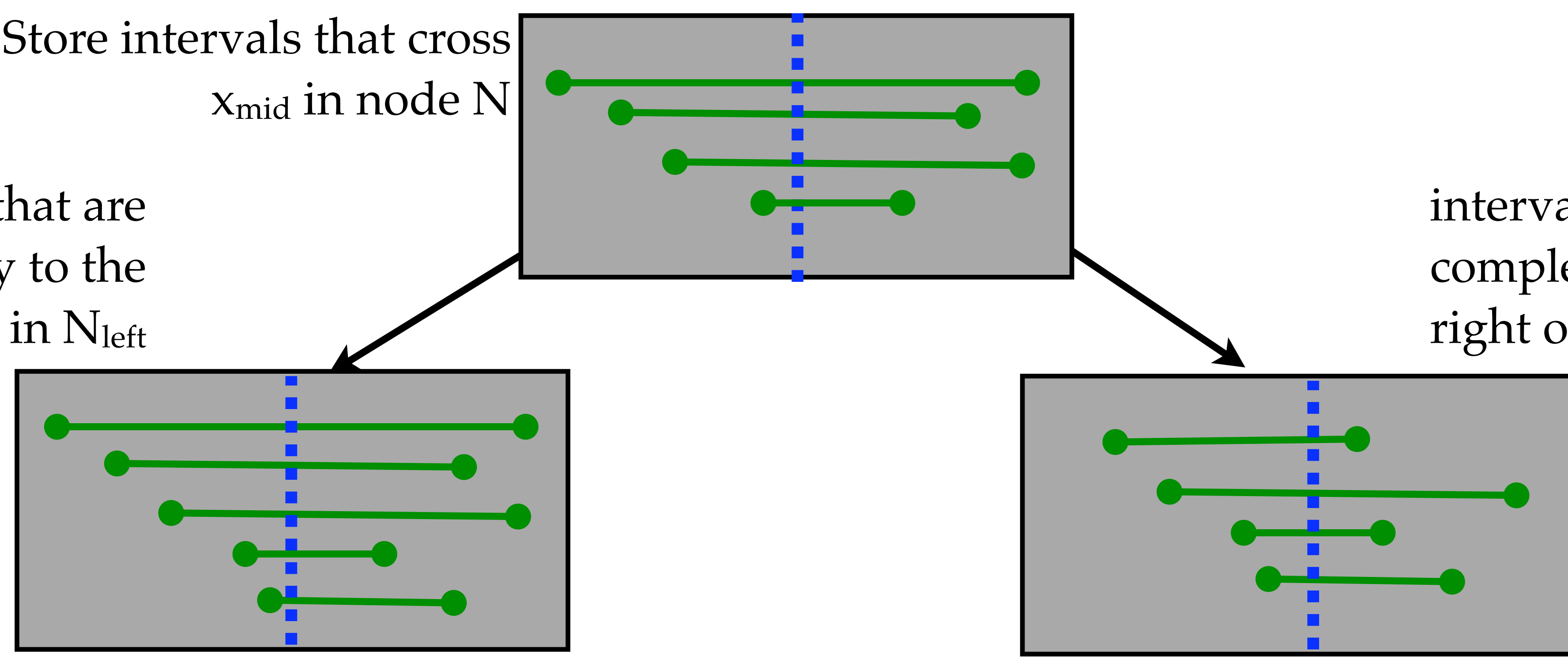
Sort the 2n endpoints

Let x_{mid} be the median point

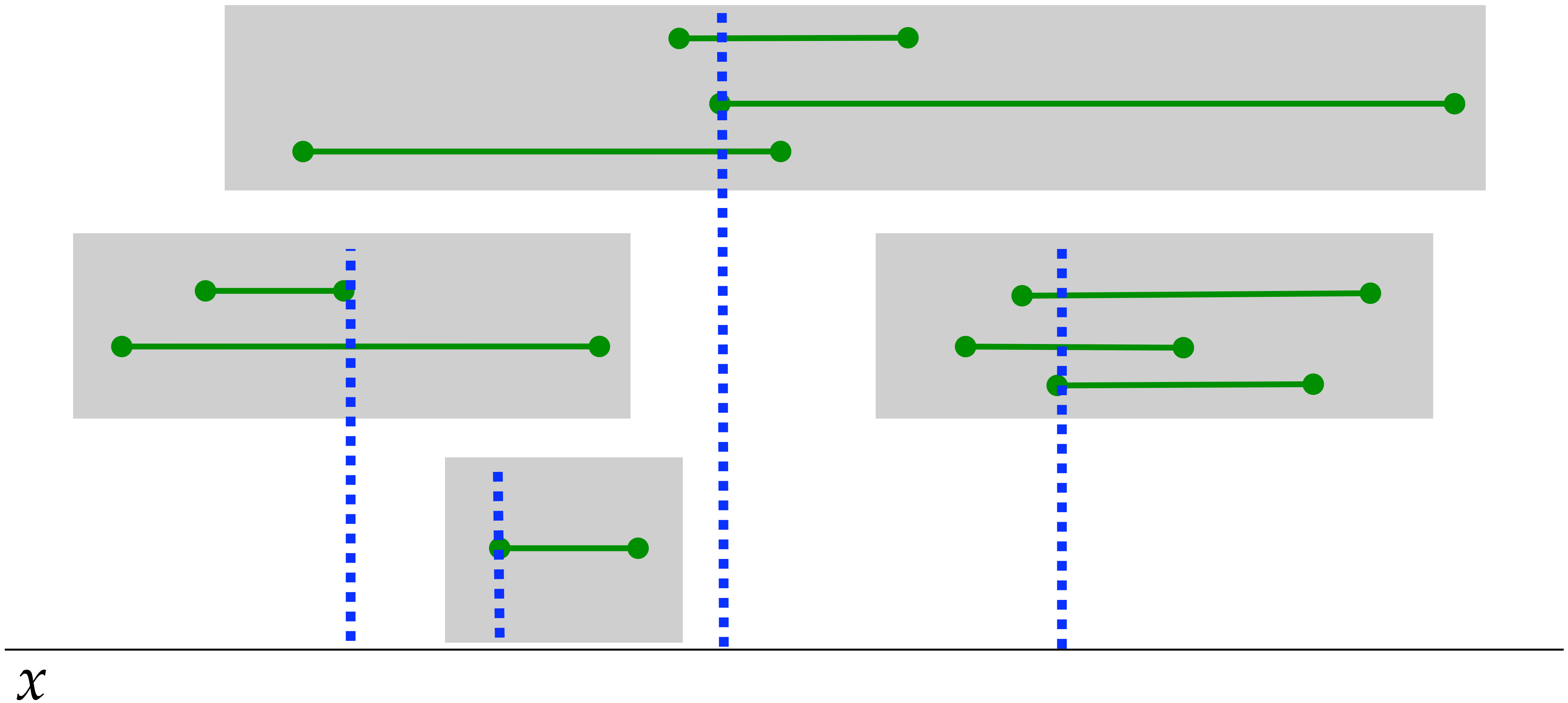
Store intervals that cross x_{mid} in node N

intervals that are completely to the left of x_{mid} in N_{left}

intervals that are completely to the right of x_{mid} in N_{right}



Another view of interval trees



Interval Trees, continued

- Will be approximately balanced because by choosing the median, we split the set of end points up in half each time
  - Depth is O(log n)

- Have to store x_{mid} with each node

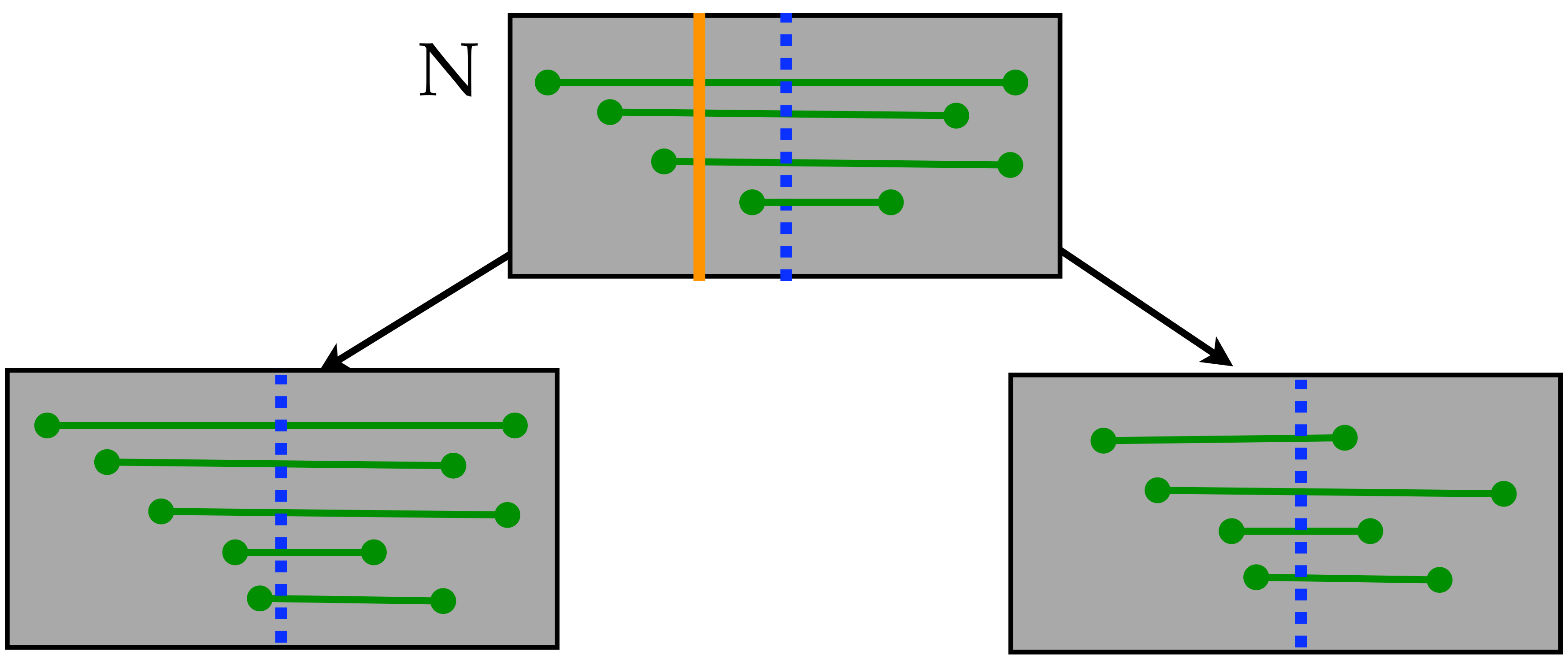
- Uses O(n) storage
  - each interval stored once, plus
  - fewer than n nodes (each node contains at least one interval)

- Can be built in O(n log n) time.

- Can be searched in O(log n + k) time
  [k = # intervals output]

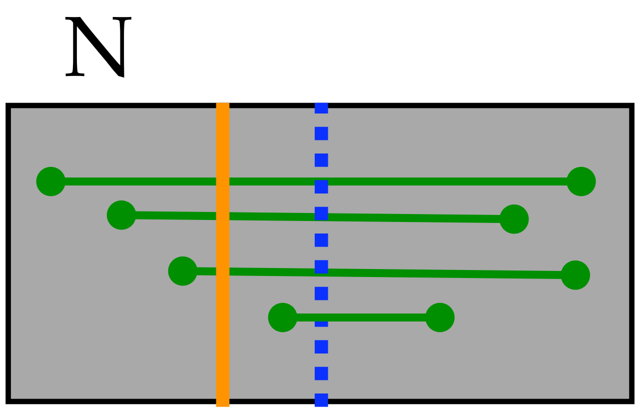
Interval Tree Searching

- Query: vertical line (aka x_q)
- Suppose we're at node N:
  - if $x_q < x_{med}$, then can eliminate right subtree
  - if $x_q \geq x_{med}$, then can eliminate left subtree
  - Always have to search the intervals stored at current node => leads to another trick (next slide)



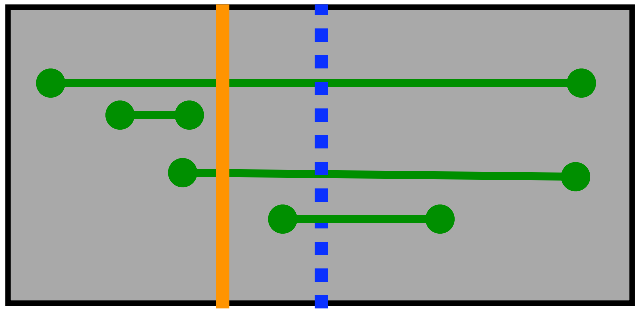
Searching intervals at current node

- Store each interval in *two* sorted lists stored at node:
  - List L sorted by increasing left endpoint
  - List R sorted by decreasing right endpoint



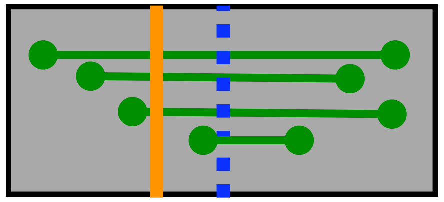
- Search list depending on which side of x_{med} the query is on:
  - If $x_q < x_{med}$ then search L, output all until you find a left endpoint $> x_q$.
  - If $x_q \geq x_{med}$ then search R, output all until you find a right endpoint $< x_q$.

- Only works because we know each segment intersects x_{med}.

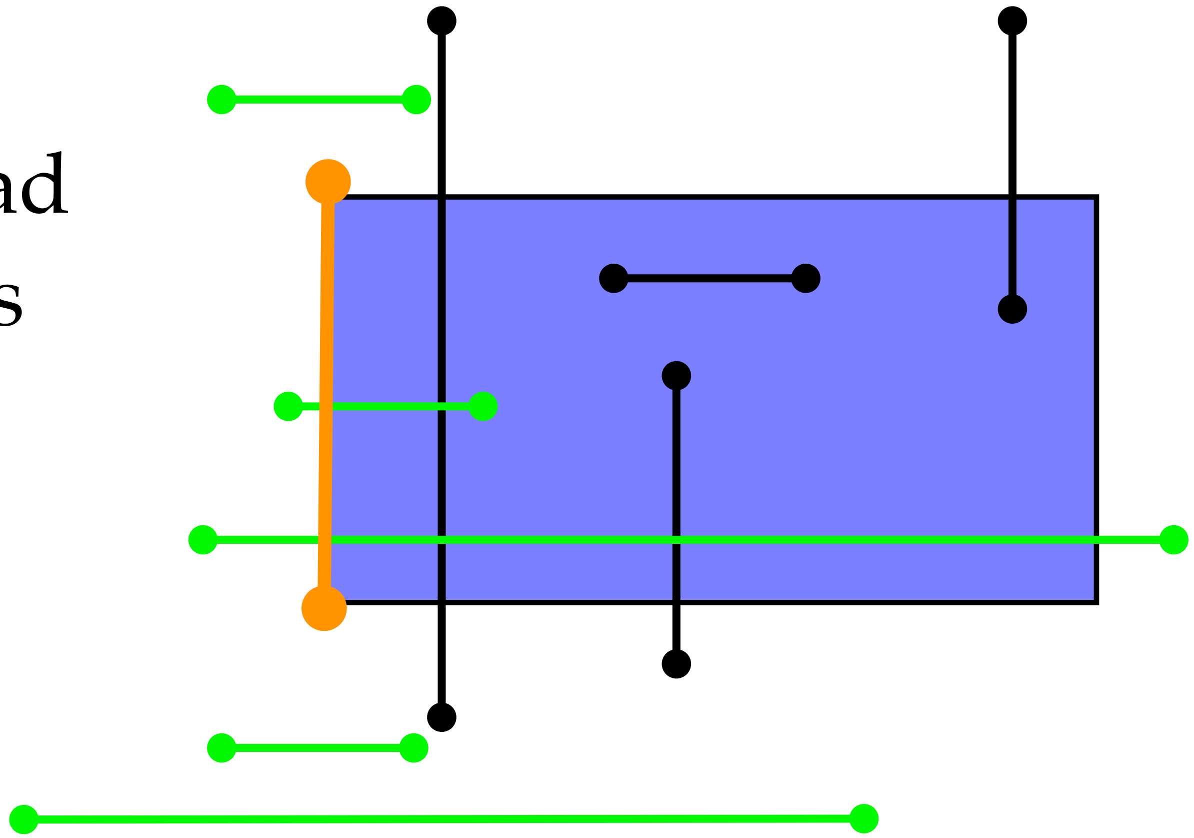


Vertical *SEGMENT* searching

- Instead of infinite vertical lines, we have finite segments as a query
- Start with same idea:
  - Interval trees => candidates
  - But somehow have to remove the ones that don't satisfy the y-constraints

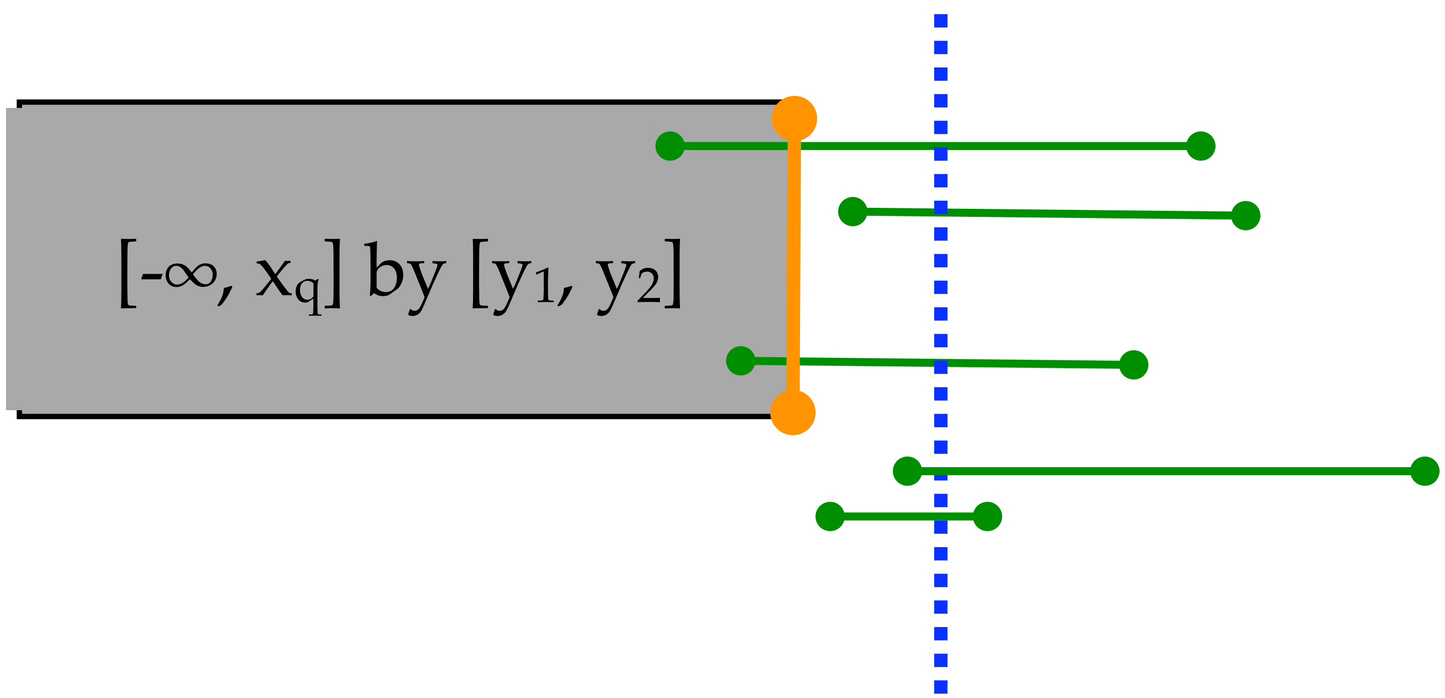


- **Idea**: use 2-d range trees instead of sorted lists to hold segments at each node



Vertical Segment Searching

- Consider the segments stored at a given node and a query segment:



- Execute a range query on a semi-infinite range on the 2d-range tree on the end points stored at each node of the interval tree.
  - optimization: keep two range trees R_{left} and R_{right} that store points to the left and to the right of x_{mid}.

Vertical Segment Queries: Runtime & Space

- Query time is $O(\log^2 n + k)$:
  - $\log n$ to walk down the interval tree.
  - At each node v have to do an $O(\log n + k_v)$ search on a range tree (assuming your range trees use fractional cascading)
- $O(n \log n)$ space:
  - each interval stored at one node.
  - Total space for set of range trees holding $\leq 2n$ items is $O(n \log n)$.
- Priority search trees reduce the storage to $O(n)$

Priority Search Trees

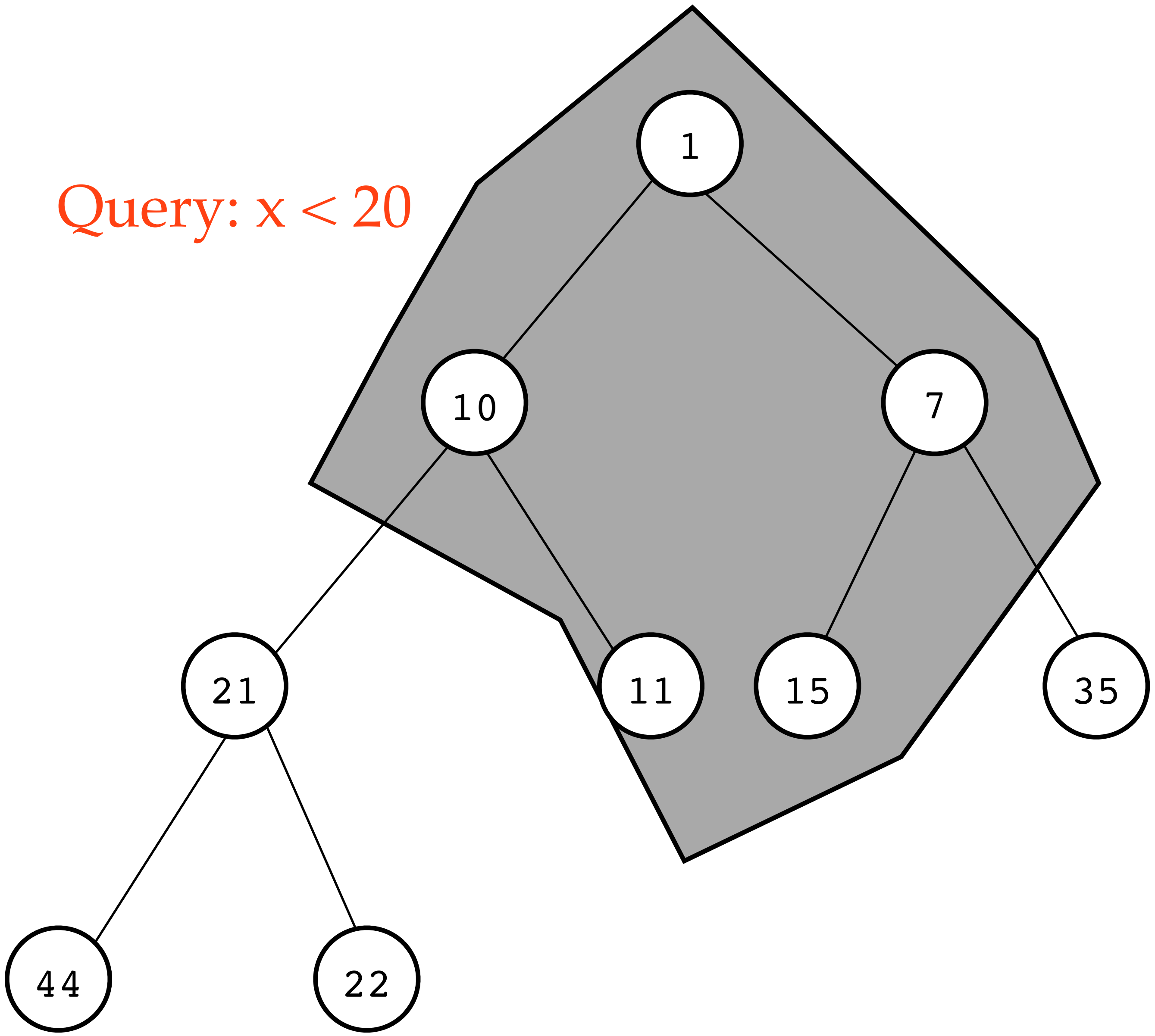
Handling queries that are unbounded on one side

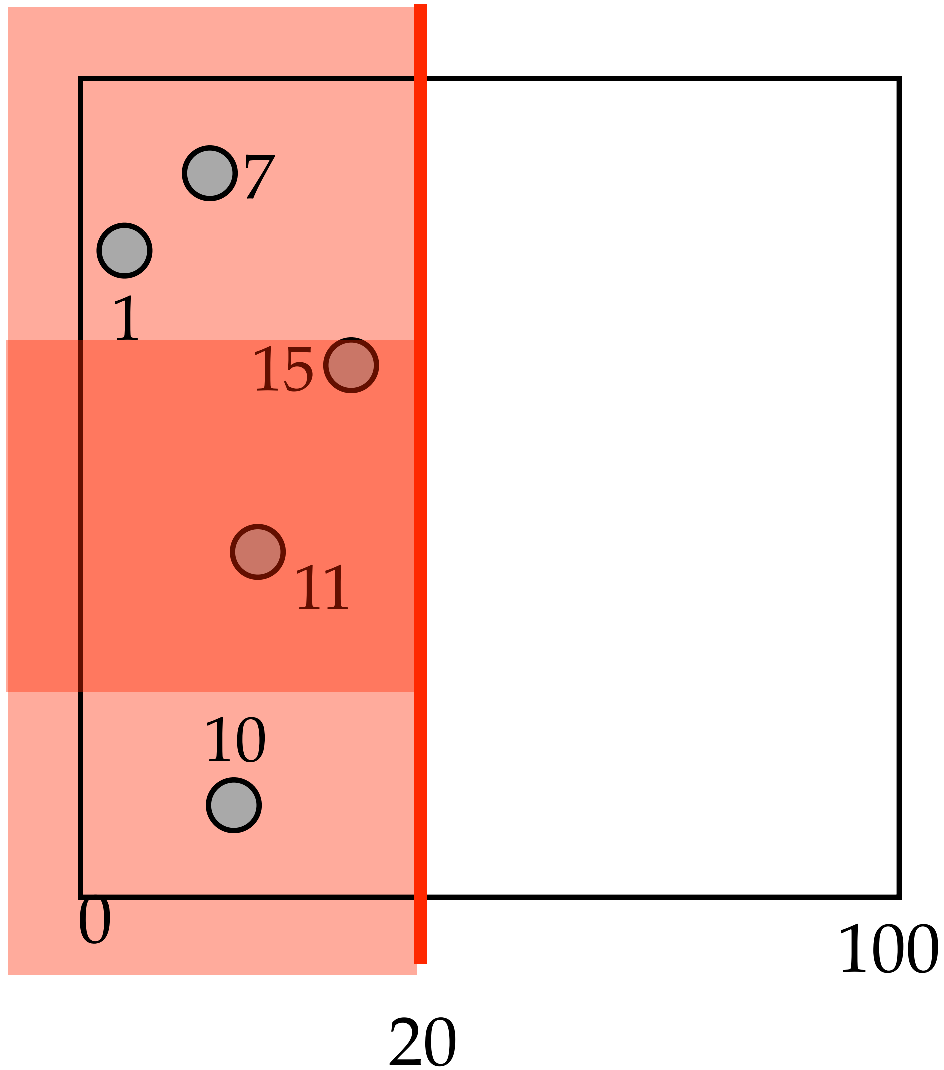
- Easy in the 1-d case:
  - just walk sorted list from left to right or right to left
- But then how long does an insert take?
  - Can we do better?

1-sided Range Queries in 1-d

Heap on x-values:

Query: x < 20





2-d case:

$x < 20$ **AND**

$25 < y < 70$

Any ideas?

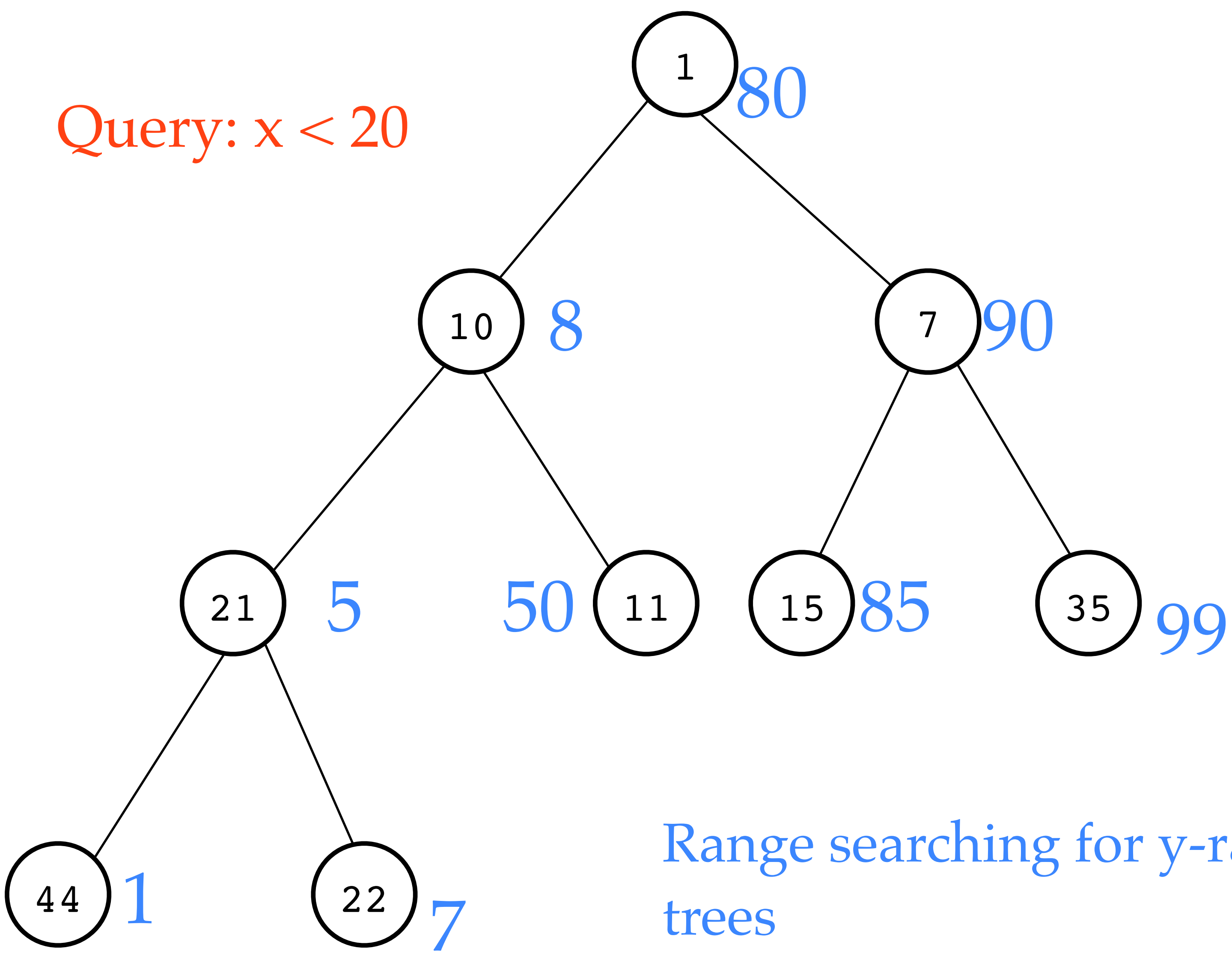
Unbounded range queries in 2d

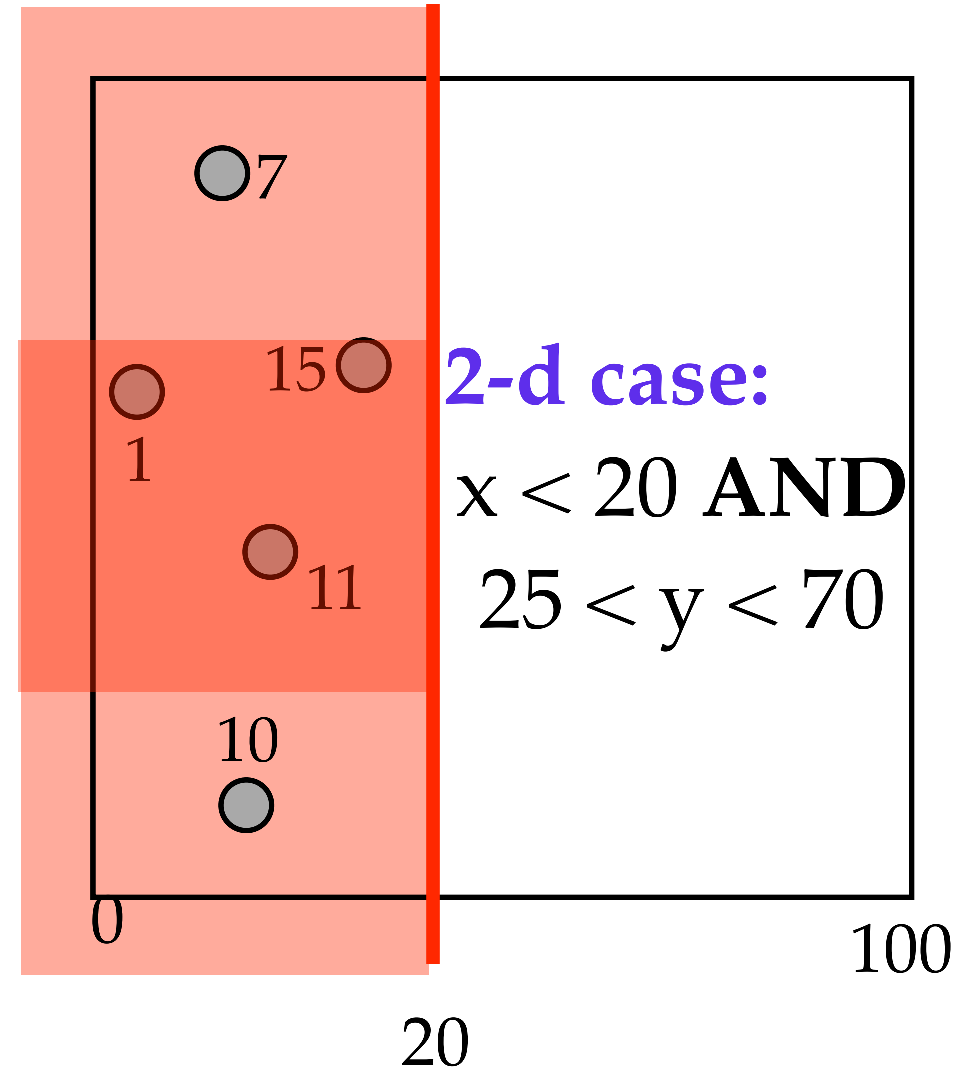
- In 2d-case:
  - Want to find points with *low* x-values
  - Within *a range* of y-values
- Idea:
  - Find low values ---> heap
  - 1-d range queries (on y-values) --> BST
- Combine them:
  - Priority Search Trees

1-sided Range Queries in 2-d

Heap on x-values:

Query: x < 20





2-d case:

x < 20 **AND**

25 < y < 70

Heap on the x-values
BST on the y-values

Range searching for y-range can be done as in 1-d range trees

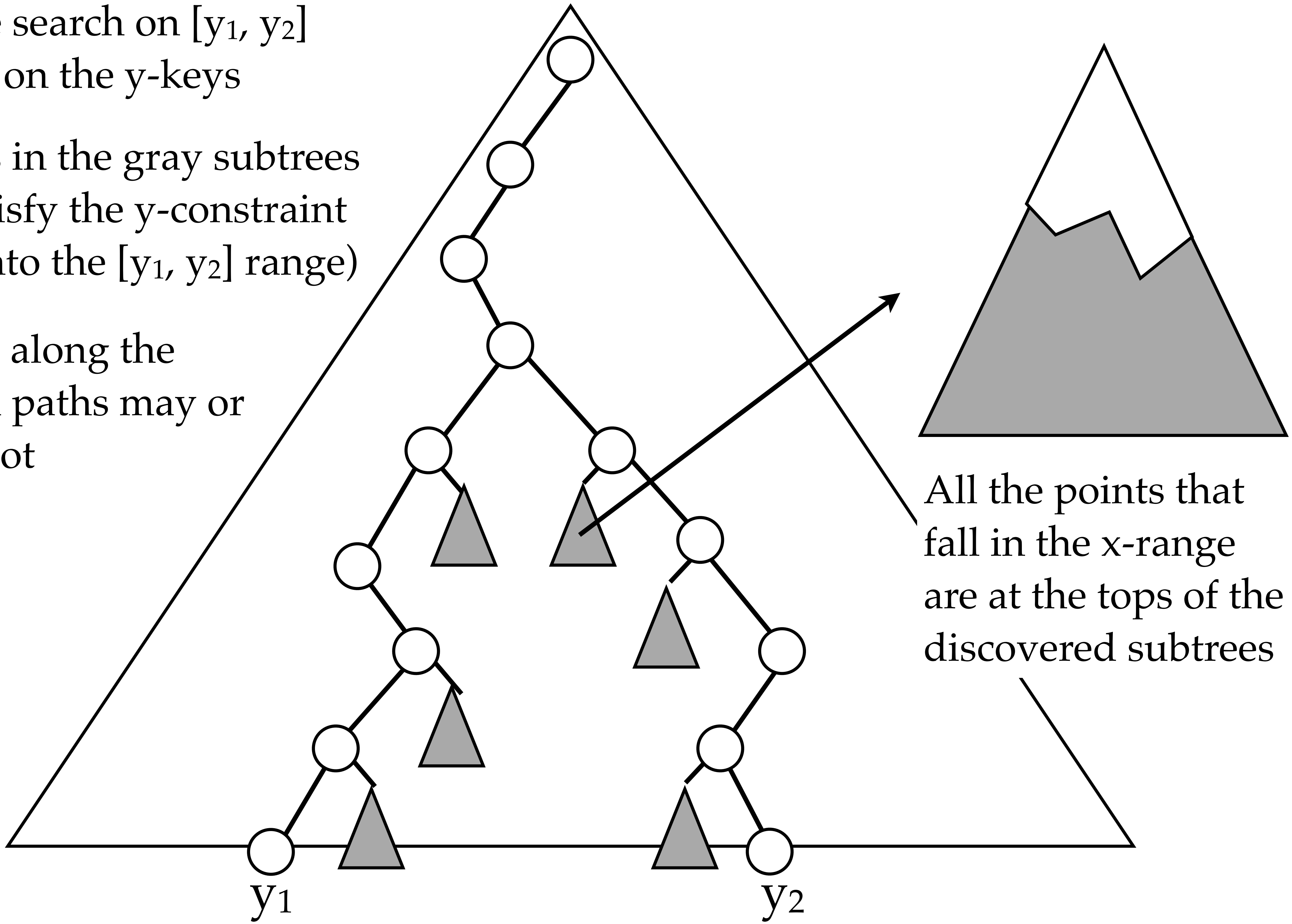
Then each of the subtrees found in that 1-d range search is a heap, so you just output the "top" of the heap.

2-d range queries with one unbounded side, cont.

Range search on $[y_1, y_2]$ based on the y-keys

Points in the gray subtrees all satisfy the y-constraint (fall into the $[y_1, y_2]$ range)

Points along the search paths may or may not



PST Searching:

- Query: $[-\infty, x]$ by $[y_1, y_2]$
- Range search on $[y_1,y_2]$
- Then output "tops" of each subtree between the paths found during the range search.
- Also, must check each node along both paths because they store points.

- Time: $O(\log n)$ to find trees + $O(k)$ to output their tops.
  - faster than the $O(\log^2 n + k)$ time required if you use range trees with fractional cascading
  - Also simpler

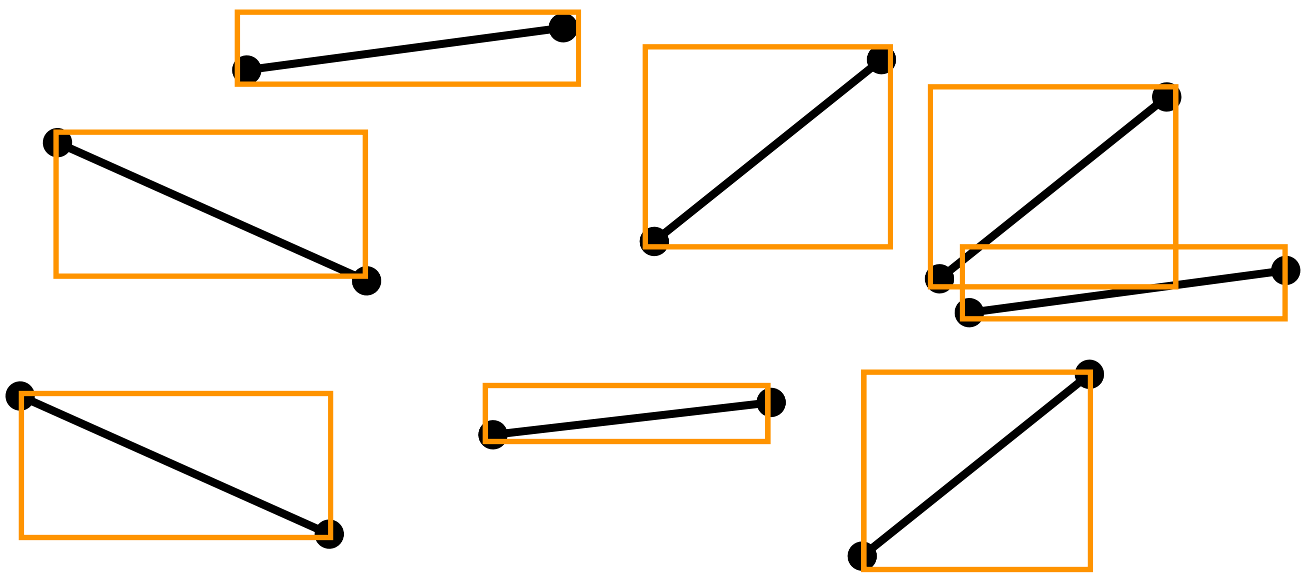
Recursive Definition of PST

- Given a set of points P, let
  - point P_{minx} = one with smallest x
  - y_{mid} = median of the y-coordinates of $P \setminus \{P_{minx}\}$
- Store point P_{minx} and y_{mid} in node a N.
  - note that y_{mid} need not correspond to point P_{minx}.
- Split the points up by y-coordinate:
  - $P_{left} = \{p \text{ in } P \setminus \{P_{minx}\} : p.y < y_{mid}\}$
  - $P_{right} = \{ p \text{ in } P \setminus \{P_{minx}\} : p.y \geq y_{mid} \}$
- Recursively built left and right subtrees of N on each of these children sets.
- => O(n log n) algorithm to build PST

Segment Trees

Arbitrarily Oriented Segments

- No longer assume that segments are parallel to the x- or y-axis.



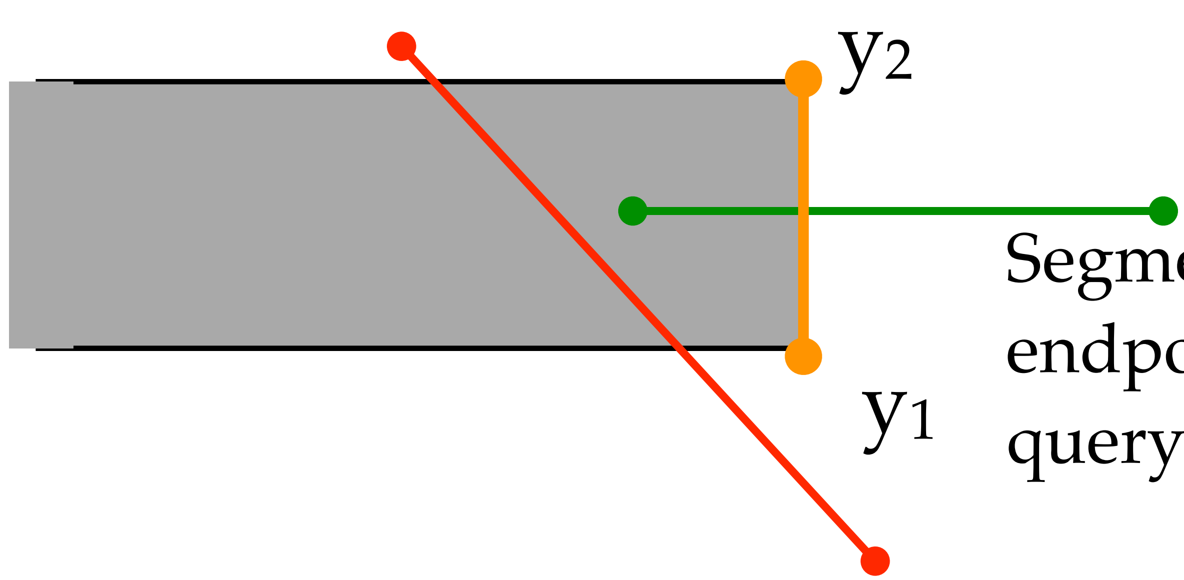
- One trick: store the bounding boxes of each segment as a collection of 4 axis-parallel segments.
  - Know how to handle range queries on these kinds of segments
  - If a vertical line crosses a segment, it crosses its bounding box (good)
  - It may be that a vertical line crosses a bounding box but doesn't cross the segment (bad)

- Interested in Vertical Segment Stabbing Queries:
- Return all segments that intersect a vertical query segment
- (Assume segments don't cross)

Why don't interval trees work?

Interval trees answer vertical segment stabbing queries for axis-parallel datasets, so why don't they work for slanted segments?

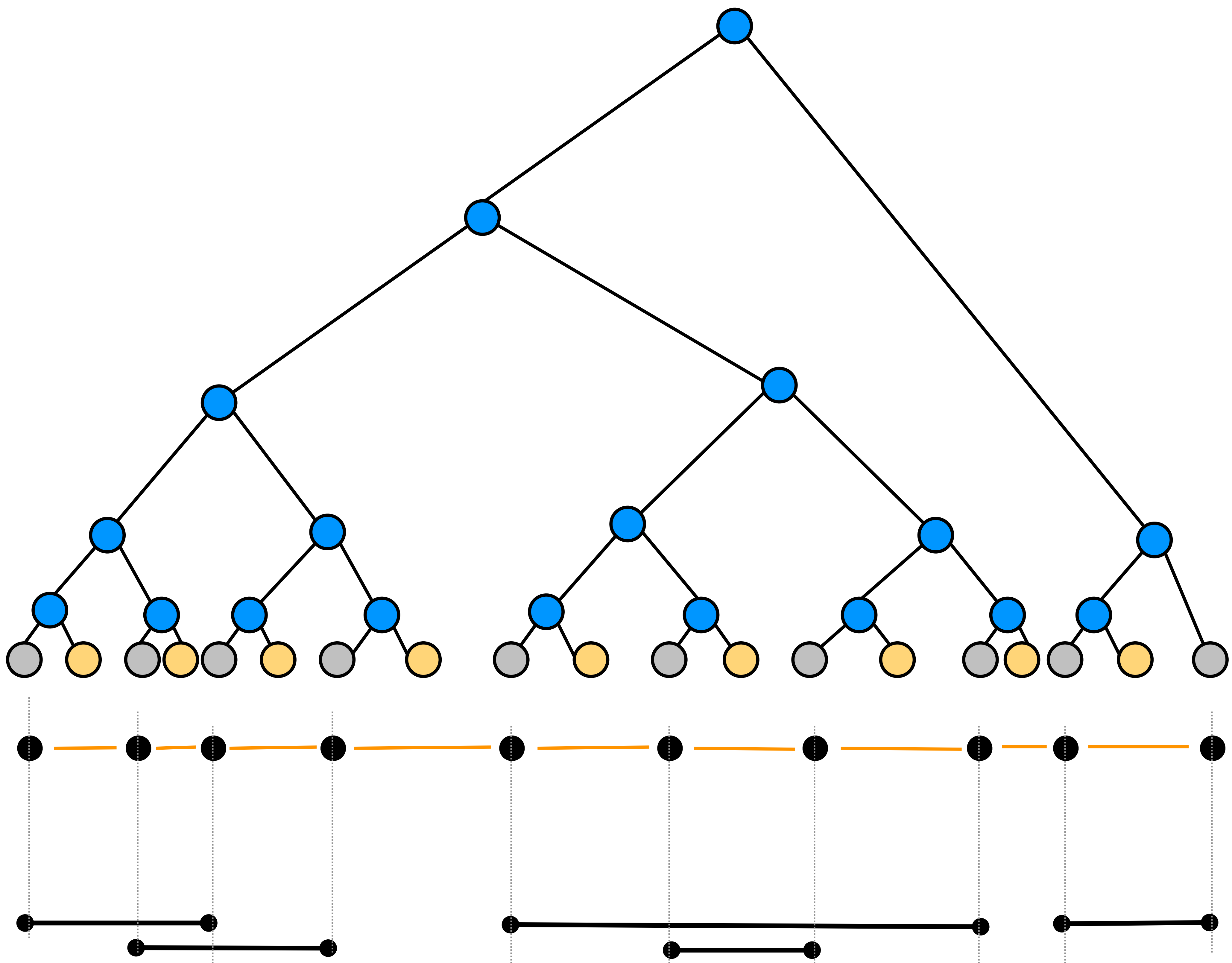
- No longer true that a query like $[-\infty, x]$ by $[y_1, y_2]$ will find the endpoints of satisfying segments:



Segment intersects range and endpoint falls in half-infinite query (as in interval trees)

Segment intersects query, but there are no endpoints in the range

Again, we consider 1-d case



Build a balanced BST on that *partitioning*

Partitioning = open intervals & endpoints

Induce a partitioning of the line

1-d segments

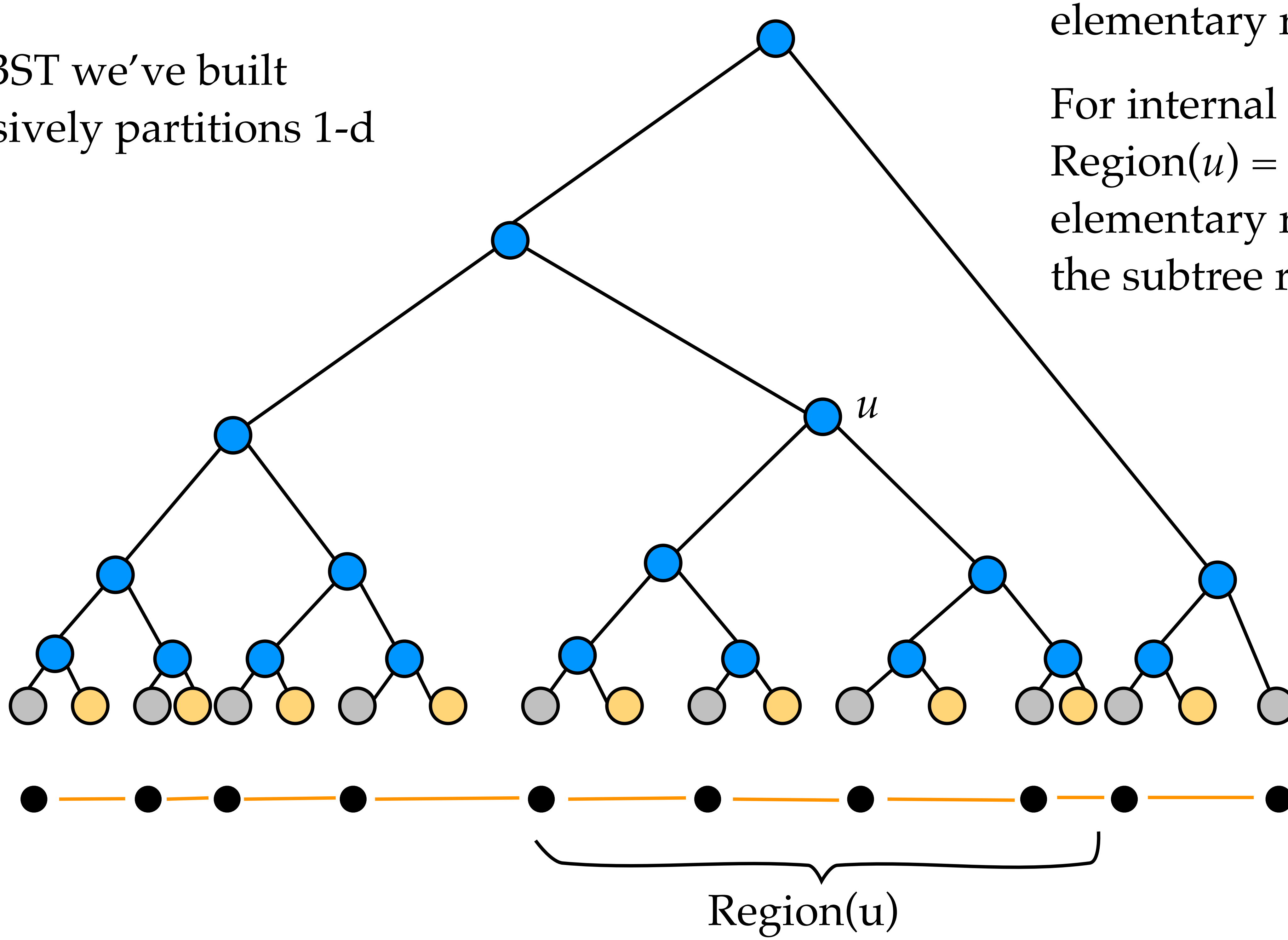
Segment Trees

Forget for a moment the segments we're trying to store.

This BST we've built recursively partitions 1-d space

Leaves store an elementary region

For internal node u, Region(u) = union of elementary regions in the subtree rooted at u.



So,

- We've divided up space into a set of basic "building-block" units.

- Subdivision of space is customized to our needs:
  - Every segment we want to store is the union of some set of these basic building block units (elementary regions)

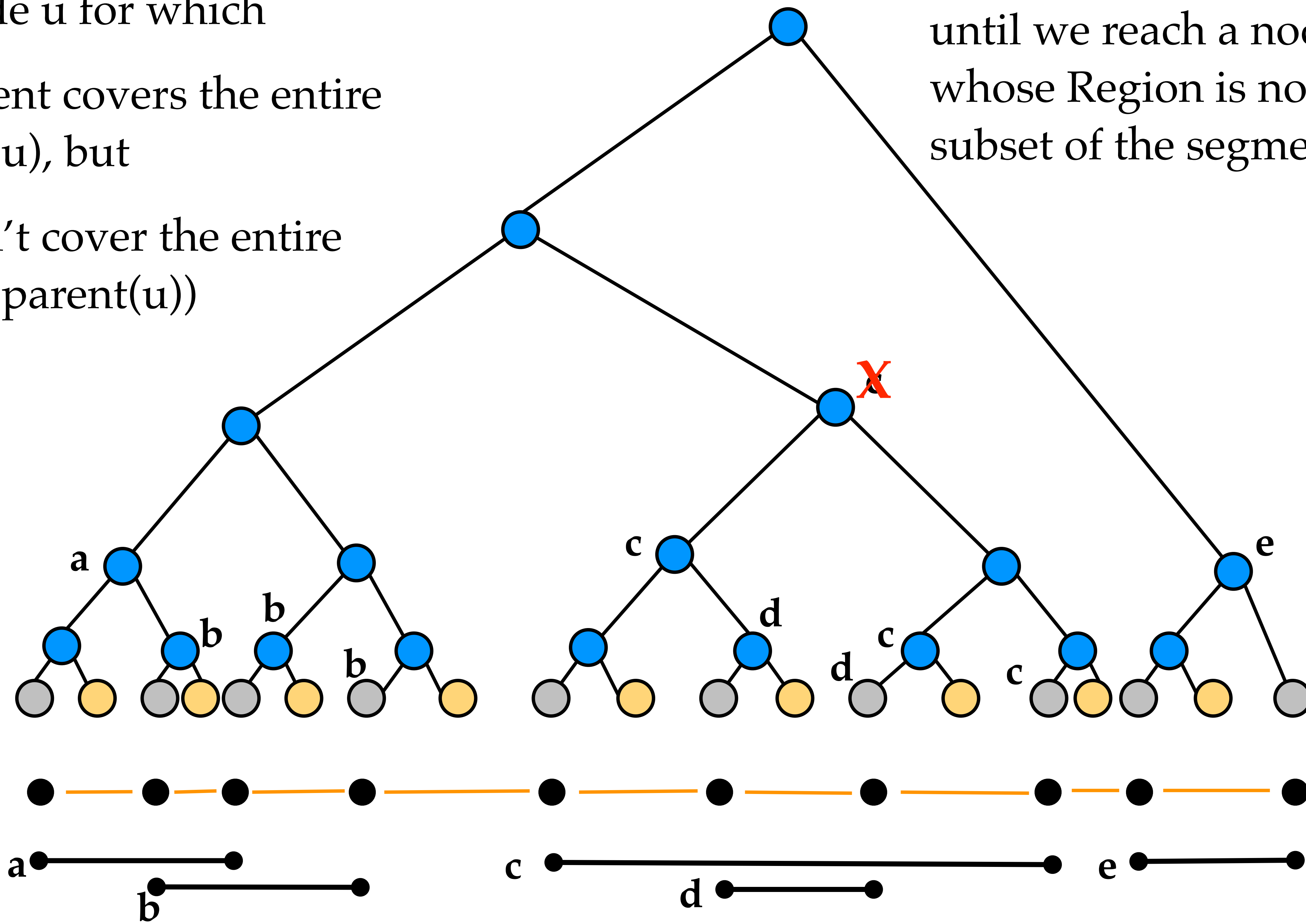
- How do we store the actual set of intervals?

Where to store segments

Rule: store segment s at any node u for which

- segment covers the entire Region(u), but
- doesn't cover the entire Region(parent(u))

(in other words, we propagate segments up until we reach a node whose Region is not a subset of the segment)



Space usage:

- Segments may be stored at several nodes, but...
- Each segment is stored at most twice at each level
  - if it where stored 3 times, there would be a parent should contain it
  - contradicts that intervals are not stored at both a child and its parent
- O(log n) height because tree is balanced.
- Therefore: O(n log n) total space.

Searching with vertical line queries

- Find segments that intersect a given x.
  - Binary Search traversal of tree
  - At each step: Output *every* segment stored at the current node *u* (x must intersect them all because they all span Region(*u*))
  - Note that Region(u) = Region(leftchild(u)) UNION Region(rightchild(u)).
  - If x falls into Region(leftchild(u)), take the left branch
  - If x falls into Region(rightchild(u)), take the right branch

- O(log n + k) time: follow a path of O(log n) nodes down to a leaf. Output all k segments encountered along the way.

Segment Tree Construction

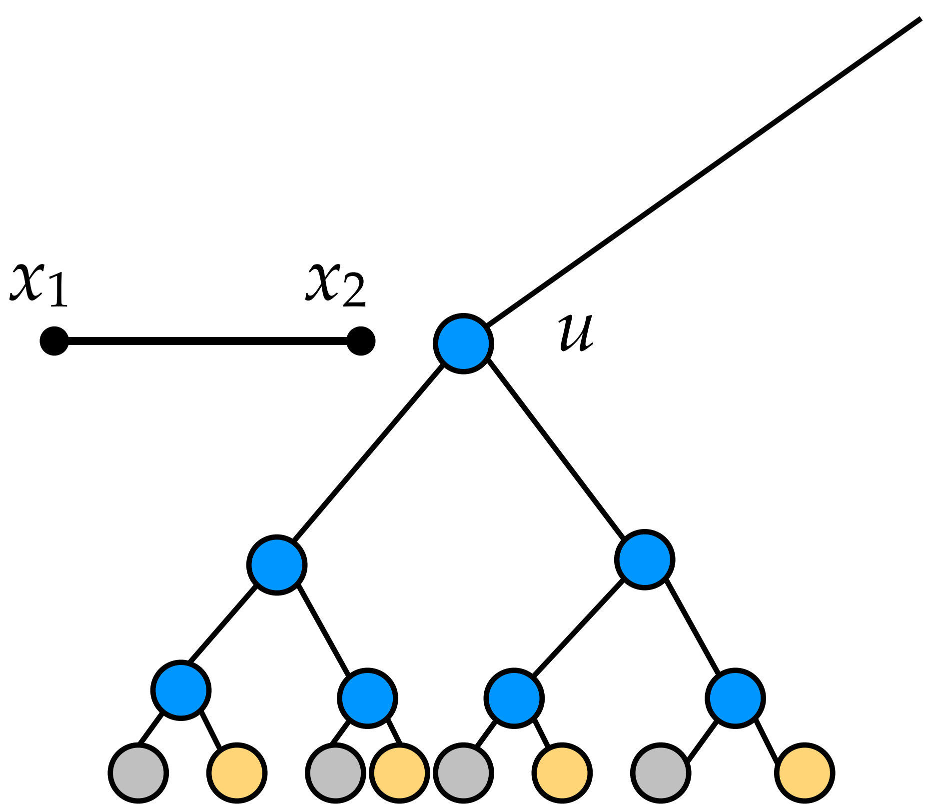
- Build the tree:
  - Sort segments
  - Break into elementary building blocks
  - Building balanced BST on these building blocks
- For every segment to insert:

```
def InsertSegment(u, x1, x2):
   // if the interval spans the region represented by u
   // store it in the linked list "segs"
   if Region(u) subset of [x1, x2]:
      u.segs.append(x1, x2)
   else:
      // otherwise, walk down both subtrees
      if [x1,x2] intersects Region(u.left):
         InsertSegment(u.left, x1, x2)
      if [x1,x2] intersects Region(u.right):
         InsertSegment(u.right, x1, x2)
```

Why is construction O(n log n)?

If we visit node u while inserting, one of 3 things happen:

- interval spans Region(u) [≤ 2 nodes / level]
- Region(u) contains x_1 [≤ 1 node / level]
- Region(u) contains x_2 [≤ 1 node / level]



Therefore, ≤ 4 nodes visited per level => O(log n) nodes visited on each segment insert

Segment Trees vs. Interval Trees

- Storage:
  - Interval trees: O(n)
  - Segment trees: O(n log n)

- Construction:
  - Interval trees: O(n log n)
  - Segment trees: O(n log n)

- Vertical line queries:
  - Interval trees: O(log n + k)
  - Segment trees: O(log n + k)

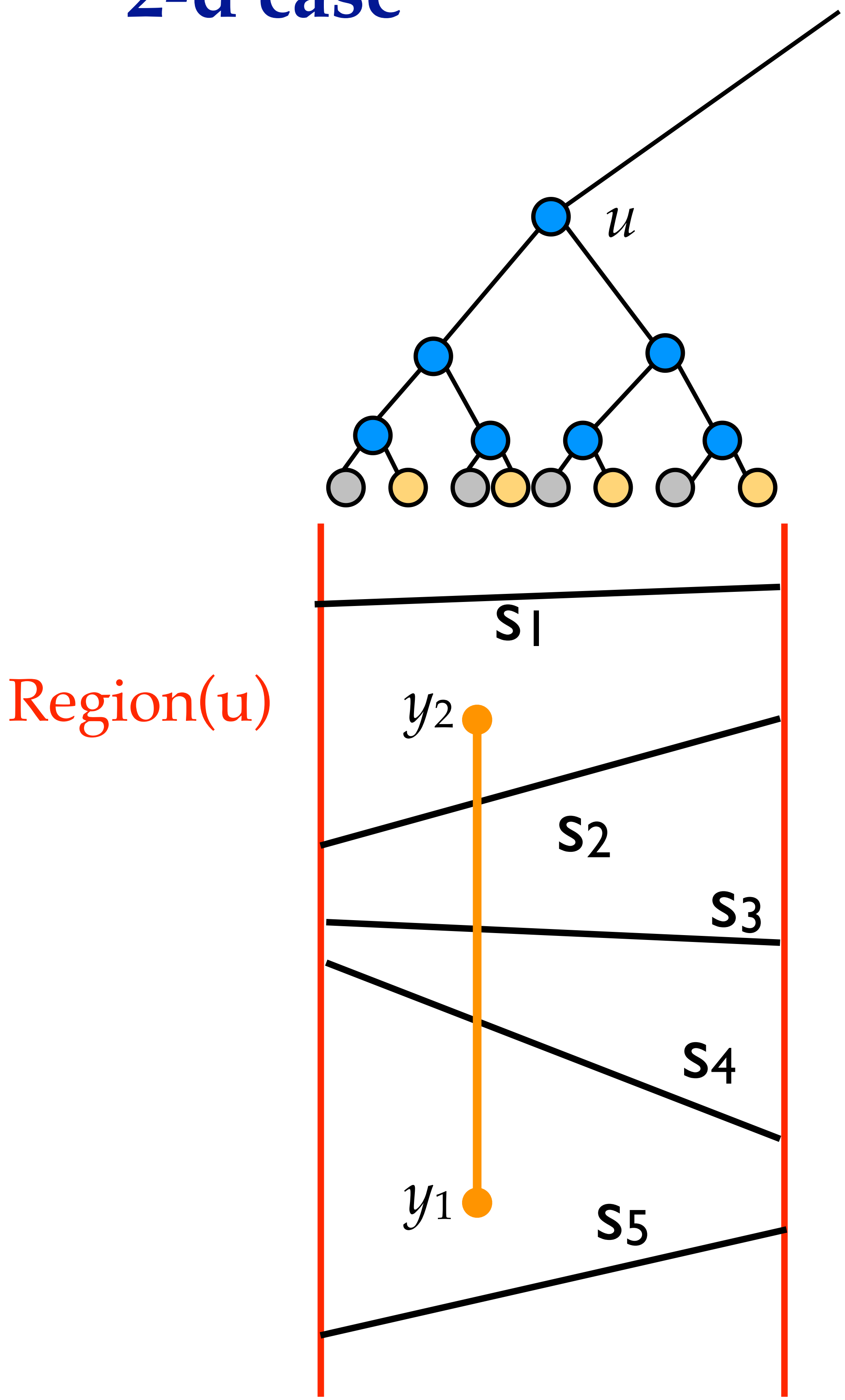
So why are segment trees interesting?

- Partition the space in a application specific manner

- *All* intervals encountered will be output

  So: instead of using aux data structure to find subset of intervals to output, we can use it for other things.)

2-d case



Segments stored at u all span Region(u) by definition.

Because we assume segments don't overlap, they can be linearly ordered from top to bottom

So, store segments in BST (aka 1-d range tree) sorted by this ordering.

Do a range search for those segments that are below y_2 and above y_1.