Nonholonomic Distance to Polygonal Obstacles for a Car-Like Robot of Polygonal Shape

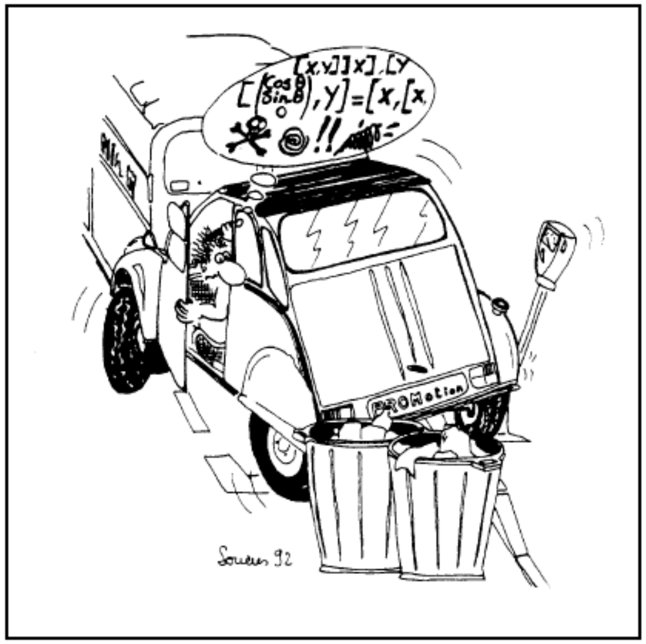
Marilena Vendittelli

joint work with

Paolo Robuffo Giordano — Università di Roma "La Sapienza"

Jean-Paul Laumond
Philippe Souères — LAAS - CNRS, Toulouse

…otherwise known as….



The Optimal Car Crashing Problem

Outline

- Introduction
- Shortest paths for the Reeds and Shepp car
- Problem decomposition
- Distance definition
- Necessary conditions for optimality
- Family refinement

Introduction

motivation

- motion planning algorithms rely on the obstacles distance computation; the more accurate the distance, the more efficient the planning
- euclidean distance is not appropriate for nonholonomic robots
- Reeds & Shepp shortest paths induce a metric in the configuration space [Laumond and Souères, 1993]

definition

- the distance from a car-like robot configuration to an obstacle is the length of the shortest feasible path bringing one point on the robot boundary in contact with the obstacle

related work

- sufficient family of shortest feasible paths linking two configurations

  [Reeds and Shepp, 1990; Sussmann and Tang, 1991; Boissonnat *et al*, 1992; Souères and Laumond, 1996]

- shortest paths to polygonal obstacles for a point robot [Vendittelli and Laumond, 1996]

- shortest paths to polygonal obstacles for a polygonal robot [Mirtich *et al*, 1996; Vendittelli *et al*, 1999]

contribution

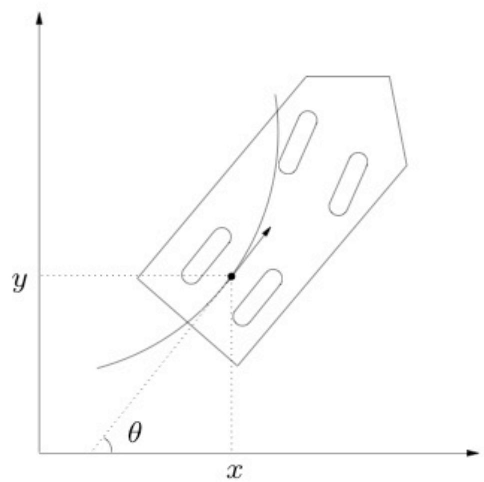
- characterization of the shortest paths to polygonal obstacles for a car-like robot of polygonal shape
- definition of an algorithm for distance computation

tool

- Pontryagin Maximum Principle + transversality conditions

Shortest paths for the Reeds & Shepp car

RS model



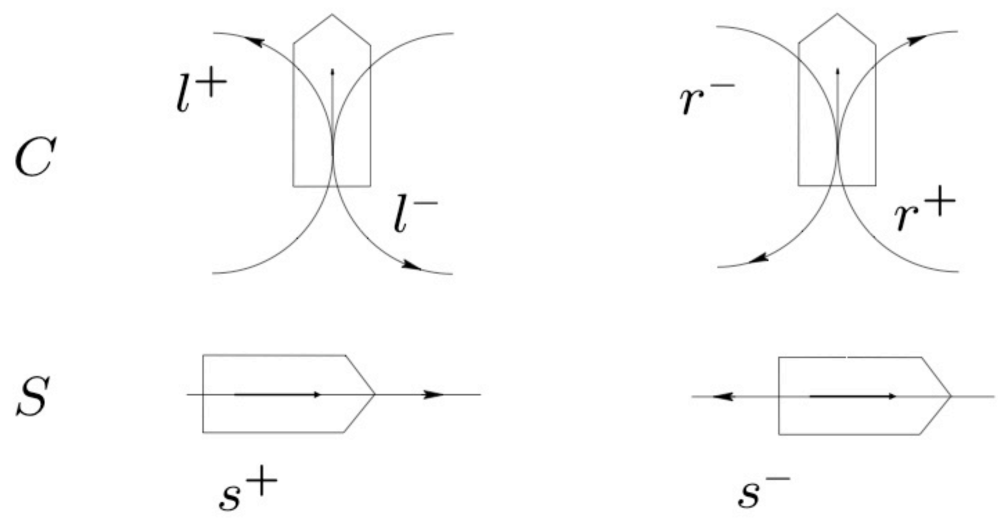
$$\dot{\xi} = f(\xi, u) = g_1(\xi)u_1 + g_2(\xi)u_2$$

$$\xi(t) = (x(t), y(t), \theta(t))$$

$$g_1(\xi) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \qquad g_2(\xi) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$|u_1(t)| = 1 \qquad |u_2(t)| \leq 1$$

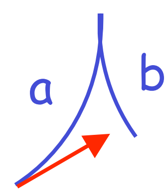
notation



C, S elementary path types

$l_a^{+(-)}$, $r_a^{+(-)}$, $s_a^{+(-)}$ elementary paths of length a

example: path $l_a^+ l_b^-$



is of type C|C

cost functional

$$J = \int_{t_i}^{t_f} \mathrm{d}t$$

Hamiltonian

$$H = \langle \psi, f \rangle = \psi_1 \cos\theta u_1 + \psi_2 \sin\theta u_1 + \psi_3 u_2$$

with

$$\dot{\psi}(t) = -\frac{\partial H}{\partial \xi}(\psi(t), \xi(t), u(t))$$

PMP: consider an admissible u(t) and the corresponding trajectory ξ(t); a necessary condition for ξ(t) to be optimal is that there exist a nontrivial ψ(t) and a constant $\psi_0 \leq 0$ s.t.

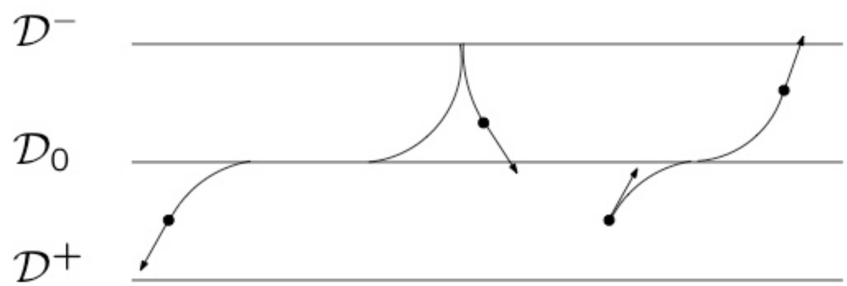
$$\begin{aligned} -\psi_0 &= \langle \psi(t), g_1(\xi(t)) \rangle u_1 + \langle \psi(t), g_2(\xi(t)) \rangle u_2 \\ &= \max_{v=(v_1,v_2)\in U} (\langle \psi(t), g_1(\xi(t)) \rangle v_1 + \langle \psi(t), g_2(\xi(t)) \rangle v_2) \end{aligned}$$

$t \in [t_i, t_f]$

the sufficient family of optimal paths contains

A paths C|C|…|C with length(C) ≤ π for any C

B paths lying between two parallel lines $\mathcal{D}^+$ and $\mathcal{D}^-$



straight line segments and inflection points are on $\mathcal{D}_0$

cusps lie perpendicularly on $\mathcal{D}^+$ and $\mathcal{D}^-$

with

$$\begin{aligned}
\mathcal{D}_0 : &\quad \psi_1 y(t) - \psi_2 x(t) + \psi_3(t_0) = 0 \\
\mathcal{D}^+ : &\quad \psi_1 y(t) - \psi_2 x(t) + \psi_3(t_0) + \psi_0 = 0 \\
\mathcal{D}^- : &\quad \psi_1 y(t) - \psi_2 x(t) + \psi_3(t_0) - \psi_0 = 0
\end{aligned}$$

- ψ_1 and ψ_2 constant
- $\psi_1/\psi_2 =$ slope of $\mathcal{D}^+$, $\mathcal{D}^-$ and $\mathcal{D}_0$
- $\psi_3(t) = \psi_1 y(t) - \psi_2 x(t) + \psi_3(t_0) \qquad (\Rightarrow \mathcal{D}_0 : \psi_3(t) = 0)$

the sufficient family $\mathcal{F}$ of optimal paths

(I) $l_a^+ l_b^- l_e^+$ or $r_a^+ r_b^- r_e^+$

$0 \le a \le \pi,\ 0 \le b \le \pi,\ 0 \le e \le \pi$

$(II)(III)$ $C_a | C_b C_e$ or $C_a C_b | C_e$

$0 \le a \le b,\ 0 \le e \le b,\ 0 \le b \le \pi/2$

(IV) $C_a C_b | C_b C_e$

$0 \le a \le b,\ 0 \le e \le b, 0 < b \le \pi/2$

(V) $C_a | C_b C_b | C_e$

$0 \le a \le b,\ 0 \le e \le b, 0 < b \le \pi/2$

a, b, e: RS path parameters

(VI) $C_a | C_{\pi/2} S_b C_{\pi/2} | C_e$

$0 \le a \le \pi/2,\ 0 \le b,\ 0 \le e < \pi/2$

$(VII)(VIII)$ $C_a | C_{\pi/2} S_b C_e$ or $C_a S_b C_{\pi/2} | C_e$

$0 \le a \le \pi,\ 0 \le b,\ 0 \le e \le \pi/2$

(IX) $C_a S_b C_e$

$0 \le a \le \pi/2,\ 0 \le b,\ 0 \le e \le \pi/2$

to every path p_i is associated a smooth function

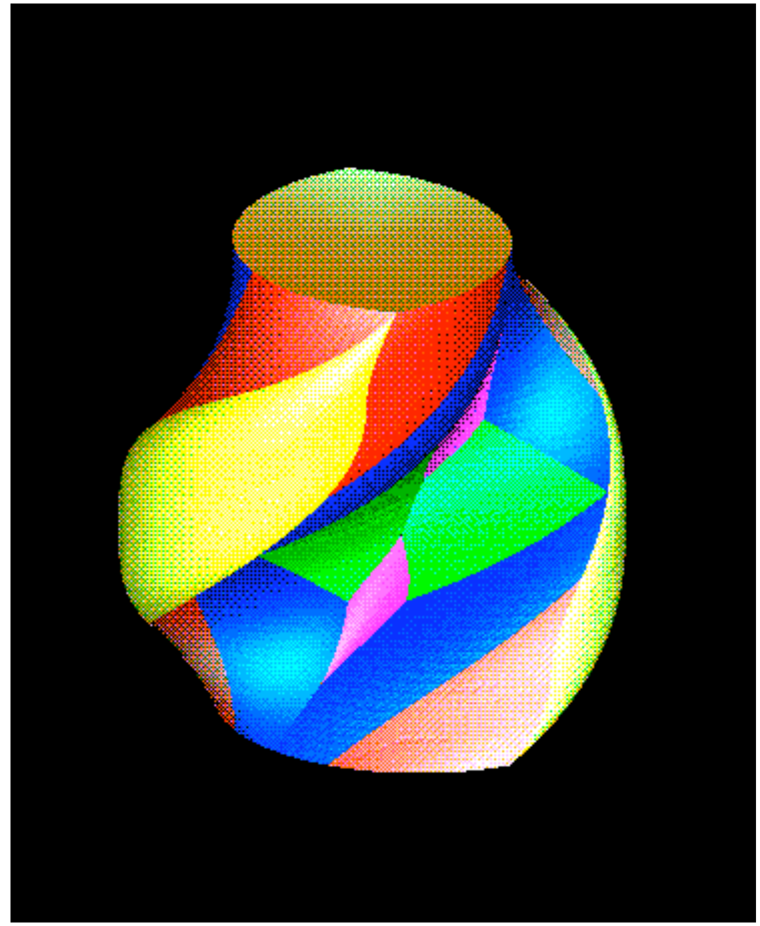
$$\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} = W_i(a,\, b,\, e) = \begin{pmatrix} X_i(a,\, b,\, e) \\ Y_i(a,\, b,\, e) \\ \Theta_i(a,\, b,\, e) \end{pmatrix}$$

mapping the three RS parameters into the configuration space
[Mirtich *et al*, 1996]

- it is possible to build a "sphere" (the RS ball) of radius ℓ around a given robot configuration ξ_0 [Laumond and Souères, 1993]

- points on the sphere are reachable by paths of length ℓ

- points inside the sphere are closer than ℓ to ξ_0



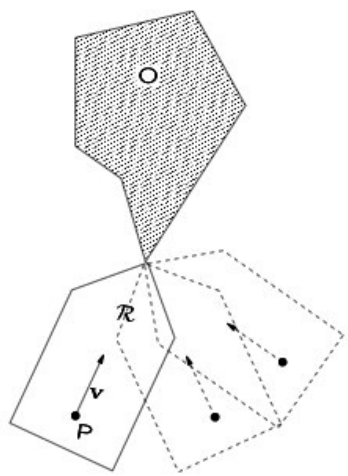
Problem decomposition

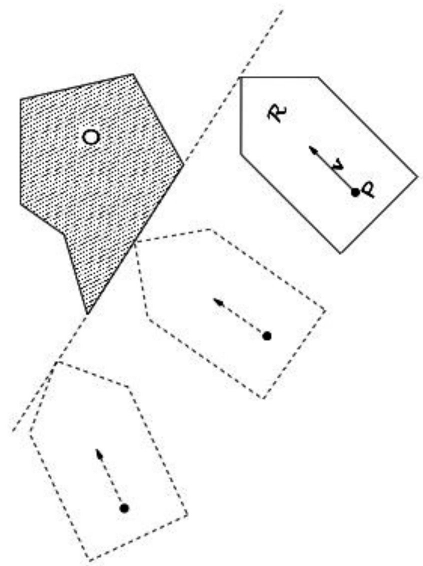
it is sufficient to solve the three subproblems of bringing in contact

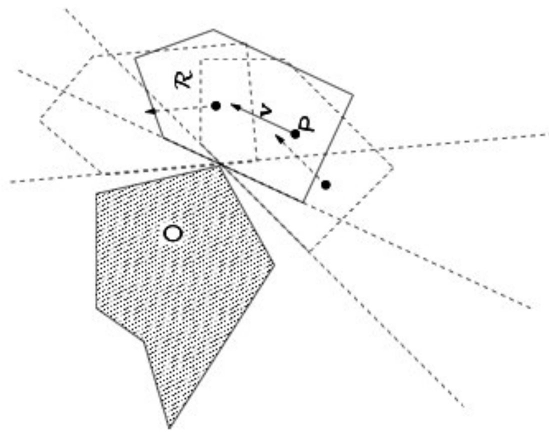
a robot vertex with
an obstacle vertex
(VV)

a robot vertex with
the line supporting
an obstacle edge
(VE)

the line supporting
a robot edge with
an obstacle vertex
(EV)







idea

- a "contact manifold" can be associated to each sub problem and each vertex/edge of the robot/obstacles

$$M_c(x,y,\theta)=\{(x,y,\theta)\,|\,\text{there is a VV, VE, EV contact}\}$$

- the distance is determined by the RS ball of minimum radius tangent to one of the M_c's

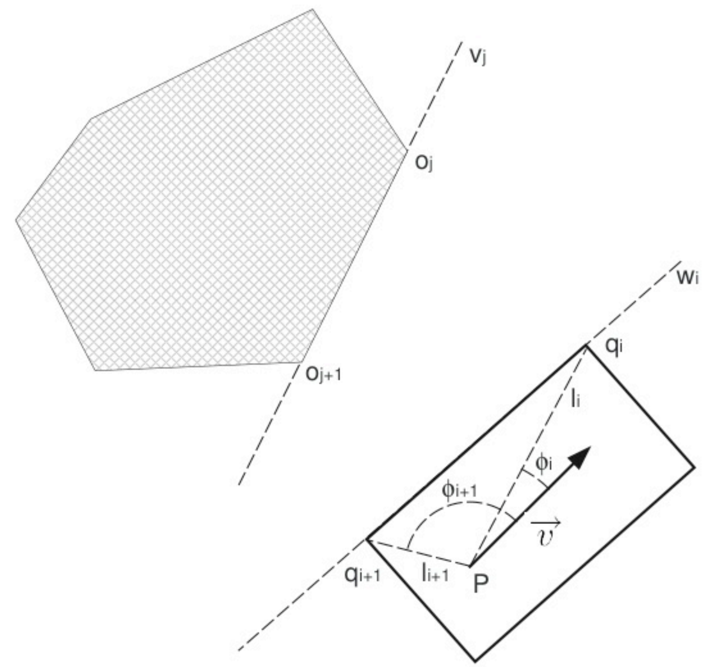
distance function definition

given the optimal solutions to each of the three sub problems

1. $L^{VV}: \quad \mathbb{R}^4 \rightarrow \mathbb{R}$
   $\qquad (q_i,\, o_j) \rightarrow L^{VV}(q_i,\, o_j)$

2. $L^{VE}: \quad \mathbb{R}^4 \rightarrow \mathbb{R}$
   $\qquad (q_i,\, v_j) \rightarrow L^{VE}(q_i,\, v_j)$

3. $L^{EV}: \quad \mathbb{R}^4 \rightarrow \mathbb{R}$
   $\qquad (w_i,\, o_j) \rightarrow L^{EV}(w_i,\, o_j)$



the distance function is

$$
\begin{aligned}
d() \quad &: \quad \mathbb{R}^8 \rightarrow \mathbb{R} \\
d() \quad &= \quad \min\{\min_{i,j} L^{VV}(q_i,\, o_j), \min_{i,j} L^{VE}(q_i,\, v_j),\, \min_{i,j} L^{EV}(w_i,\, o_j)\}
\end{aligned}
$$

proposed approach to the solution of the three sub problems

- definition of the contact manifold

$M_c(x,y,\theta)$ -- > constraint on the final state $\chi(x_f, y_f, \theta_f) = 0$

- for each path $p_i \in \mathcal{F}$, closed-form solution in the RS parameter space

$$\chi(W_i(a,b,e)) = 0 \qquad (*)$$

the problem is underconstrained: the number of constraints are < 3

VV

one-dimensional M_c --> (*) is a system of 2 equations with 3 unknowns

VE and EV

two-dimensional M_c --> (*) defines 1 equation in 3 unknowns to make the problem "square"

- type A paths
  - geometric properties of the RS ball

the boundary of the C|C|C domain coincides

with the level curve $\ell=|\theta|$

- type B paths
  - transversality conditions

$$\psi_f = \frac{\partial \chi(\xi_f)}{\partial \xi_f}\zeta$$

where

$$\psi_f = (\psi_1(t_f), \psi_2(t_f), \psi_3(t_f))^T$$

$$\xi_f = (x(t_f), y(t_f), \theta(t_f))^T$$

Vertex-Vertex

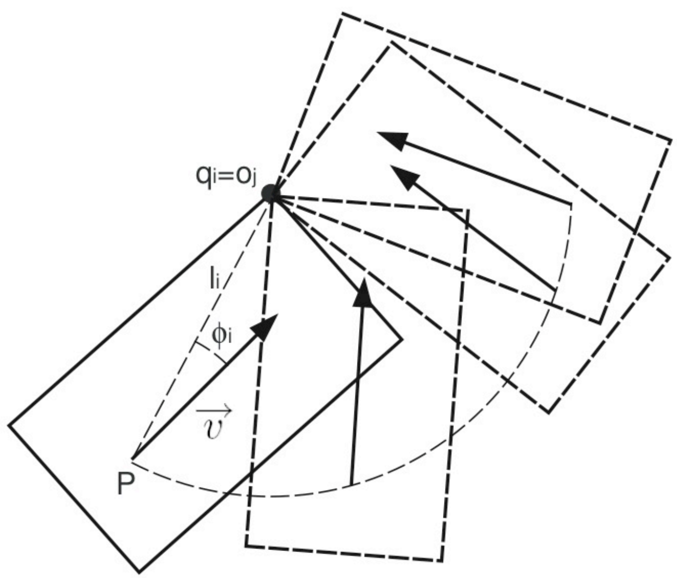
- contact manifold

$$\mathcal{M}_c(x,y,\theta) = \{(x,y,\theta) | q_i = o_j\}$$

$$q_i = (x + l_i\cos(\theta+\phi), y + l_i\sin(\theta+\phi))$$

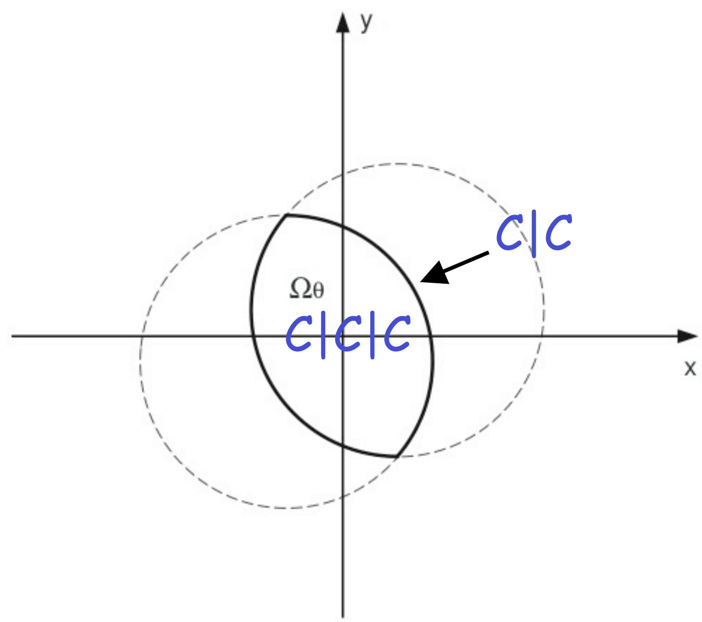
$$o_j = (o_{j_x}, o_{j_y})$$

$$\chi(x_f, y_f, \theta_f) = \begin{pmatrix} x_f - o_{j_x} + l_i\cos(\theta_f+\phi_i) \\ y_f - o_{j_y} + l_i\sin(\theta_f+\phi_i) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



- type A paths

  if a path of type A is optimal for the VV problem then it is a C|C path



--> $$\chi(W_{p_k}(0, b, e)) = 0$$ is square

- type B paths

  if a path of type B is optimal for the VV problem then the corresponding line $\mathcal{D}_0$ passes through the point obstacle o_j

$$\psi_f = \frac{\partial \chi(\xi_f)}{\partial \xi_f}\zeta$$

$$\frac{\partial \chi(\xi_f)}{\partial \xi_f} = \begin{pmatrix} 1 & 0 & -l_i \sin(\theta_f + \phi_i) \\ 0 & 1 & l_i \cos(\theta_f + \phi_i) \end{pmatrix}$$

$$\begin{cases} \psi_1 & = & \zeta_1 \\ \psi_2 & = & \zeta_2 \\ \psi_3(t_f) & = & -l_i \sin(\theta_f + \phi_i)\zeta_1 + l_i \cos(\theta_f + \phi_i)\zeta_2 \end{cases}$$

$$\psi_3(t) = \psi_1 y(t) - \psi_2 x(t) + \psi_3(t_0)$$

$$\psi_3(t_0) = -\psi_1 o_{j_y} + \psi_2 o_{j_x}$$

$$\boxed{\psi_3(t) = \psi_1(y - o_{j_y}) - \psi_2(x - o_{j_x}) = 0}$$

- to each path $p_k \in \mathcal{F}$ we associate a map $VV_{pk}(q_i, o_j)$ which solves VV for the couple (q_i, o_j) using p_k; it is determined by the solution of

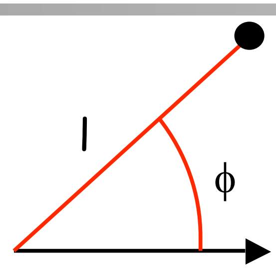
$$\begin{cases} \chi(W_k(a,\, b,\, e)) = 0 \\ \psi_1 \cdot (\overline{Y}_k(a,\, b,\, e) - o_{j_y}) - \psi_2 \cdot (\overline{X}_k(a,\, b,\, e) - o_{j_x}) = 0 \end{cases}$$

where $\overline{Y}_k(a,\, b,\, e)$ and $\overline{X}_k(a,\, b,\, e)$ represent the position of the robot on the line $\mathcal{D}_0$ computed via $W_k(a,\, b,\, e)$

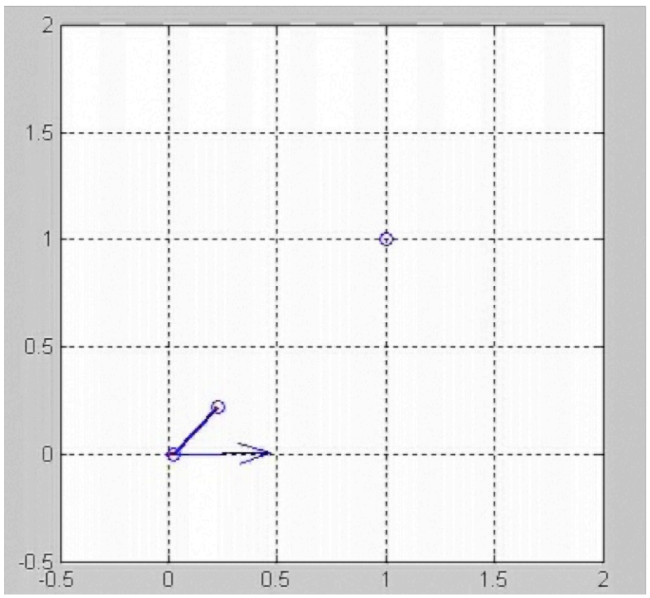
- the optimal solution to the problem VV is

$$L^{VV}(q_i,\, o_j) = \min_{p_k \in \mathcal{F}} \text{length } VV_{p_k}(q_i,\, o_j)$$

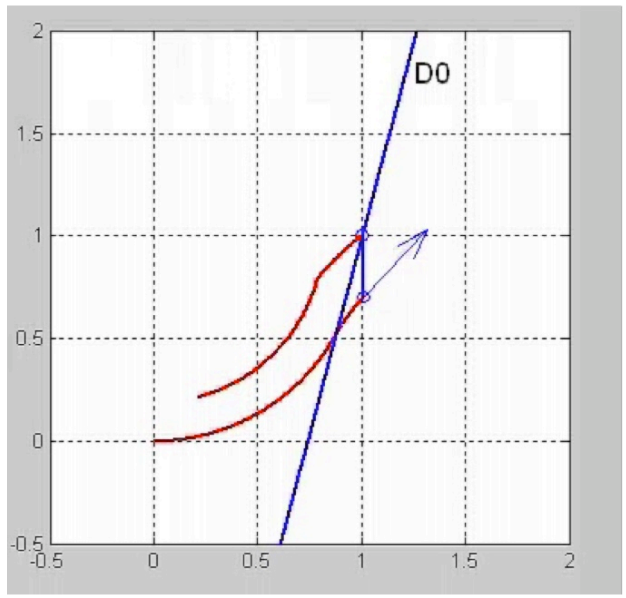
VV - example 1



l = 0.3, $\phi=\pi/4$, obstacle in (1,1)



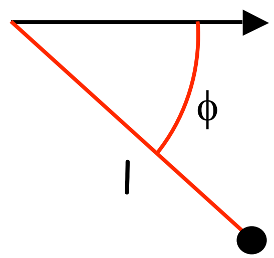
start



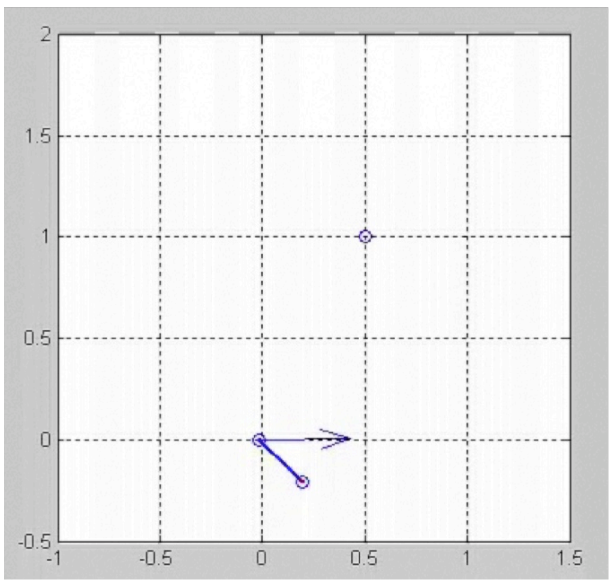
end

$C_a C_b$ type

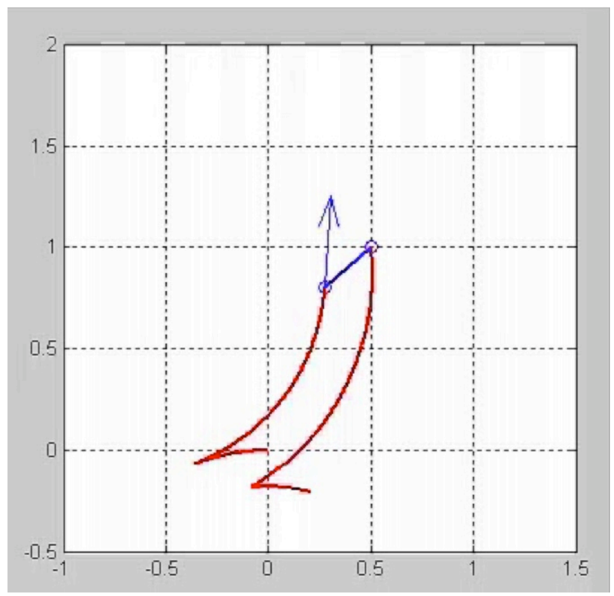
VV - example 2



l = 0.3, φ=-π/4, obstacle in (0.5,1)



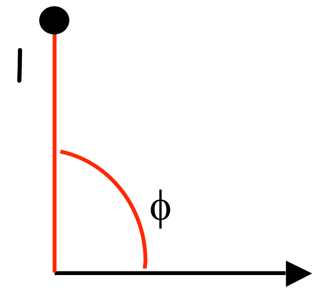
start



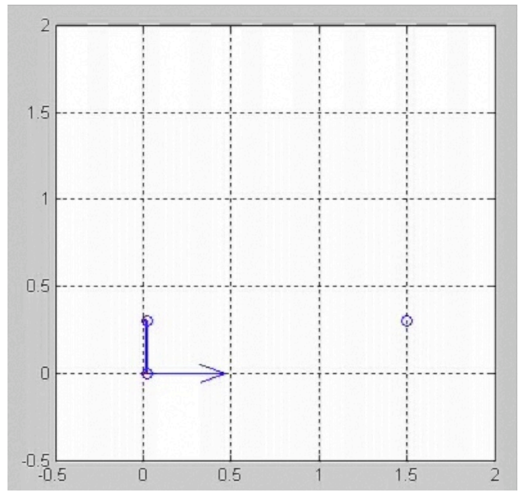
end

$C_a | C_b$ type

VV - example 3

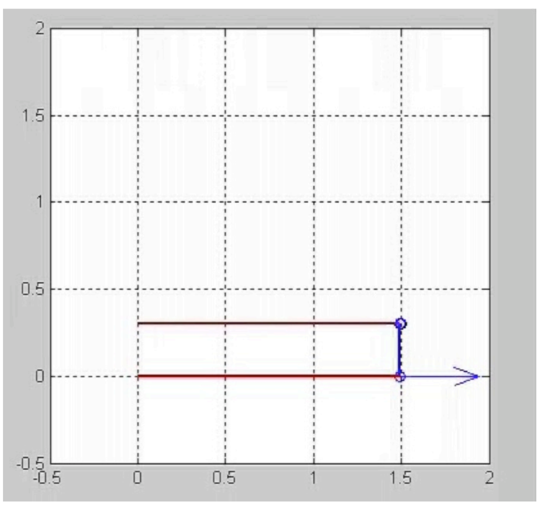


$l = 0.3, \quad \phi = \pi/2, \quad$ obstacle in (1.5,0.3)



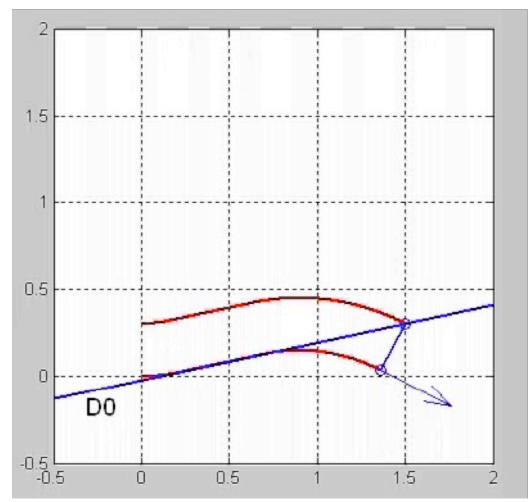
start

length=1.5



path 1

length=1.39



path 2

Vertex-Edge

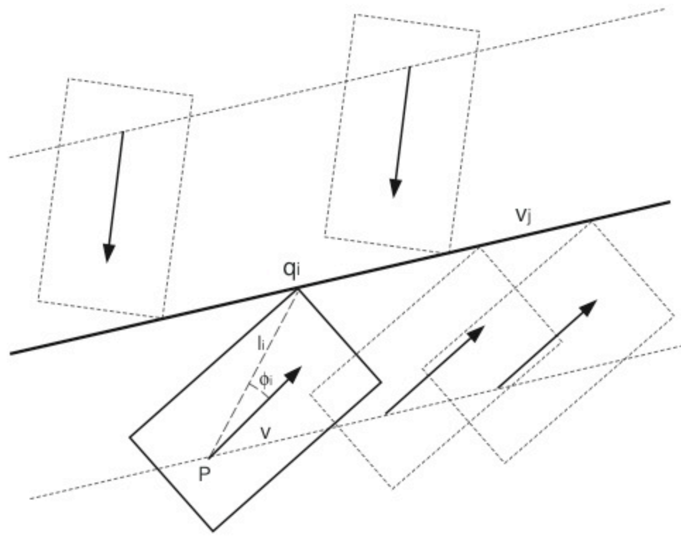
- contact manifold

$$\mathcal{M}_c(x, y, \theta) = \{(x, y, \theta) |\ q_i \in v_j\}$$

$$q_i = (x + l_i \cos(\theta + \phi), y + l_i \sin(\theta + \phi))$$

$$v_j :\ \bar{y} = m_i \bar{x} + n_j$$

$$\chi(x, y, \theta) = y - m_j x - n_j - l_i m_j \cos(\theta + \phi_i) + l_i \sin(\theta + \phi_i) = 0$$

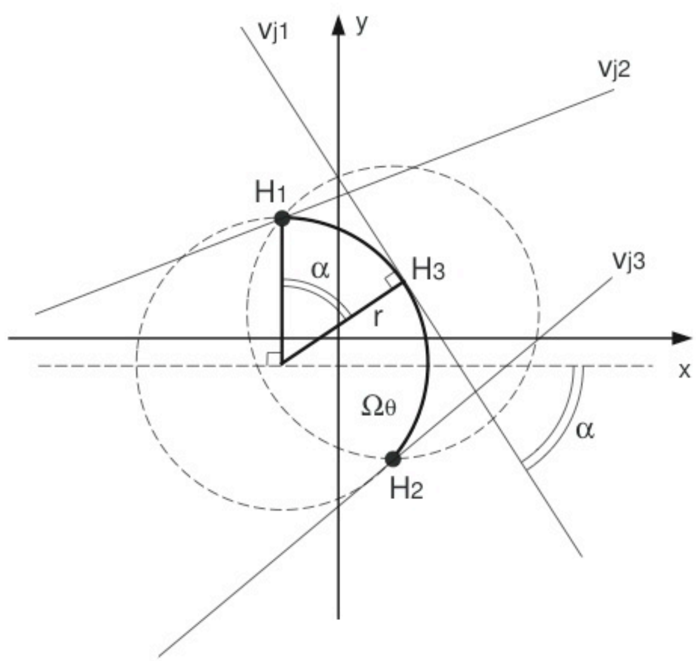


- type A paths

  if a path of type A is optimal for the VV problem then it is a C|C path

  but $\chi(W_{p_k}(0,b,e))=0$ is now underspecified

an "ad hoc" analysis is needed to obtain a square system of equations



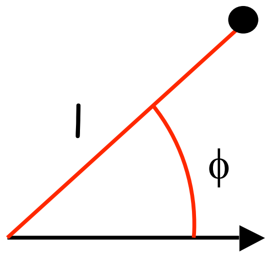
- type B paths

  if a path of type B is optimal for the VE problem then

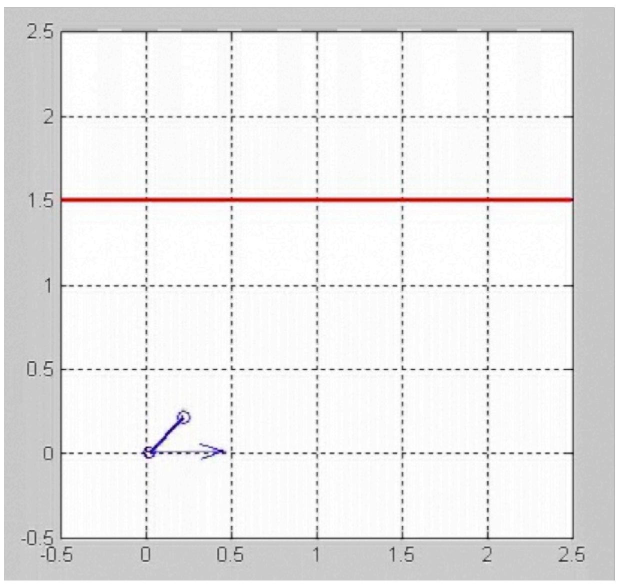
  1. the corresponding line $\mathcal{D}_0$ is perpendicular to the line v_j
  2. the contact point lies at the intersection between $\mathcal{D}_0$ and v_j

  - the contact constraint and the above two necessary conditions allow to define a system of three equations in the three unknowns (a, b, c)
  - a solution is sought for each path in the sufficient family $\mathcal{F}$
  - as in the VV case, the optimal solution to the VE problem is obtained by choosing the path of minimum length among all the paths in $\mathcal{F}$

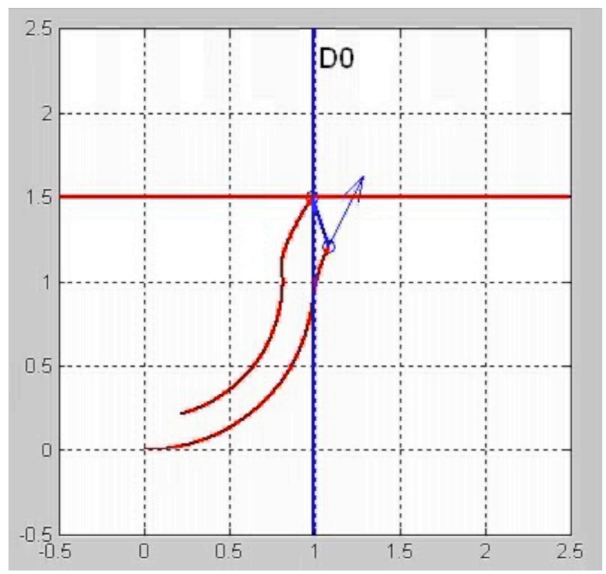
VE - example



$l = 0.3, \quad \phi = \pi/4, \quad v_j: y=1.5$



start



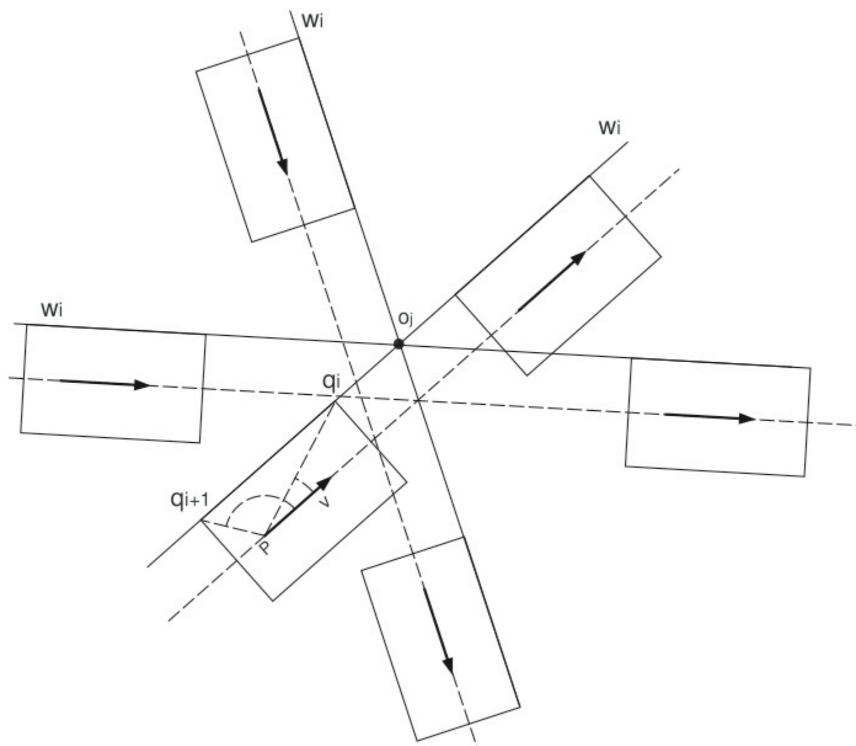
end

$C_a C_b$ type

Edge-Vertex

- contact manifold

$$\mathcal{M}_c(x, y, \theta) = \{(x, y, \theta) | o_j \in w_i\}$$



- type A paths

  if a path of type A is optimal for the VV problem then it is a C|C path

  but $\chi(W_{p_k}(0,b,e)) = 0$ is underspecified

  the same analysis of VE apply to obtain a square system of equations

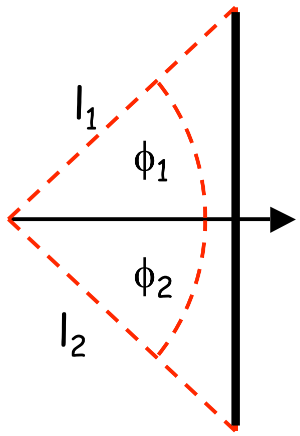
- type B paths

  if a path of type B is optimal for the EV problem then

  1. the corresponding line $\mathcal{D}_0$ is perpendicular to the edge w_j at the end of the path
  2. the contact point lies at the intersection between $\mathcal{D}_0$ and w_j

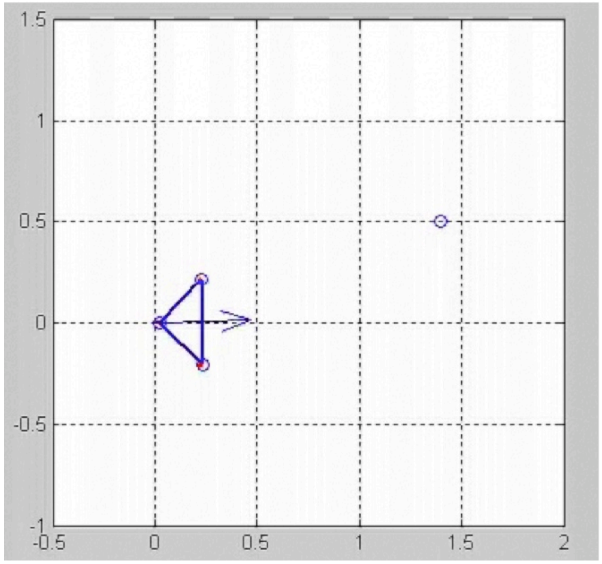
optimal solution to the EV problem : as before

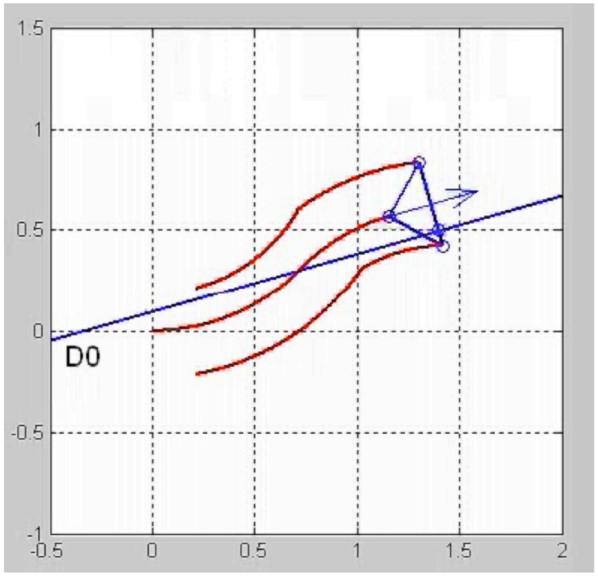
EV - example



$l_1 = 0.3$, $\phi_1 = \pi/4$, $l_2 = 0.3$, $\phi_2 = -\pi/4$

obstacle in (1.4, 0.5)





$C_a C_b$ type

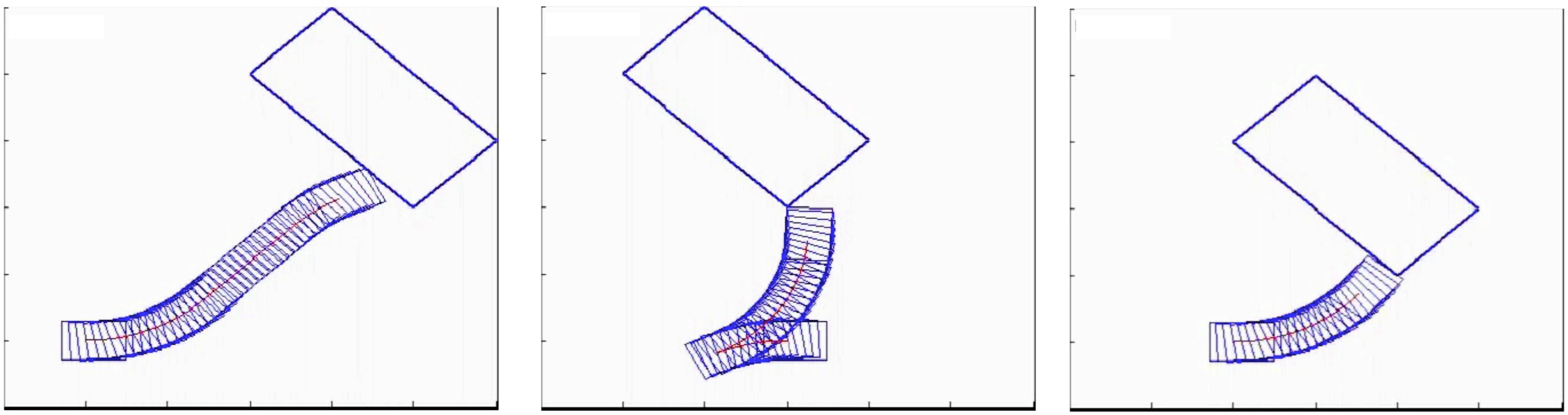
Family refinement

using continuity of the RS path parameters (a, b, e) w.r.t. the robot parameter l_i, it is possible to show that

paths types (IV), (V), (VI), (VIII) are never optimal solutions of the problems VV, VE, EV

the search can be restricted to 26 path types

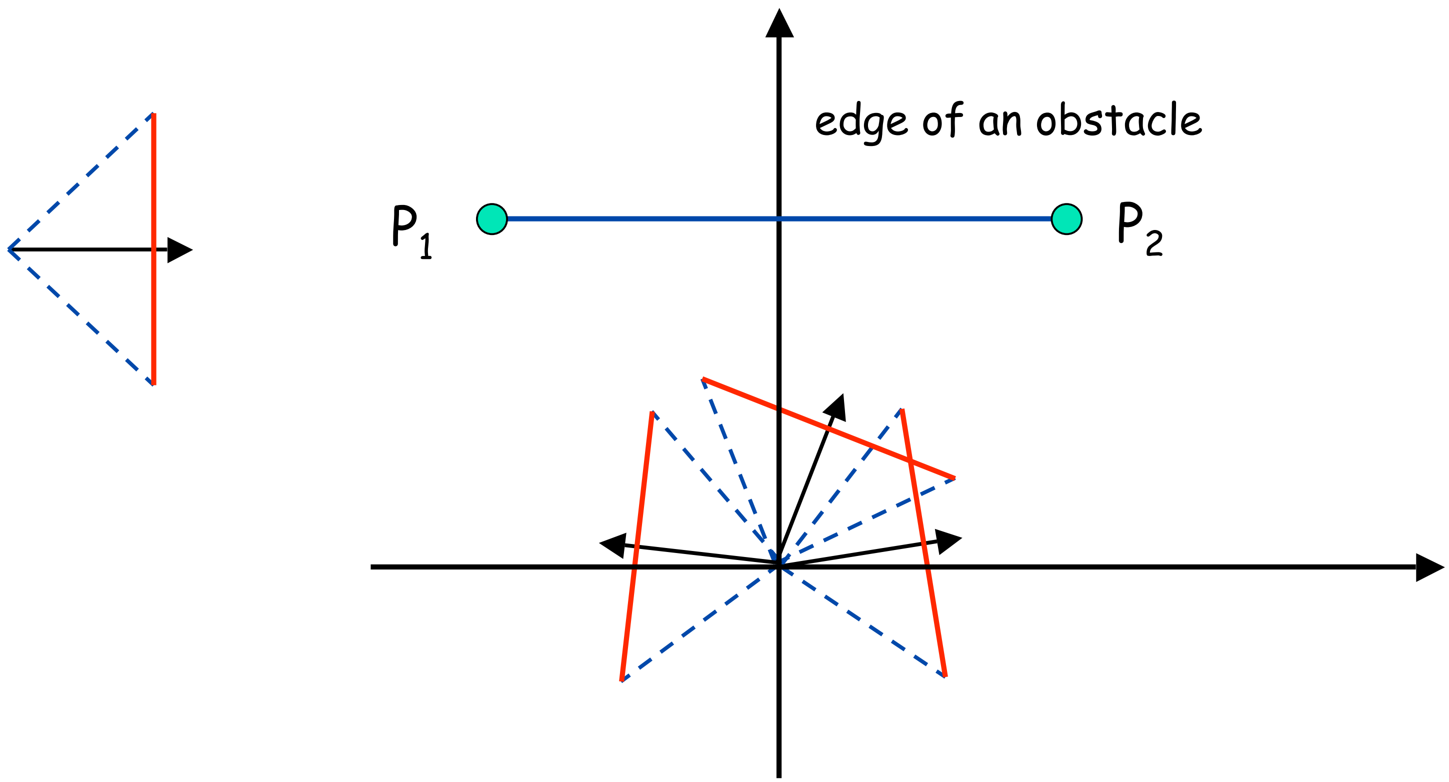
examples



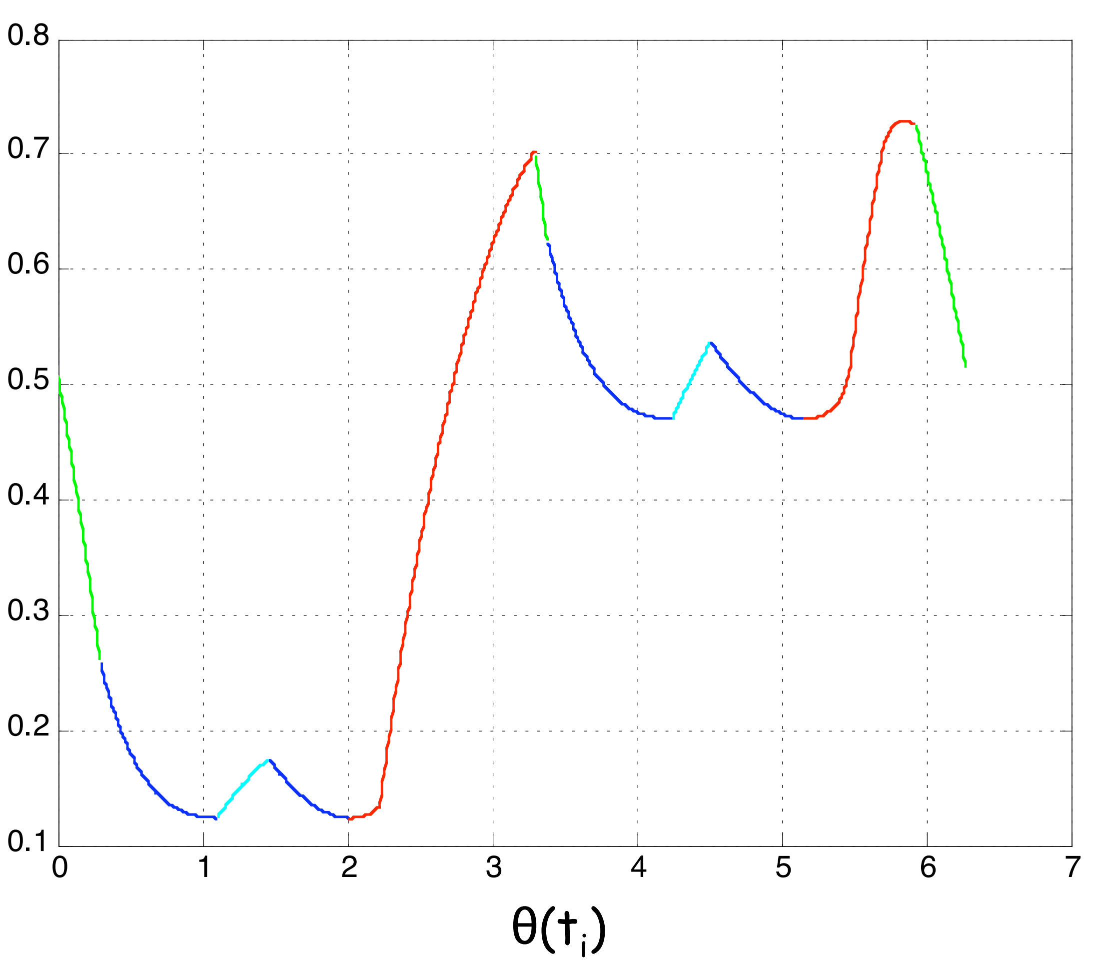
Vertex-Vertex Vertex-Edge Edge-Vertex

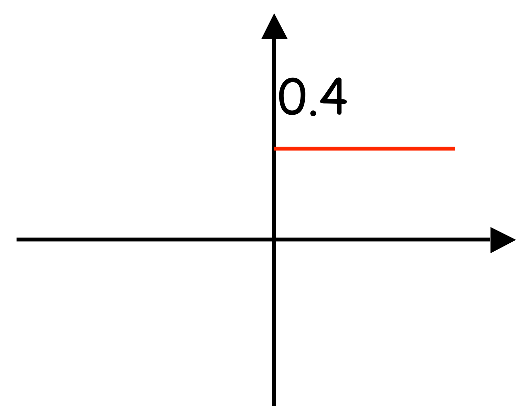
smoothness

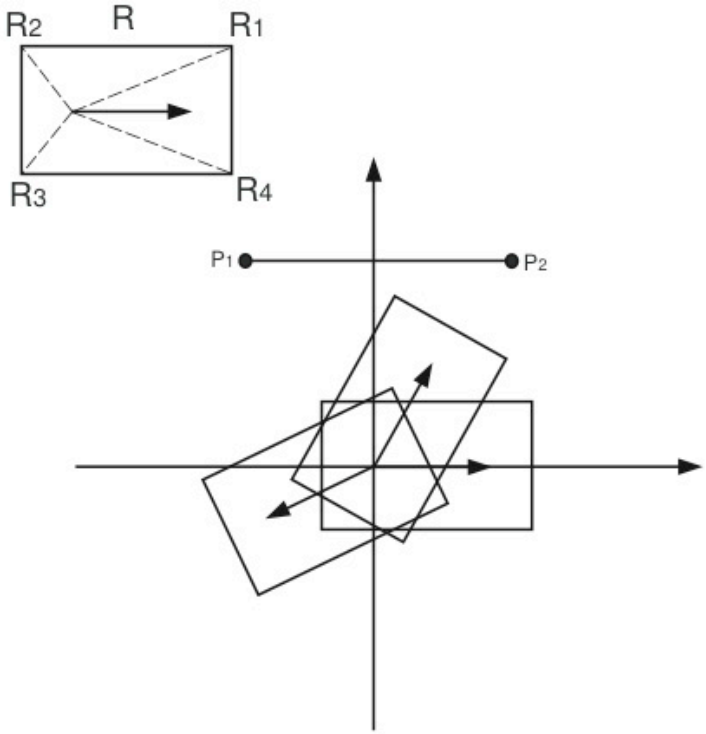
the defined distance function is piecewise smooth; the not derivable points are located at the switches between the L^{VV}, L^{VE} and L^{EV} functions

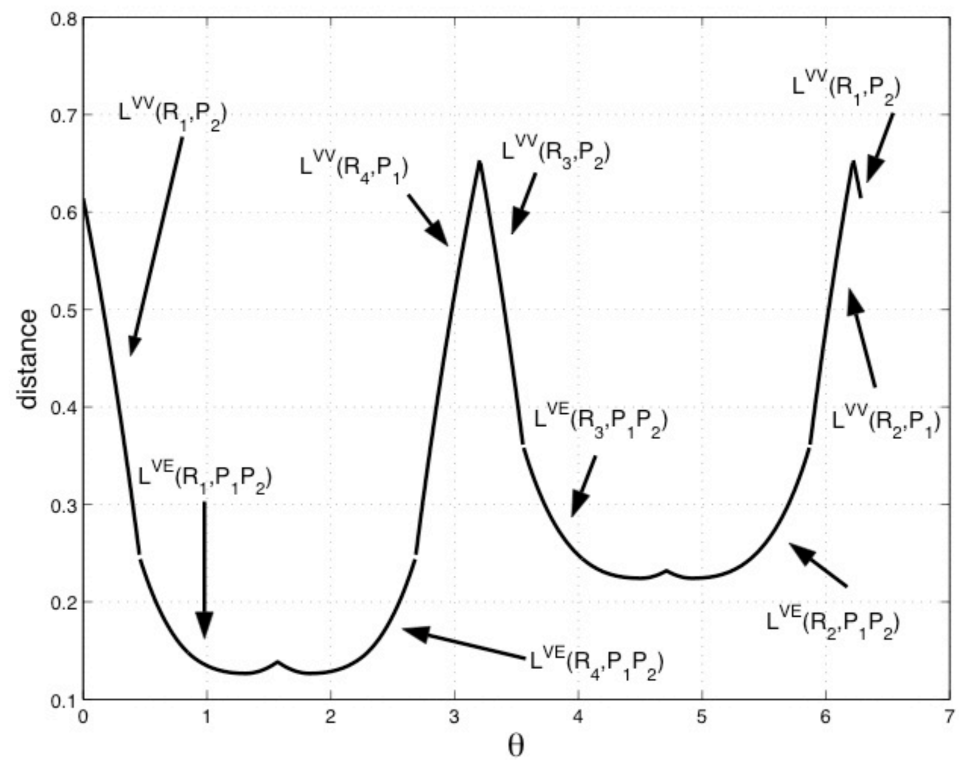


$P_1(0, 0.4)$ $P_2(0.3, 0.4)$



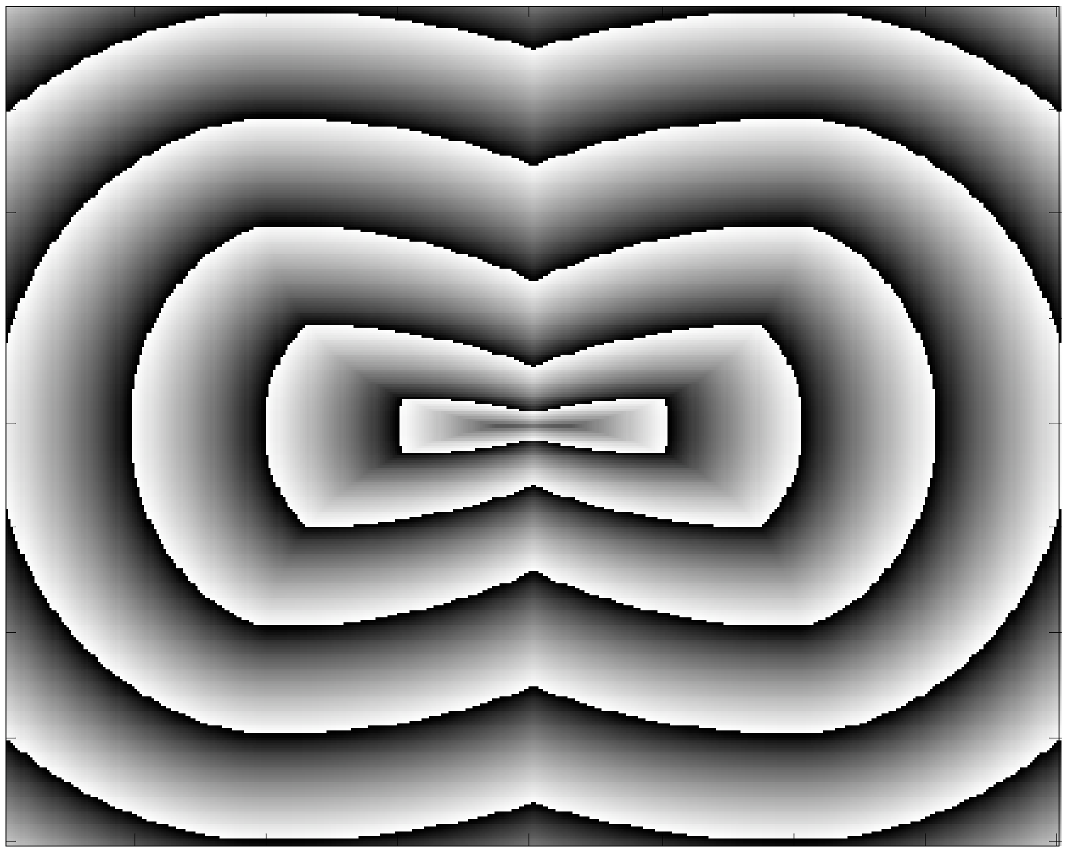


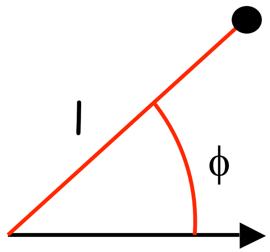

R2
R
R1
R3
R4
P1
P2


distance
θ
L^VV(R_1,P_2)
L^VV(R_4,P_1)
L^VV(R_3,P_2)
L^VV(R_1,P_2)
L^VE(R_3,P_1P_2)
L^VV(R_2,P_1)
L^VE(R_1,P_1P_2)
L^VE(R_2,P_1P_2)
L^VE(R_4,P_1P_2)

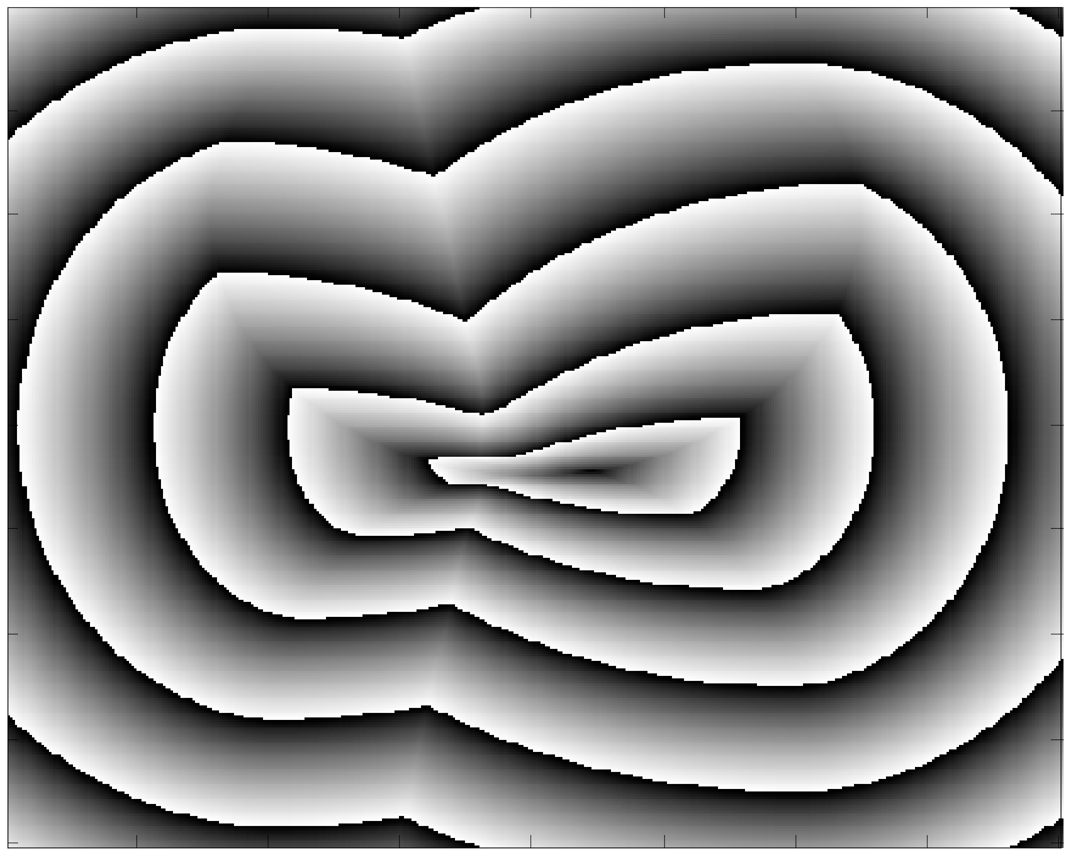
level curves

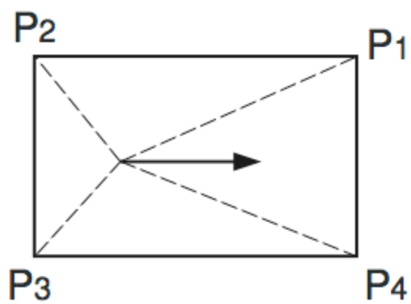
point robot: $l=0, \quad \phi=0$



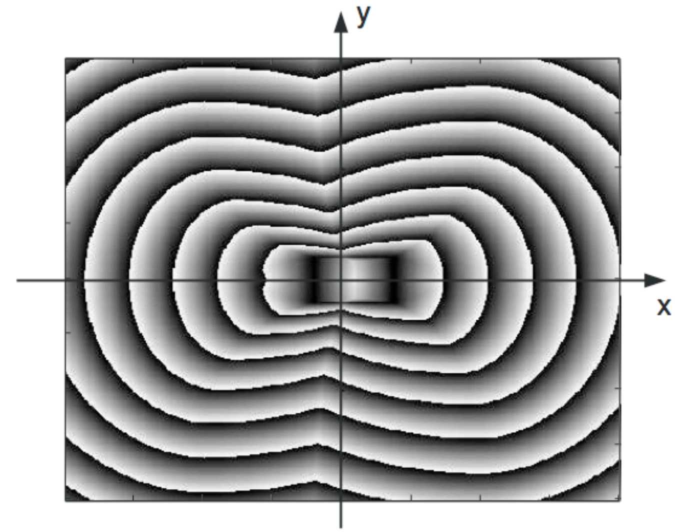


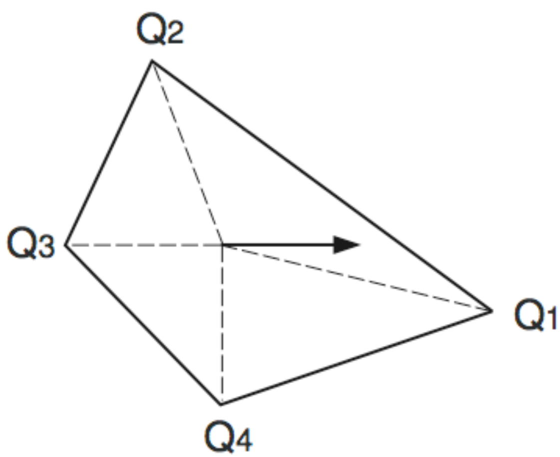
$l=0.3,\quad \phi = \pi/4$



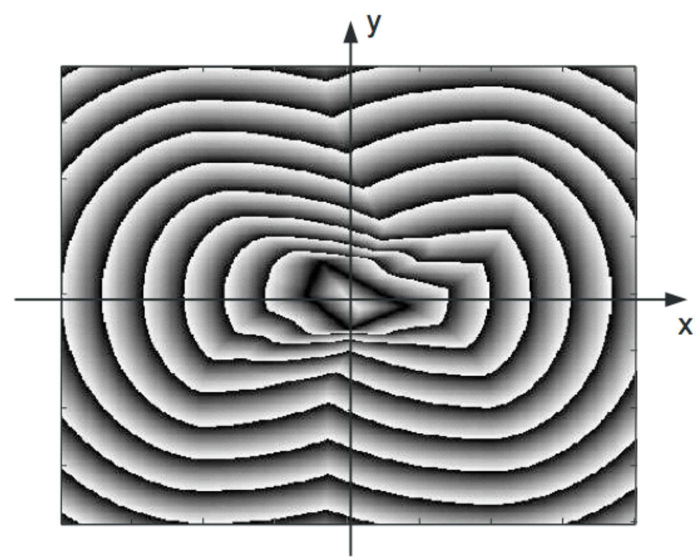


R1





R2



Conclusion

main result

characterization of the shortest paths between polygonal RS car-like robot and polygonal obstacles

- analytic expression of the distance function

- reduction of the sufficient family of optimal paths

future work

- extend to general manifolds in configuration space
- apply to motion planning