1 Priority Arguments and Post's Problem

Recall that A' is the Turing jump of A:

$$A' = \{\, e \mid \{e\}^A(e)\downarrow \,\}$$

As we have seen, the jump produces a set of strictly higher Turing degree. Iterating the jump operation generates a sequence of sets of strictly increasing Turing degree. In fact, we can extend this operation to transfinite levels:

$$\emptyset <_T \emptyset' <_T \emptyset'' <_T \emptyset^{(3)} <_T \ldots <_T \emptyset^{(\omega)} <_T \emptyset^{(\omega+1)} <_T \ldots$$

Here $\emptyset^{(\omega)} = \{\, \langle n, x\rangle \mid x \in \emptyset^{(n)} \,\}$ is the disjoint union of all the finite jumps $\emptyset^{(n)}$, $\emptyset^{(\omega+1)} = (\emptyset^{(\omega)})'$ and so forth. Thus $\emptyset^{(n)} <_T \emptyset^{(\omega)}$ for all $n \geq 0$. Considering this chain of increasing Turing degrees, one might suspect that the Turing degrees form a linear order: given any two sets A and B either A contains at least as much information as B or the other way round.

We will see shortly that the actual picture is quite a bit more complicated: there are incomparable degrees. In other words, there are sets A and B such that neither one can be used as an oracle to decide membership in the other. Moreover, one can construct such sets of relatively low complexity: both can be made to be recursively enumerable:

$$\text{there exist } A, B \text{ r.e.} : A \not\leq_T B \text{ and } B \not\leq_T A$$

This result is quite surprising in a way. Experience shows that r.e. sets that appear in "nature," meaning in other areas of mathematics, always turn out to be either decidable or complete; there appears to be a 0/1-law. For example, the collection of theorems of an axiomatizable theory is always r.e. and for all standard examples in, say, algebra they turn out to be decidable or complete. Or consider a natural r.e. set like the collection of all solvable Diophantine equations:

$$D = \{\, P(\vec{x}) \in \mathbb{Z}[\vec{x}] \mid \exists\, \vec{x}\, P(\vec{x}) = 0 \,\}$$

By Matiyasevic's theorem, D is complete. The proofs may be very difficult, but in the end the complexity settles down at one end of the spectrum or the other. Alas, with enough effort one can construct r.e. sets that are incomparable and thus in particular neither decidable nor complete. The question of whether there are natural examples for this phenomenon is open (and, of course, difficult since a solution would require a precised definition of what is meant by a problem being natural).

The first proof below gives a weaker result and outlines the technique used in the construction. The idea is to build the incomparable sets in stages. At each stage, some elements may be added to A or B subject to constraints, so-called *requirements*. The requirements make sure that neither set can be computed using the other as an oracle.

A refinement of this technique then yields a second, stronger result: there are two incomparable recursively enumerable degrees. Note that this settles the question of whether there are any intermediate degrees, i.e., whether there is a set A such that $\emptyset <_T A <_T \emptyset'$. The simple set S constructed earlier is of no help here: it turns out to be Turing complete.

1.1 The Use Principle

It is convenient to identify a set $A \subseteq \mathbb{N}$ and its characteristic function $\chi_A : \mathbb{N} \to \{0, 1\}$: thus we write $A(x) = \chi_A(x) \in \{0, 1\}$. Since any convergent computation using A as an oracle can perform only finitely many queries to A, we can replace the oracle for a specific computation by a finite approximation to A (essentially a bit-vector) of sufficient length. This idea is captured in the next definition.

For a finite Boolean function $F : \{0, \ldots, s-1\} \to \{0, 1\}$ write $F \sqsubset A$ to indicate that F agrees with A on its domain:

$$F \sqsubset A \iff \forall x < s\big(F(x) = A(x)\big).$$

We write $s = lh(f)$ for the length of this segment. For any F as above define $\{e\}^F(x) \simeq y$ iff $\{e\}^A(x) \simeq y$ for some set A such that $F \sqsubset A$ and all queries to the oracle A in the computation are less than $lh(F)$. Since every convergent computation can include only boundedly many queries to the oracle we have

$$\{e\}^A(x) \simeq y \iff \exists\, F \sqsubset A\,\big(\{e\}^F(x) \simeq y\big).$$

For the construction below it will be convenient to have a notation for the size of the initial segment of A we have to know in order to compute the eth recursive function on input x with oracle A. Let

$$\mathsf{use}(e, x, A) \simeq \min\big(\, lh(F) \mid F \sqsubset A \wedge \{e\}^F(x)\downarrow\big).$$

Thus use is a function that is partial recursive in A. The corresponding recursive approximation is

$$\mathsf{use}(e, x, A, \sigma) \simeq \min\big(\, lh(F) \mid F \sqsubset A \wedge \{e\}^F_\sigma(x)\downarrow\big)$$

and $\lim_\sigma \mathsf{use}(e, x, A, \sigma) = \mathsf{use}(e, x, A)$ in the discrete topology.

1.2 Two Incomparable Sets below $\emptyset'$

As a warm-up exercise we will build two incomparable sets that narrowly miss being recursively enumerable. The two sets are constructed in stages $\sigma < \omega$. At each stage, simple rules determine which elements should be added to A or B. No elements are ever removed from either set. The construction rules are simple, but fail to be recursive, so the resulting sets are not quite recursively enumerable. During the construction, we have to satisfy the following requirements:

(R_e) $A \neq \{e\}^B$ insure $A \not\leq_T B$

(R'_e) $B \neq \{e\}^A$ insure $B \not\leq_T A$

If all requirements are indeed satisfied, then neither set can be used to compute the other one. The principal problem in the construction is that we have to deal with infinitely many requirements, and the individual requirements may well clash with each other. For example, we may wish to add some element x to A at stage σ to make sure that requirement R_e is satisfied: adding x would cause $A(x) = 1 \neq 0 = \{e\}^B(x)$. But adding x to A might change the value of some computation $\{g\}^A(y)$, and thereby affect the requirement R'_e. We resolve these clashes by dealing with the requirements in order, and preserving computations by changing the oracles only outside of a fixed initial segment (Use Principle).

Theorem 1.1 *There exist two incomparable Turing degrees below $\emptyset'$: there are sets A, $B \leq_T \emptyset'$ such that neither $A \leq_T B$ nor $B \leq_T A$.*

Proof. To construct A and B in stages we will use finite functions of the form $A_\sigma, B_\sigma : \{0, \ldots, s-1\} \to \{0, 1\}$ as approximations to A and B: $A_\sigma \sqsubset A$, $B_\sigma \sqsubset B$ and $A = \lim A_\sigma$, $B = \lim B_\sigma$. We let $A_{<\sigma}$ be the part of A constructed prior to stage σ (and likewise for B).

The Construction

Stage $\sigma = 0$: Let $A_0 = B_0 = \emptyset$.

Stage $\sigma = 2e$:

We work on requirement (R_e): $A \neq \{e\}^B$. Let $n = lh(A_{<\sigma})$, $n' = lh(B_{<\sigma})$ be the lengths of the parts of the sets constructed so far. At stage σ, we will determine whether n and n' are placed into A and B, respectively. Our actions will depend on whether some finite extension of $B_{<\sigma}$ produces a computation of $\{e\}$ on n, the least number for which membership in A is as yet undetermined, with a Boolean value. Note that if no such extension exists, then the requirement is satisfied no matter whether we place n into A or not: $\{e\}^B$ cannot be a characteristic function. In this case we simply set $A_\sigma(n) = 0$, $B_\sigma(n') = 0$. So suppose that

$$\exists\, F, t \left(n' < lh(F) \wedge B_{<\sigma} \sqsubset F \wedge \{e\}^F_t(n) \in \{0, 1\}\right)$$

Pick F and t minimal such and set $B_\sigma = F$, $A_\sigma(n) = 1 - \{e\}^F_t(n)$.

Stage $\sigma = 2e + 1$
Exchange A, B, (R_e) and (R'_e).

Note that by construction $A = \lim A_\sigma$, $B = \lim B_\sigma$ are indeed total, so that membership is settled for each natural number.

Claim: Every requirement is satisfied.

Let us consider only (R_e), the other case is entirely similar. Suppose for the sake of a contradiction that $A = \{e\}^B$. Note that the function $\{e\}^B$ is then necessarily total and Boolean. Then at stage $\sigma = 2e$ the condition in the construction must have been satisfied: there is a finite extension that satisfies the condition above.

But then our construction makes sure that $\{e\}^B(n) = \{e\}^{B_\sigma}(n)$: the part of B below $lh(B_\sigma)$ will not be changed. It follows from the Use Principle that

$$n \in A \Leftrightarrow 0 = \{e\}_t^{B_\sigma}(n) = \{e\}^B(n) = A(n) \Leftrightarrow n \notin A,$$

a contradiction.

Also note that to determine the existence of F and t requires no more than a $\emptyset'$ (or any other complete recursively enumerable set) as an oracle: we can find F and t by using unbounded search with a primitive recursive predicate. □

Since the construction uses $\emptyset'$ as an oracle, the two incomparable sets unfortunately fail to be recursively enumerable. However, they are not far removed from being r.e. either.

Lemma 1.1 *$A \leq_T \emptyset'$ iff A is Δ_2-definable.*

Proof. Suppose that $A = \{e\}^K$ for some index e where K is a complete recursively enumerable set. Then $x \in A \Leftrightarrow A(x) = 1 \Leftrightarrow \{e\}^K = 1 \Leftrightarrow \exists\, \sigma, F \sqsubset K \left(\{e\}_\sigma^F(x) \simeq 1\right)$.

The predicate $\{e\}_\sigma^F(x) \simeq 0$ is primitive recursive and therefore certainly Δ_1-definable.

Now $F \sqsubset K \Leftrightarrow \forall x < lh(F)\big(x \in F \Rightarrow x \in K\big) \wedge \forall x < lh(F)\big(x \in K \Rightarrow x \in F\big)$.

Replacing K by a Σ_1 definition we see that $F \sqsubset K$ is Δ_2-definable and so A is Σ_2-definable. But the same argument also holds for $\mathbb{N} - A$, thus A is Δ_2-definable.

On the other hand suppose A is Δ_2-definable, say $x \in A \Leftrightarrow \exists s \forall t \phi(x, s, t)$. Then there is a $\emptyset'$-computable function f such that

$$f(x,s) = \begin{cases} 0 & \text{if } \forall t \phi(x,s,t), \\ 1 & \text{otherwise.} \end{cases}$$

For $f(x,s) = 1 \Leftrightarrow \exists t \neg \phi(x,s,t)$ which is a r.e. question and thus trivial for oracle $\emptyset'$. But then A is r.e. in K: $A_\sigma = \{\, x \mid \exists s < \sigma (f_\sigma(x,s) = 0)\,\}$. □

1.3 Two Incomparable r.e. Sets

Now for the main goal in this section: the construction of incomparable sets that are still recursively enumerable. The construction is quite similar to the previous one, but this time we cannot afford to assume knowledge about the existence of the finite extension F in the last proof. Instead, at stage σ, we only consider computations of length at most σ using whatever part of the oracle has already been constructed: $\{e\}_{\sigma}^{B_{<\sigma}}(x)$ is all the information we have available. For small values of σ this computation will not converge, and we have no resulting value, but for sufficiently large values of σ we will obtain a value (provided the computation converges at all). Accordingly we can then either place x into A, or try to prevent x from entering A so as to guarantee $A(x) \neq \{e\}^{B}(x)$.

The crucial problem is that the value of the computation may well change since other requirements will place elements into B at later stages. But since we are trying to construct recursively enumerable sets, we cannot remove x from A in order to respond to a changed value: once an element has entered the set, it has to stay forever. The construction below resolves this conflict by linearly ordering the requirements and giving preference to requirements of higher priority. As it turns out, each requirement will ultimately be satisfied after finitely many steps, so that requirements of lower priority also have a chance to be satisfied.

Theorem 1.2 *Friedberg-Muchnik*

There exist two incomparable r.e. Turing degrees: there are two r.e. sets A, B such that neither $A \leq_T B$ nor $B \leq_T A$.

Proof. Again we try to satisfy the requirements

$$(R_{2e}) \qquad A \neq \{e\}^{B}$$

$$(R_{2e+1}) \qquad B \neq \{e\}^{A}.$$

By symmetry, it suffices to consider the even-numbered requirements. To satisfy (R_{2e}) we will try to find a *witness* x such that $A(x) = 1$ but $\{e\}^{B}(x) \simeq 0$.

The case where $\{e\}_{\sigma}^{B_{\sigma}}(x) \not\simeq 0$ for all stages σ is easy: R_{2e} is satisfied as long as we keep x out of A.

Now suppose at some stage σ we find out that $\{e\}_{\sigma}^{B_{<\sigma}}(x) \simeq 0$. Then we throw x into A_{σ} and try to preserve the part of B that is used in the computation of $\{e\}_{\sigma}^{B_{<\sigma}}(x) \simeq 0$ so as to make sure that the value of the computation does not change in the future. This is done by means of a *restraint function*:

$$r(2e, \sigma) = \mathsf{use}(e, x, \sigma; B_{<\sigma}) + 1$$

If we succeed in keeping elements from entering this part of B we are done since $A(x) = 1 \neq 0 = \{e\}_{\tau}^{B_{\tau}}(x) = \{e\}^{B}(x)$, for all $\tau \geq \sigma$.

Note that we have to find a way to work on all requirements simultaneously. We will say that (R_i) has *higher priority* than (R_j) iff $i < j$. Working on a requirement of higher priority may destroy a witness of a requirement of lower priority, hence we may have to change the witness on occasion. To this end we will pick a witness for (R_{2e}) in a special, reserved set of potential witnesses: $\mathbb{N}_e = \{ \langle x, e \rangle \mid x \geq 0 \}$. Let $x(e, \sigma)$ be the witness for (R_e) at (the end of) stage σ. Note that a witness $\langle x, e \rangle$ is put into A only in an attempt to satisfy R_{2e}.

(R_{2e}) is said to *require attention* at stage σ iff

$$\{e\}_\sigma^{B_{<\sigma}} (x(2e, <\sigma)) \simeq 0 \text{ and } r(2e, <\sigma) = 0.$$

Construction

Stage 0:
Let $A_0 = B_0 = \emptyset$, $x(e, 0) = \langle 0, e \rangle$, $r(e, \sigma) = 0$.

Stage $\sigma > 0$:
Pick the requirement of highest priority that requires attention, say, (R_{2e}).

Set $x = x(2e, \sigma) = x(2e, <\sigma)$ and put x into A_σ,

Define $r(2e, \sigma) = \mathsf{use}(e, x, B_{<\sigma}, \sigma)$.

For $i < 2e$ set $x(i, \sigma) = x(i, <\sigma)$ and $r(i, \sigma) = r(i, <\sigma)$.

For $i >$ 2e set $x(i, \sigma) = \min\big(x \in \mathbb{N}_i \mid x \notin A_\sigma \cup B_\sigma \wedge x > \max\big(r(j, \sigma) \mid j \leq 2e \big) \big)$ and $r(i, \sigma) = 0$.

If no such requirement exists do nothing.

The requirement (R_{2e}) as above is said to *receive attention* at stage σ. Note that the witnesses and restraints of all requirements of higher priority are preserved, but all requirements of lower priority are clobbered: a new potential witness is selected, and the restraint function is set to 0.

Claim: Every requirement receives attention at most finitely often. Furthermore, all requirements are eventually satisfied.

Proof. By induction on e'. By IH we may pick a stage σ such that no requirement (R_i), $i < e'$, receives attention at $\tau \geq \sigma$. Let us assume $e' = 2e$.

Case 1: R_{2e} never receives attention after stage σ.

Since all higher order requirements are already satisfied, that can happen only because R_{2e} never requires attention after σ. Then $x = x(2e, \sigma) = x(2e, \tau)$, $\tau \geq \sigma$, is not in A: since the sets $\mathbb{N}_i$ are all disjoint, only R_{2e} could put x into A, and only if R_{2e} received attention. But $\{e\}^{B_\tau} (x) \not\simeq 1$, done.

Case 2: R_{2e} receives attention at some stage $\tau \geq \sigma$.

Then $x(2e, \tau)$ is put into A and a restraint is established for B at the maximum number queried in $\{e\}_{\tau}^{B_{<\tau}}(x(2e, <\tau)) \simeq 0$. But no requirements of priority $i \leq 2e$ ever become active after τ, hence $x = x(2e, \tau) = x(2e, \tau')$, $\tau' \geq \tau$, and $\{e\}^{B_{\tau'}}(x) \simeq 0 \neq A(x) = 1$ for all $\tau' \geq \tau$. Hence R_{2e} is satisfied and we are done. $\square$

The construction of the last theorem is called a *finite-injury* argument, since each of the requirements is violated at most finitely often (because a requirement of higher priority receives attention). There are analogous infinite-injury arguments that can be used to establish more complicated results.

For example, there is a density theorem that says that between any two r.e. sets there is a third:

$$\forall\, A, B \ r.e., A <_T B \,\exists\, C \ r.e. \,(A <_T C <_T B).$$

Also, for all intermediate r.e. sets one can find an incomparable one:

$$\forall\, A \ r.e., \emptyset <_T A <_T \emptyset' \,\exists\, B \ r.e. \,(A, B \text{ incomparable }).$$

Note that the construction in the theorem depends heavily on a universal Turing machine. In fact, Soare has shown that the disjoint sum $A \oplus B$ is complete, so the lack of completeness of A really comes from hiding B. In an intuitive sense the whole construction process is still complete; it is really because of information hiding that we are able to produce incomplete sets.