CDM

Numbers, Ordinals and Cardinals

KLAUS SUTNER

CARNEGIE MELLON UNIVERSITY

In mathematics, there are many examples of operations that need to be repeated "infinitely" often in order to construct some object.

To make real sense out of this, one needs to generalize the natural numbers to transfinite ordinals.

Here is a classic example of the need for such an infinitary operation, due to Cantor, in his early work on the limits of Fourier analysis.

Surprisingly, the same machinery is also used in his later work on the continuum hypothesis.

Suppose f is a smooth real-valued function over $(-\pi, \pi)$.
Then we can write f as a trigonometric series

$$f(x) = b_0/2 + \sum a_n \sin(nx) + b_n \cos(nx).$$

Fourier discovered in 1807 how to calculate the coefficients via an integral.

As it turns out, this representation is unique: f determines the coefficients uniquely.

Cantor proved the following.

Theorem (Cantor 1870)

Every real function has most one representation by a trigonometric series.

This comes down to showing that if a trigonometric series converges pointwise to 0, all the coefficients must be 0.

Cantor knew that one can relax the condition, convergence may fail on finitely many points. So the question is how far one can relax the hypothesis.

A set of uniqueness is a set U such that convergence on

$$(-\pi, \pi) - U$$

is enough.

Cantor set out to find general sets of uniqueness.

So this question is deeply involved with the structure of sets of reals, a horribly complicated domain as we now know.

Definition

Let $A \subseteq \mathbb{R}$ closed. A point $a \in \mathbb{R}$ is isolated (in A) if there exists a neighborhood of a that is disjoint from A. Otherwise a is a limit point or accumulation point. A is perfect if it has no isolated points.

Perfect sets can be constructed from arbitrary closed sets by removing isolated points. Write

$$X' = \{\, x \in X \mid x \text{ limit point} \,\}$$

for the derived set of X.

Unfortunately, $A' \subseteq A$ may not be perfect, either. So we repeat the process: A'' , A''' and so on. More precisely, we form a sequence $A^{(0)} = A$, $A^{(n+1)} = A^{(n)'}$ for all $n \in \mathbb{N}$.

First Species

Cantor called a set A a set of first species iff $A^{(n)} = \emptyset$ for some natural number n.

Theorem (Cantor 1871)

Every set of first species is a set of uniqueness.

As it turns out, this is just the tip of an iceberg.

There is no guarantee that $A^{(n)}$ will be perfect for any $n \in \mathbb{N}$.

How do we continue our pruning process beyond the first infinitely many stages?

It's a fair guess that we should consider $\bigcap A_n$ next. So we define

$$A^{(\infty)} = \bigcap_{n<\omega} A^{(n)}$$

Alas, $A^{(\infty)}$ may not be perfect either.

We could continue by setting

$$A^{(\infty+1)} = \big(A^{(\infty)}\big)'$$

and similarly $A^{(\infty+2)}$ and $A^{(\infty+n)}$ in general. As one might suspect, we wind up with $A^{(\infty+\infty)}$ and still no end in sight.

At this point one should worry about the actual, precise meaning of expressions like $\infty + \infty$, 5∞, $\infty \cdot \infty$ and so on.

Ordinals

The goal is to define a collection of numbers called ordinals that extends the naturals and comes equipped with

- a natural order, and
- arithmetic operations like addition, multiplication, exponentiation.

Everything should be a strict generalization of the natural numbers.

This extension will be radically different from others like $\mathbb{Q}$ or $\mathbb{R}$, the emphasis is on order properties rather than arithmetic.

Ordinals, Intuitively

Think of the naturals as a total order $\langle \mathbb{N}; < \rangle$.

$$0, 1, 2, \ldots, n, n{+}1, \ldots$$

There are exactly two kinds of elements in this order:

- 0, the minimum element
- successors

Here y is the successor of x (and x the predecessor of y) if

$$x < y \wedge \neg \exists\, z\, (x < z < y)$$

We append a new ordinal ω at the end and we get[†]

$$0, 1, 2, \ldots, n, n{+}1, \ldots\ \omega$$

Note that ω is neither 0 nor a successor, it is the first limit ordinal.

The ordinals $\alpha < \omega$ are exactly all the natural numbers.

[†] The ellipsis is one of the most consistently abused symbols in all of math. Here the first one stands for finitely many steps, but the second one for infinitely many.

Once we have reached level ω, we can take one more step and get to the successor, written $\omega + 1$.

And then to $\omega + 2$, . . . , $\omega + n$, and so on. If we keep going, we finally wind up at

$$0, 1, 2, \ldots, n, \ldots \ \omega, \omega{+}1, \omega{+}2, \ldots, \omega{+}n, \ldots \mid$$

The new limit ordinal is called $\omega + \omega = \omega \cdot 2$.
So we have two infinite blocks, one after the other†.

† Careful, $2 \cdot \omega = \omega$ when one defines ordinal arithmetic formally, see below.

And More . . .

You guessed it, we can also get $\omega + \omega + \omega = \omega \cdot 3$.

In fact, we can get an infinite sequence of increasing limit ordinals

$$\omega, \omega \cdot 2, \omega \cdot 3, \dots, \omega \cdot n, \dots \mid$$

We denote this level by $\omega \cdot \omega = \omega^2$: ω many blocks of size ω each.

And we can get ω^k for any natural number $k^{\dagger}$.

[†] In fact, this is just the very tip of the iceberg, but it's enough for our purposes.

Let's ignore the problem of arithmetic and just consider the order types[†].

The orders ω^k can be obtained cheaply by sorting $\mathbb{N}^k$ in the standard lexicographic way.

E.g., $\mathbb{N}^2$ produces the same order as ω^2:

$$\begin{aligned} (u,x) &< (u,y) \qquad \text{for all } x < y \\ (u,x) &< (v,y) \qquad \text{for all } u < v \end{aligned}$$

[†]Classes of all order-isomorphic linear orders.

Alternatively

We could also rearrange $\mathbb{N}$ to obtain orders that are isomorphic to (some small) ordinals.

E.g., if we sort $\mathbb{N}$ first by even/odd and then by the standard order, we get $\omega + \omega$:

$$0, 2, 4, \ldots, 2n, \ldots \mid 1, 3, 4, \ldots, 2n{+}1, \ldots \mid$$

We can get $\omega + \omega + \omega$ by

$$0, 3, 6, \ldots, 3n, \ldots \mid 1, 4, 7, \ldots, 3n{+}1, \ldots \mid 2, 5, 8, \ldots, 3n{+}2, \ldots \mid$$

and similarly $\omega \cdot n$ for any n.

Exercise

Find a rearrangement for ω^2.

It is not hard to show that every non-empty perfect set has cardinality $2^{\aleph_0}$.

Cantor could prove that every uncountable closed set of reals has a non-empty perfect subset, obtained by transfinite iteration of his derivative.

The process removes only countably many points from the set, so this result shows that the continuum hypothesis holds for closed sets.

The key property of the class of ordinals is that it is a well-order with respect to $\in$. To find an implementation of ordinals in set theory it is thus natural to try to find a set N_α for each ordinal α that represents α in some natural way.

By natural we mean that the structure

$$\langle N_\alpha, \in \rangle$$

should be a well-order that is order-isomorphic to α.

So, for any well-order $\langle A, \lhd \rangle$ there is an isomorphic order of the form $\langle N_\alpha, \in \rangle$, we don't need any order relation more complicated than $\in$.

J. von Neumann
Zur Einführung der transfiniten Zahlen
Acta Litt. Acad. Sci. Hung., 1 (1923) 199–208

J. von Neumann
Eine Axiomatisierung der Mengenlehre
J. Reine und Angewandte Math., 154 (1925) 219–240

Von Neumann carried out this project in the 1920s.

Well-Orders

Definition

A structure $\langle A, \lhd \rangle$ is a well-order if $\lhd$ is a total order on A and every non-empty subset of A has a $\lhd$-minimal element.

All the ordinal examples from above are well-orders. The integers and the positive rationals are the standard counterexample.

Lemma (ZFC)

$\langle A, \lhd \rangle$ is a well-order iff there are no infinite descending chains.

In other words, we must not have a sequence

$$a_0 \rhd a_1 \rhd a_2 \rhd \ldots \rhd a_n \rhd \ldots$$

Do They Exist?

A key question in the early development of set theory was whether every set can be well-ordered. In fact, Hilbert placed well-ordering the reals on top of his famous list of open problems in 1900.

Cantor thought this was "self-evident" because we construct the required order in stages. To well-order A construct a sequence a_α in stages

$$a_\alpha = \text{ pick some } x \in (A - \{\, x_\beta \mid \beta < \alpha \,\})$$

Two problems: one needs to explain exactly what the "stages" are, and how the "pick some" operations is supposed to work: A is an abstract set and there is no clear mechanism how this choice should be made. We need the right set theory axiom: the Axiom of Choice.

The following principles are equivalent to the Axiom of Choice, but often easier to apply in concrete situations.

- The Well-Ordering Principle: every set can be well-ordered.
- Zorn's lemma: every partial order in which every chain has an upper bound contains a maximal element.
- Hausdorff's Maximality Principle: every partial order has a maximal chain.
- Every equivalence relation has a set of representatives.

Zorn's lemma was popularized in particular by Bourbaki (who was allergic to most of logic).

Is it True?

> The Axiom of Choice is obviously true,
> the Well-Ordering Principle obviously false,
> and who can tell about Zorn's Lemma?
>
> Jerry Bona

Logic sometimes coexists uneasily with psychology.

Transitivity

If we use $\in$ as the underlying order we must have the following (order relations are transitive):

$$z \in y \in x \text{ implies } z \in x.$$

This property warrants a definition of its own.

Definition

A set x is transitive if $z \in y \in x$ implies $z \in x$.

Thus, x is transitive iff $\bigcup x \subseteq x$ iff $x \subseteq \mathfrak{P}(x)$.

The empty set is transitive, $x \cup \{x\}$ is transitive whenever x is, and the union of transitive sets is transitive.

Transitive Closure

As an aside, we note that for any set x we can construct the least transitive superset y, by induction on $\in$.

Definition

The transitive closure of x is defined to be

$$\mathsf{TC}(x) = \bigcap \{\, y \supseteq x \mid y \text{ transitive} \,\}.$$

We can define the transitive closure operator in by recursion along $\in$:

$$\begin{aligned} \mathsf{TC}(\emptyset) &= \emptyset \\ \mathsf{TC}(x) &= x \cup \bigcup_{z \in x} \mathsf{TC}(z) \end{aligned}$$

Exercise

Show that $y \subseteq x$ *implies* $\mathsf{TC}(y) \subseteq \mathsf{TC}(x)$.

Exercise

Show that the recursive definition works as advertised.

We define the von Neumann ordinals N_α by induction:

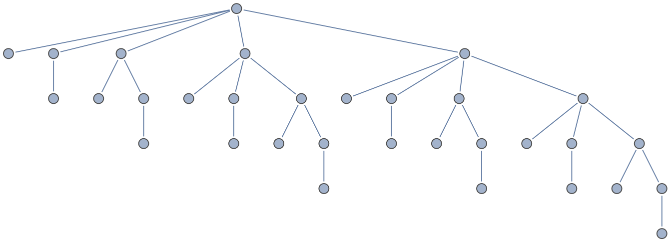
$$\begin{aligned} N_0 &= \emptyset \\ N_{\alpha+1} &= S(N_\alpha) \\ N_\lambda &= \bigcup_{\alpha<\lambda} N_\alpha \end{aligned}$$

We write $On = \{\, \alpha \mid \alpha \text{ ordinal} \,\}$ for the collection of all ordinals.

It follows that

$$N_a = \{\, N_\beta \mid \beta < \alpha \,\}$$

a fairly natural idea.



The leaves correspond to $\emptyset$, the edges indicate membership.

Characterization

Theorem

An ordinal is a set that is transitive and is well-ordered by the element-of relation $\in$.

We could have used this as a concise definition. Alas, this would be rather too opaque.

Ordinals are well-ordered, so the classical principle of induction on $\mathbb{N}$ extends to On.

Definition

Let $\Phi(x)$ be some property of ordinals. Φ is inductive if

$$\forall\, \beta{<}\alpha\ \Phi(\beta) \Rightarrow \Phi(\alpha)$$

Theorem

If $\Phi(x)$ is inductive, then $\Phi(\alpha)$ holds for all $\alpha \in On$.

Transfinite Recursion

Likewise, we can define "recursive" functions on the ordinals, using recursion much in the same way as for natural numbers.

Theorem (von Neumann, 1923, 1928)

Given a function $F(x, y)$ defined on sets, there is a unique function $f : V \times On \to V$ defined by

$$f(x, \alpha) = F(x, \lambda z f(x, z) \restriction \alpha).$$

Note that this definition works for all ordinals α. In practice, it may still be convenient to distinguish between arguments 0, successor ordinals and limit ordinals.

Example: An Order Isomorphism

Consider the well-ordering $A = \langle \mathbb{N} \times \mathbb{N}, < \rangle$ with the usual lexicographic order. We can define an order isomorphism $f : \lambda \to A$ for some limit ordinal λ by transfinite recursion:

$$f(\nu) = \min_{<}(A - \{ f(\alpha) \mid \alpha < \nu \})$$

The right hand side is well-defined as long as $A \neq \{ f(\alpha) \mid \alpha < \nu \}$. In other words, we can think of $f : On \nrightarrow A$ as a partial function whose domain is some initial segment λ of On.

This ordinal λ is the order type of our well-order; λ must be a limit ordinal since A has no largest element.

Order Type

In this case it is not hard to given an explicit description of f and λ.

Note that any ordinal below ω^2 can be written uniquely in the form $\alpha = \omega^n + m$ for $n, m < \omega$. Then

$$f(\alpha) = (n, m)$$

and $\lambda = \omega^2$: we have $A = \{ f(\alpha) \mid \alpha < \omega^2 \}$ and the recursion stops.

Exercise

What is the order type of $\mathbb{N}^k$ under lexicographic order?

Another Well-Order

Here is another well-order: $B = \langle \mathbb{N}_+, \prec \rangle$ where

$$n \prec m \iff \nu_2(n) < \nu_2(m) \vee \nu_2(n) = \nu_2(m) \wedge n < m.$$

Here $\nu_2(x)$ is the dyadic valuation of x, the highest power of 2 which divides x. Thus

$$1 \prec 3 \prec 5 \prec \ldots 2 \prec 6 \prec 10 \prec \ldots 4 \prec 12 \prec 20 \prec \ldots$$

Exercise

Determine the order type of B.

$$
\begin{aligned}
\alpha + 0 &= \alpha \\
\alpha + \beta' &= (\alpha + \beta)' \\
\alpha + \lambda &= \sup\{\, \alpha + \beta \mid \beta < \lambda \,\} \\[1ex]
\alpha \cdot 0 &= 0 \\
\alpha \cdot \beta' &= \alpha \cdot \beta + \alpha \\
\alpha \cdot \lambda &= \sup\{\, \alpha \cdot \beta \mid \beta < \lambda \,\} \\[1ex]
\alpha^0 &= 0' \\
\alpha^{\beta'} &= \alpha^\beta \cdot \alpha \\
\alpha^\lambda &= \sup\{\, \alpha^\beta \mid \beta < \lambda \,\}
\end{aligned}
$$

For clarity, we have written α' for the successor of α.

Properties

The arithmetic operations follow the Dedekind definitions on the natural numbers.

One can show that addition and multiplication of ordinals are both associative, but they fail to be commutative.

For example, letting $1 = 0'$, $2 = 1'$, we have

$$1 + \omega = \omega \neq \omega + 1$$

Similarly

$$2 \cdot \omega = \omega \neq \omega \cdot 2 = \omega + \omega$$

It may happen that $\omega^\alpha = \alpha$.

Apart from the analogy to Dedekind's approach, it is worth pointing out that they faithfully represent certain natural operations on well-orders.

Suppose A and B are well-orders of order type α and β, respectively. Let

$$C = \{\, (0,a) \mid a \in A \,\} \cup \{\, (1,b) \mid b \in B \,\}$$

be the disjoint union of A and B. Order C by "first A, then B". Then C is a well-order of order type of C is $\alpha + \beta$.

Likewise, setting $C = A \times B$ and ordering this set lexicographically produces a well-order of order type $\alpha \cdot \beta$.

Fixed Points

Ordinals have a very important fixed point property.

Suppose $f : On \to On$ is a normal function:

- strictly increasing: $f(\alpha) < f(\beta)$ for all $\alpha < \beta$
- continuous: $f(\lambda) = \sup(f(\alpha) \mid \alpha < \beta$

Claim: There is an ordinal γ such that $f(\gamma) = \gamma$.

To see why, define $\alpha_0 = 0$, $\alpha_{k+1} = f(\alpha_k)$ and set $\gamma = \sup \alpha_k$.

One important type of fixed are so-called ε-numbers, the fixed points of the map $\alpha \mapsto \omega^\alpha$.

The least ε-number is epsilon naught:

$$\varepsilon_0 = \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}$$

Note that an ε-number is closed with respect to exponentiation.

There are lots of ε-numbers, in fact just as many as there are ordinals.

Cantor Normal Form

For ordinals $\alpha < \varepsilon_0$, Cantor has established the following normal form:

$$\alpha = \omega^{\alpha_1} + \omega^{\alpha_2} + \ldots + \omega^{\alpha_k}$$

where $\alpha > \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k$.

Given this constraint on exponents, the normal form is indeed unique.

Without the condition $\alpha > \alpha_1$ the normal form holds in general, but is typically less useful.

Enumerating a Well-Order

Unlike arbitrary total orders, well-orders are always comparable in a very strict sense.

Suppose $\langle A, \lhd \rangle$ is a well-order. We can enumerate the elements of A by constructing a partial function $f : On \to A$ defined by

$$
\begin{aligned}
f(0) &= \min_{\lhd}(A) \\
f(\alpha+1) &= \min_{\lhd}(A - \{f(0), \ldots, f(\alpha)\}) \\
f(\lambda) &= \min_{\lhd}(A - \{\, f(\nu) \mid \nu < \lambda \,\}
\end{aligned}
$$

Note the domain of f is an initial segment of On, so we get an order isomorphism

$$f : \{\, \alpha \in On \mid \alpha < \beta \,\} \to A$$

Length of a Well-Order

The ordinal β is uniquely determined by A, so we can propose the following measure of the length of a well-order.

Definition

The ordinal β such that that there is an order isomorphism from $\{\, \alpha \in On \mid \alpha < \beta \,\}$ to A is the order type or length of A.

The length of a well-order is a successor ordinal if the order has a largest element, and a limit ordinal otherwise.

Note that any two well-orders are comparable in the sense that one must be isomorphic to an initial segment of the other.

One application of ordinals is to show termination of recursive functions.

For simple functions like the factorial plain induction on $\mathbb{N}$ is sufficient. But consider the Ackermann function $A : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by a double recursion:

$$\begin{aligned} A(0, y) &= y^+ \\ A(x^+, 0) &= A(x, 1) \\ A(x^+, y^+) &= A(x, A(x^+, y)) \end{aligned}$$

The standard totality proof for A is by induction on x and subinduction on y. This is fine, but a bit clunky.

Why Bother?

It may seem that any recursive definition along the lines of Ackermann's is automatically guaranteed to produce a well-defined, total function. Alas, that's most emphatically not the case.

$$\begin{aligned} f(0,0) &= 0 \\ f(x+1,y) &= f(x,y+1) \end{aligned}$$

$$\begin{aligned} g(x,0) &= x+1 \\ g(x,y+1) &= g(x+1,y) \\ g(x+1,y+1) &= g(x,g(x,y)) \end{aligned}$$

Exercise

Explain what goes wrong in the definitions of f and g above.

Recall that ω^2 can be modeled by ordering $\mathbb{N} \times \mathbb{N}$. lexicographically:

$$(a, b) \prec (c, d) \iff (a < c) \vee (a = c \wedge b < d).$$

It is easy to see that whenever a call to (x', y') is nested inside a call to (x, y) we have $Y \prec X$:

$$\begin{aligned} (x, 0) &\succ (x - 1, 1) \\ (x, y) &\succ (x, y - 1), (x - 1, z) \end{aligned}$$

Thus termination is guaranteed: there are no infinite descending chains.

Multiple Recursion

General recursive functions as defined by Herbrand-Gödel can have any number k of recursion variables; Ackermann is an example for $k = 2$.

To prove termination, if it holds at all, one can either use deeply nested subinductions, or one can use induction over ω^k.

This can easily be formalized in Dedekind-Peano arithmetic, so the proofs are not exceedingly complicated.

But: one cannot establish induction to ε_0 since this would contradict Gödel's incompleteness theorem (it would prove consistency.

Fundamental Sequences

We can exploit Cantor normal form to define, for each limit ordinal $\lambda < \varepsilon_0$, a "natural" increasing sequence $\lambda[n]$, where $n < \omega$, such that $\lambda = \lim \lambda[n]$.

- For $\lambda = \omega^{\alpha_1} + \omega^{\alpha_2} + \ldots + \omega^{\alpha_k}$ we set

  $\alpha[n] = \omega^{\alpha_1}[n] + \omega^{\alpha_2}[n] + \ldots + \omega^{\alpha_k}[n]$.

- For $\lambda = \omega^{\alpha+1}$ we set $\lambda[n] = \omega^{\alpha} \cdot n$.

- For $\lambda = \omega^{\alpha}$, α limit, we set $\lambda[n] = \omega^{\alpha[n]}$.

We can now define functions $W_\alpha : \mathbb{N} \to \mathbb{N}$ for all $\alpha < \varepsilon_0$.

$$W_0(x) = x + 1$$
$$W_{\alpha+1}(x) = W_\alpha^x(x)$$
$$W_\lambda(x) = W_{\lambda[x]}(x)$$

The first ω levels are quite similar to the classical Ackermann function. But then all hell breaks lose.

Note that one can define W_{ε_0} in a similar fashion, one just needs to fix a fundamental sequence for ε_0.

Wainer versus Dedekind-Peano

For $\alpha < \beta$, W_β dominates W_α: for all sufficiently large x we have $W_\beta(x) > W_\alpha(x)$. Much larger, indeed.

All the W_α are computable, and can be shown to be total in (DPA).

W_{ε_0} is also total, but this is no longer provable in (DPA).

A first attempt to define cardinals in terms of sets is to set

$$|A| = \{ x \mid x \approx A \}$$

Unfortunately, this collection of sets fails to be a set itself, it is proper class (it has the same size as V, the collection of all sets).

One can work around this issue by using Scott's trick: instead of collecting all sets of the same size, only use those of minimal rank. Still, one might try to find a better description of cardinality.

The following approach is due to von Neumann.

We can use ordinals to define cardinals.

Clearly, any well-order of order type ω requires a countably infinite carrier set. But countable sets suffice to build well-orders of higher order types such as $\omega + \omega$, $\omega \cdot \omega$, ω^ω. Even ε_0 can easily be squeezed into a countable carrier set.

This suggests the following definition.

Definition

An ordinal κ is a cardinal if the carrier set of any well-order of length κ is larger than the carrier set of any well-order of length $\alpha < \kappa$.

This may sound a bit complicated, but it all comes down to insisting that there is no injection

$$\kappa \to \alpha$$

for any $\alpha < \kappa$ in On.

Informally, cardinalities jump when the length of a well-order reaches κ.

Thus, all countable ordinals map to ω, but the first uncountable one does not and is thus a cardinal.

What can we say about cardinals $Card \subseteq On$?

First off, there are the finite cardinals which consist of all ordinals less than ω – if you like, you can identify these with the natural numbers.

The first infinite cardinal is ω, though in its capacity as a cardinal rather than just plain ordinal it is often written

$$\aleph_0 \qquad \text{aleph naught}$$

This notation is, of course, due to Cantor.

But things do not end there. In fact, one can show that $Card$ is a well-ordered subclass of On, so we actually have a sequence

$$\aleph_0, \aleph_1, \ldots \aleph_\omega, \ldots \aleph_{\omega+\omega}, \ldots \aleph_{\omega^2}, \ldots \aleph_{\varepsilon_0}, \ldots, \aleph_{\aleph_1}, \ldots$$

If you find this vertigo-inducing you are quite right.

Note that for our examples, $\aleph_\alpha$ is always much, much larger than α.

Alas, one can prove in ordinary set theory that there is a cardinal κ such that

$$\aleph_\kappa = \kappa$$

To wit, κ can be chosen to be the least ordinal larger than $f(n)$ for all $n < \omega$ where $f(0) = 0$ and $f(n+1) = \aleph_{f(n)}$.

Take a moment to figure out what this means. This number is mind-numbingly large.

Actually, not all that big . . .

In modern set theory, κ would be considered fairly small.

It is standard to study inaccessible cardinals $\alpha = \aleph_\alpha$:

- for all $\beta < \alpha$: $2^\beta < \alpha$,
- α is regular: no sequence of length $\lambda < \alpha$ has limit α.

Note that our κ from the last slide is the limit of a sequence of puny length ω.

One huge difference here is that ZFC cannot prove the existence of an inaccessible cardinal (otherwise ZFC could prove its own consistency, contradicting Gödel's theorem).

This never-ending stream of alephs is important in axiomatic set theory, but for us only the first few items are relevant.

The natural numbers (aka finite ordinals) form the first number class. The countable ordinals form the second number class. Thus $\aleph_1$ is the least uncountable ordinal, the least ordinal that does not belong to the second number class.

One can show that

$$\aleph_1 \leq |\mathbb{R}|$$

but equality, Cantor's famous Continuum Hypothesis, the other top entry on Hilbert's list, cannot be settled in the framework of standard set theory. So it is safe to assume that $\aleph_1 = |\mathbb{R}|$ or that $\aleph_1 < |\mathbb{R}|$. Unlike with the Axiom of Choice, neither option seems to be particularly natural or consequential, so the Continuum Hypothesis has not been adopted as a standard axiom.

One should note the arithmetic of cardinals is different from the arithmetic of ordinals. We won't give a detailed definition of the arithmetic operations; they are supposed to represent the effect on cardinality of disjoint union, Cartesian product and function space formation. For example, $\aleph_0 + \aleph_0 = \aleph_0$: it is easy to construct a bijection between $\mathbb{N}$ and two disjoint copies of $\mathbb{N}$.

One can show that addition and multiplication of cardinals are associative, commutative operations. Moreover

Lemma

Let λ and κ be two cardinals, at least one of them infinite and neither 0. Then

$$\lambda + \kappa = \lambda \cdot \kappa = \max(\lambda, \kappa).$$

Note that exponentiation is not mentioned here; in fact $2^\kappa > \kappa$ for all cardinals κ as Cantor's diagonal argument shows.

One can also derive some basic properties of inequalities between cardinalities.

Lemma

Let $\lambda \leq \kappa$ *and* $\lambda' \leq \kappa'$*. Then*

$$\lambda + \lambda' \leq \kappa + \kappa' \text{ and } \lambda \cdot \lambda' \leq \kappa \cdot \kappa'$$

Unless $\lambda = \lambda' = \kappa = 0 < \kappa'$ *we also have*

$$\lambda^{\lambda'} \leq \kappa^{\kappa'}.$$

There are also infinitary version of the arithmetic operations. Here is one famous result. Let I be an arbitrary index set.

Theorem (König 1904)

Let $\lambda_i < \kappa_i$ for all $i \in I$. Then

$$\sum_i \lambda_i < \prod_i \kappa_i$$

This requires the Axiom of Choice.

One interesting property of the second number class is that all its elements can be approximated by sequences of length ω.

Definition

Define the cofinality of a limit ordinal α to be the length of a shortest sequence that is cofinal in $\{ \beta \in On \mid \beta < \alpha \}$. Ordinal α is regular if it has cofinality α.
In symbols: $\operatorname{cof} \alpha$.

Note that $\omega + \omega$, ω^2 and the like all have cofinality ω.

Lemma

All limit ordinal in the second number class have cofinality ω. But $\aleph_1$ is regular.

Cofinality and König

Lemma

Let $\aleph_0 \leq \kappa$ *and* $2 \leq \lambda$. *Then*

1. $\kappa < \kappa^{\mathrm{cof}\,\kappa}$
2. $\kappa < \mathrm{cof}\,\lambda^{\kappa}$

Proof.

Let $\alpha_\nu < \kappa$, $\nu < \mathrm{cof}\,\kappa$, be a sequence of ordinals with limit κ. By König

$$\kappa = \sum \alpha_\nu < \prod \kappa = \kappa^{\mathrm{cof}\,\kappa}$$

For the secone part, set $\mu = \lambda^\kappa$ and assume $\mathrm{cof}\,\mu \leq \kappa$. Then by part (1)

$$\mu < \mu^{\mathrm{cof}\,\mu} \leq \mu^\kappa = (\lambda^\kappa)^\kappa = \lambda^\kappa = \mu$$

contradiction. $\square$

One of the reason the last lemma is interesting is that it shows

$$\aleph_\omega \neq 2^{\aleph_0}$$

The Continuum Hypothesis asks about the value of $\mathfrak{c} = 2^{\aleph_0}$.

Solovay has shown that this constraint (and analogous ones obtained in a similar manner) are the only obstructions to choosing a value for $\mathfrak{c}$, anything else can be realized in suitable models for ZFC.

So far we have carefully avoided explaining how to represent ordinals and cardinals as sets and instead used concepts such as "stage of construction", "well-ordering" and so on. Here is one way to represent ordinals as ordinary pure sets due to von Neumann. All the assertions we have made above can then be proven in set theory.

We want to define a set N_α for each ordinal α that represents α in some natural way.

First off, what is meant by natural here is that the structure

$$\langle N_\alpha, \in \rangle$$

ought to be a well-ordering of order type α.

So, for any well-ordering $\langle A, < \rangle$ there is an isomorphic order of the form $\langle N_\alpha, \in \rangle$, we don't need any order relation more complicated than $\in$.

Transitivity

If we use $\in$ as the underlying order we must have the following (order relations are transitive):

$$z \in y \in x \text{ implies } z \in x.$$

This property warrants a definition of its own.

Definition

A set x is transitive if $z \in y \in x$ implies $z \in x$.

Thus, x is transitive iff $\bigcup x \subseteq x$ iff $x \subseteq \mathfrak{P}(x)$.

Structures that can be represented by transitive sets together with $\in$ play an important role in set theory, but we won't pursue the issue here.

As an aside, we note that for any set x we can construct the least transitive superset y, by induction on $\in$.

Definition

The transitive closure of x is defined to be

$$\mathsf{TC}(x) = \bigcap \{\, y \supseteq x \mid y \text{ transitive} \,\}.$$

We can define the transitive closure operator in by recursion along $\in$:

$$\begin{aligned} \mathsf{TC}(\emptyset) &= \emptyset \\ \mathsf{TC}(x) &= x \cup \bigcup_{z \in x} \mathsf{TC}(z) \end{aligned}$$

Exercise

Show that $y \subseteq x$ *implies* $\mathsf{TC}(y) \subseteq \mathsf{TC}(x)$.

Exercise

Show that the recursive definition works as advertised.

So, what should the von Neumann ordinals look like?

Needless to say, we start with $N_0 = \emptyset$.

The successor function on sets is defined by

$$S(x) = x \cup \{x\}$$

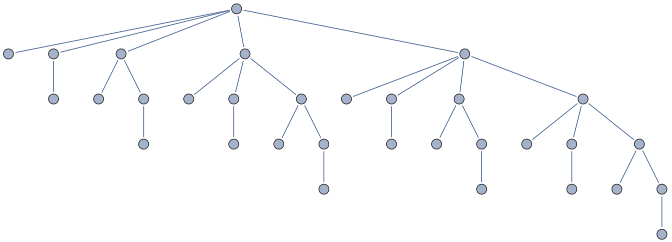
and $S(N_\alpha)$ is the successor of N_α.

So the question is what to do with limit ordinals. The answer is fairly simple: take the union of all earlier von Neumann ordinals. So

$$N_\lambda = \bigcup_{\alpha<\lambda} N_\alpha$$

Exercise

Show that the von Neumann ordinals are in fact well-ordered by $\in$.



The leaves correspond to $\emptyset$, the edges indicate membership. This tree emphatically does not share common subexpressions.

Not every transitive set represents an ordinal, but there is a nice way to characterize these sets.

Lemma

The von Neumann ordinals are precisely the transitive sets that are well-ordered by the element relationship $\in$.

Proof. An easy induction shows that all von Neumann ordinals are well-ordered by $\in$.

For the opposite direction use induction on α to show that the element in $\langle A, \in \rangle$ of rank α must be N_α (transitivity is crucial). $\Box$

Exercise

Show that every element of a von Neumann ordinal is a von Neumann ordinal using as definition "transitive and $\in$-well-ordered".

Implementation versus Definition

The last lemma is rather neat; to represent well-orders in set theory we can dispense with specific order relations and just use ε. The only thing that changes is the carrier set, which has rather nice properties itself.

We note that some authors define ordinals in terms of their set-theoretic implementation. While formally correct, this approach is rather dubious since it obscures the intended properties of ordinals – they all have to be tediously discovered after the fact. Some would also gripe that this method over-emphasizes set theory.

At any rate, the crucial idea is to generalize inductive proofs and/or definitions to well-orderings other than just the natural numbers. Ordinals represent the stages in these arguments and constructions.

Summary

- Ordinals capture the notion of stages in an inductive process, including the transfinite case.
- Alternatively, we can think of them as the order types of all well-orderings.
- Ordinals can be implemented in set theory as transitive sets that are well-ordered by ε.
- Using transitive induction, one can define ordinal arithmetic which corresponds naturally to operations on the well-orderings.
- Cardinals are special types of ordinals, and carry their own, rather messy arithmetic.