1. Büchi Arithmetic (20)

Background

In class we have seen an extension of Presburger arithmetic by a divisibility predicate that is still automatic. Here is a very similar but different extension that also preserves automaticity, referred to as Büchi arithmetic.

Let $m \geq 2$ be some fixed integer and let $V_m : \mathbb{N}_+ \to \mathbb{N}$ defined by

$$V_m(x) = m^{\max\left(k \geq 0 \mid m^k | x \right)}$$

As usual, we can also think of V_m as a binary relation. We will consider the structure $\mathfrak{N}_{V,m} = \langle \mathbb{N}; +, V_m \rangle$.

Here are two closely related functions/relations that can be expressed in Büchi arithmetic. Let $\mathsf{nxt}(x) = m^e$ iff e is minimal such that $x < m^e$. For all $0 \leq d < m$, define a binary relation $\mathsf{dig}_{m,d}(x, y)$ as follows: "$x = m^k$ is a power of m and y, written in base m, has a term $d \cdot m^k$."

Task

A. Show that the function nxt is definable in $\mathfrak{N}_{V,m}$.

B. Show that the relation $\mathsf{dig}_{m,d}(x, y)$ is definable in $\mathfrak{N}_{V,m}$.

C. Show that the first-order theory of $\mathfrak{N}_{V,m}$ is decidable.

D. How about $\mathfrak{N}_{m,m'} = \langle \mathbb{N}; +, V_m, V_{m'} \rangle$?

Comment

For the last part, no proof is necessary, just state your gut feeling. The actual argument is quite hard.

2. Automatic Interpretations (40)

Background

Recall the two universal automatic structures discussed in class (we only consider the case $m = 2$):

$$\mathfrak{N}_2 = \langle \mathbb{N}; +, |_2 \rangle$$
$$\mathsf{tree}_2 = \langle \varDelta_2{}^{\star}; \sigma_0, \sigma_1, \prec, \mathsf{el} \rangle$$

As usual, we use reverse base-2 as our numeration system. Here it is convenient to allow ε as a notation for zero, and to allow trailing zeros.

Task

A. We sketched a proof that tree_2 can be interpreted in $\mathfrak{N}_2$. Show in detail that the proposed interpretation actually works.

B. Show that all the following relations are definable in tree_2.

   (a) Length-order $|x| \leq |y|$.

   (b) Digit predicate: $\mathsf{dig}_d(x, y)$ meaning "the digit of x in position $|y|$ is d."

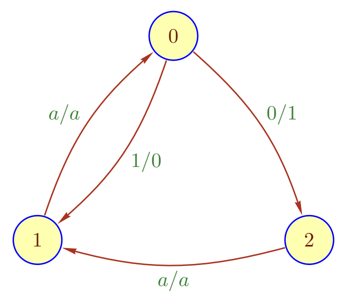
C. Show that $\mathfrak{N}_2$ can be interpreted in tree_2.

Comment The purpose of part (B) is to make it a bit easier to write down the formula for part (C). Needless to say, there are different ways to go about the details.

3. Mystery Transducer (40)

Background

Automatic structures are one way to make concrete the idea of using finite state machines to describe mathematical structures. There are other options, though, in particular in algebra. Consider the following alphabetic transducer $\mathcal{C}$ over the binary alphabet $\mathbf{2}$.



Each state p in $\mathcal{C}$ determines a transduction $\mathcal{C}(p) : \mathbf{2}^\star \to \mathbf{2}^\star$ if we think of p as the initial state and all states as terminal (these devices are also known as output modules). For simplicity we simple write $\underline{p}$ for $\mathcal{C}(p)$.

So $\langle \mathbf{2}^\star; \underline{0}, \underline{1}, \underline{2} \rangle$ is an automatic structure, but that turns out to be boring. Here is a better angle: consider the maps $\mathbf{2}^\star \to \mathbf{2}^\star$ obtained by composing the three basic maps in all possible ways. In other words, let S be the semigroup generated by $\underline{0}, \underline{1}, \underline{2}$. One can check that the $\underline{p}$ are bijections. Hence one can also define a group G generated by these maps (just add all the inverse maps $\underline{p}^{-1}$).

Task

A. Show that the maps $\underline{p}$ are indeed bijections. What would a transducer defining the inverse maps look like?

B. Show that S is commutative and therefore G must also be commutative.

C. Show that the identity $\underline{0}^2 \underline{1}^2 \underline{2} = I$ holds.

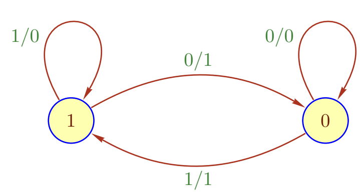
D. Conclude that S is already a group and specifically a homomorphic image of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

E. Show that the length of the orbit of any word $x \in \mathbf{2}^\star$ under any $f \in S$ is always a power of 2.

Comment

It turns out that S is actually isomorphic to $\mathbb{Z}^2$, but the proof is quite hard.

It is not clear from this simple example, but groups defined by transducers have become very important in group theory; the machines often are the most straightforward way to define various groups. Moreover, tiny automata can produce fairly complicated groups. For example,



is a famous monster and generates the Lamplighter group.