Computation and Discrete Mathematics

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1 **Administrivia**

2 **Science, Math and Crises**

3 **CDM, the Idea**

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Website and Communication

Course Website:	http://www.cs.cmu.edu/~cdm
Communication:	Ed
Syllabus:	Syllabus

Make sure to read the course syllabus carefully, I will assume you are familiar with all the rules and policies spelled out there. If you feel the instructions are not clear enough, talk to me or post on Ed.

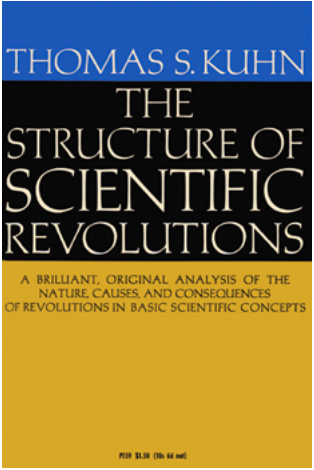
Prerequisites

There is only one official prerequisite: 15-251.

You should interpret this as meaning: "utterly comfortable with all the 251 material, and with math in general."

There is a lot of material posted on the website, if you need to brush up on some particular topic (say, Turing machines or coding functions) take a look at this stuff. And ask if there are any questions.

Digression: Thomas Kuhn



In 1962, Thomas Kuhn published an influential and hugely popular book: The Structure of Scientific Revolutions.

Kuhn takes the social and historical dimensions of science seriously and provides deep insights into the structure of scientific development.

He breaks with the idea that scientific progress is a purely cumulative process where scientific theories are simply becoming more and more refined, each layer being a logical extension of the previous one (like an onion), the refinement subsumes and improves the previous model.

Kuhn's Model

Normal Science: Puzzle-solving within an accepted framework.

Crisis: Inexplicable anomalies accumulate.

Paradigm Shift: A new paradigm replaces the old.

New Normal Science: The new becomes the old.

Instead of the onion model, things change in a discontinuous way, once the established system runs into a sufficient number of problems.

Example: Newton to Einstein

Newton's law of gravitation in *Philosophiæ Naturalis Principia Mathematica* (1687)

$$F = G\,\frac{m_1 m_2}{r^2}$$

Space and time are absolute and separate.
Gravity is a force that acts instantaneously at a distance.

This model works spectacularly well over many orders of magnitude, from falling apples to tides to projectile motion to solar system dynamics.

A huge success story, it lasted unchallenged over 200 years.

Precession of Mercury: Cannot properly account for the slow shift in Mercury's perihelion. The discrepancy is small (43 arcseconds per century) but undeniable and unexplained.

Speed of Light Constancy: Maxwell's equations (1861) implied a fixed speed of light, which is incompatible with Newtonian mechanics. The Michelson-Morley experiment (1887) failed to detect motion relative to the aether, contradicting the Newtonian model.

Conceptual Tensions: Some physicists (Leibniz, Mach, Einstein) reject instantaneous action-at-a-distance. In addition, the Newtonian model lacks a mechanism for gravity transmitting force.

General Relativity (1915)

In Einstein's model, space and time are fused into one single 4-dimensional structure spacetime, with somewhat interesting mathematical properties.

Instead of a Newtonian force, gravity arises as a curvature of spacetime.

Mass is responsible for the curvature, and the curvature is responsible for the movement of mass.

The predictive power of Einstein's model is astounding: Mercury's perihelion, gravitational lensing, gravitational redshift, gravitational time dilation, gravitational waves and black holes.

So What?

In the natural sciences, Kuhn's argument clearly has a lot of traction.

It is not so clear, though, that mathematics works the same way.
Some important differences:

- there are no pesky empirical observations that can cause a crisis,
- the mechanism of proof[†] prevents any inaccuracies and errors.

So there is never any chance for anomalies to develop, and no need for revolutions to occur. Math is an onion.

[†]There is a notion of proof in the sciences, but it is substantially different.

Or Is It?

First off, there are indeed observations, in particular in the era of cheap and fast computation.

Second, anomalies did in fact arise in the 19th century and caused quite a bit of concern among many.

Lastly, proof in the current sense is a relatively recent development (in fact a response to a crisis)[†].

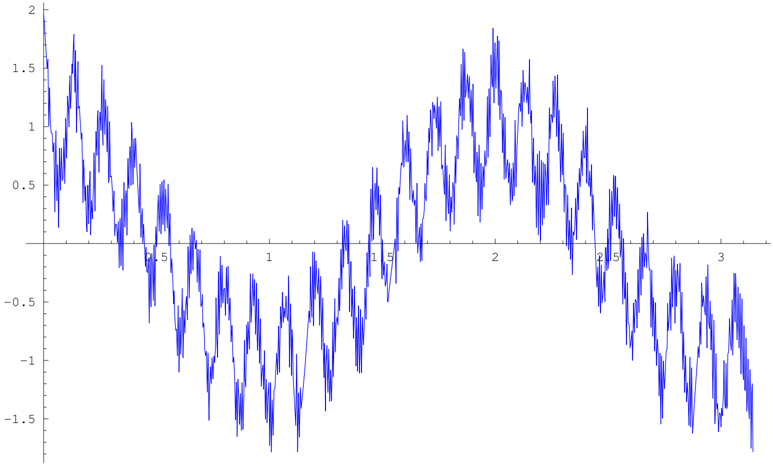
[†] "It is safe to say that no proof given at least up to 1800 in any area of mathematics, except possibly in the theory of numbers, would be regarded as satisfactory by the standards of 1900. The standards of 1900 are not acceptable today." Morris Kline

In math, anomalies are typically referred to as paradoxes or contradictions/inconsistencies.

E.g., Weierstrass showed that there are continuous functions that are nowhere differentiable. Worse, some of these can be written down as trigonometric series:

$$f(x) = \sum_{n<\infty} b^n \cos(a^n \pi x)$$

For parameter values $0 < b < 1$ and $ab > 1 + 3/2\pi$, this function f is continuous but nowhere differentiable. One can almost see from a plot how the function works.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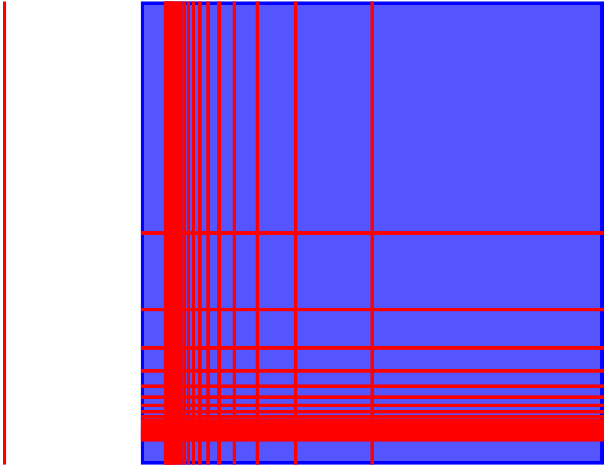


Cantor discovered that the unit-interval and the unit-square have the same size, there is a bijection between them.

Russell

$$R = \{ x \mid x \notin x \}$$

Bertrand Russell realized that unconstrained set formation can be used to produce an actual inconsistency, not just a paradoxical sets[†].

It is truly annoying that the Russell set is almost trivial to define—most sets in analysis are orders of magnitude more complicated.

[†] Thereby destroying Frege's lifework.

Poincaré and Hilbert, arguably the two leaders in the early 20th century.

> Formerly, when one invented a new function, it was to further some practical purpose; today one invents them in order to make incorrect the reasoning of our fathers, and nothing more will ever be accomplished by these inventions.

Hermite called Cantor's work "German metaphysics," not mathematics.

As usual, the establishment pushes hard to stick with the current paradigm, anomalies be damned.

Logicism: Reduce math to pure logic (Frege, Russell, Whitehead).

Formalism: Reconstruct math as a formal system, with axioms and a logical system (Peano, Hilbert).

Intuitionism: Only accept a narrow fragment of math that can be constructively verified (Brouwer, Heyting, Bishop).

Structuralism: Mathematics as the study of isomorphism-invariant structures (Bourbaki).

Russell and Whitehead

Publish Principia Mathematica in 3 volumes between 1910 and 1913.

Hugely ambitious undertaking†, some characteristics:

- Build a foundation for all of mathematics based solely on logic.
- Use a strict formal system with a clear syntax and rules.
- Use axiomatization to construct mathematics in layers.
- Avoid paradoxes via a ramified type system.

Unfortunately, the type is system is very unwieldy and ultimately compromised by the infamous axiom of reducibility.

† Russell later complained that his mind never fully recovered from the effort.

A Theorem

$*54{\cdot}43.\quad \vdash :. \alpha, \beta \in 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$

Dem.

$\vdash . *54{\cdot}26 . \supset \vdash :. \alpha = \iota`x . \beta = \iota`y . \supset : \alpha \cup \beta \in 2 . \equiv . x \neq y .$

$[*51{\cdot}231] \qquad \equiv . \iota`x \cap \iota`y = \Lambda .$

$[*13{\cdot}12] \qquad \equiv . \alpha \cap \beta = \Lambda \qquad (1)$

$\vdash . (1) . *11{\cdot}11{\cdot}35 . \supset$

$\vdash :. (\exists x, y) . \alpha = \iota`x . \beta = \iota`y . \supset : \alpha \cup \beta \in 2 . \equiv . \alpha \cap \beta = \Lambda \qquad (2)$

$\vdash . (2) . *11{\cdot}54 . *52{\cdot}1 . \supset \vdash .$ Prop

From this proposition it will follow, when arithmetical addition has been defined, that $1 + 1 = 2$.

An Issue

Principia is carefully designed to make all the paradoxes/contradictions known at the time disappear.

Alas, that leaves open the possibility that other problems might be found later. The system is far too complicated to simply rule out that possibility by some glib reference to intuition and common sense.

Also, there is the question of how far the development of PM could actually go. In particular, could it potentially cover all relevant parts of mathematics?

Hilbert proposed the use of a formal/axiomatic system as a foundation of mathematics. However, he developed a project to study formal systems with mathematical tools, including in particular proofs themselves (metamathematics, proof theory).

Consistency: Show that no contradictions arise in the system.

Completeness: Show that all true mathematical statements (in the relevant domain) are provable within the system.

Finitary Proof Methods: All the arguments can use only strictly finitary, constructive, uncontentious methods.

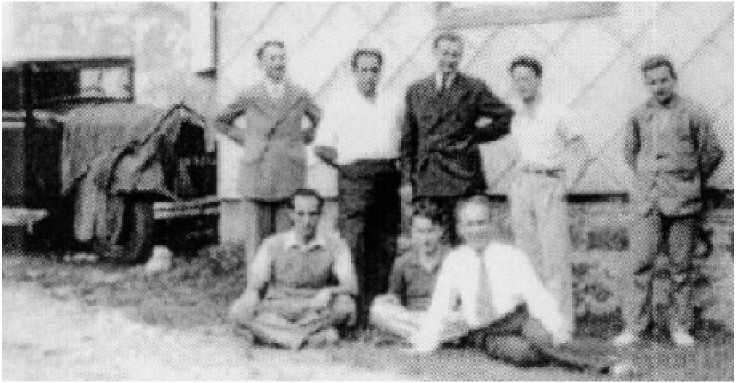
Sadly, after some initial progress, Gödel demonstrated that the program in its original form is doomed.

Intuitionism

Brouwer thinks of mathematical as a creation of the mind, ultimately arising from the intuition of space and time, and subject to strict constructive standards. It is not an objective structure independent of humans.

- An existential claim requires an explicit construction.
- The law of the excluded middle is not admissible in general.
- A mathematical truth is a proof, not a proposition that is true or false independently of being known.

One needs to reconstruct basic areas of math such as arithmetic and real analysis, adhering to these new principles.



Cartan, Dieudonné, Weil, Chevalley et al., 1935. Since then, there were some $40+$ members in total, all outstanding mathematicians.

Rebuild math on a rigorous, formal, axiomatic foundation.

Use structures as the central concept.

Present math as a unified, coherent domain.

The key difference to the other efforts:

Focus on the mathematics, not the foundational issues[†].

[†] Bourbaki pays lip-service to logic and formalism in the first volume, but ignores their own framework throughout.

Éléments de Mathématique

- I Set theory
- II Algebra
- III Topology
- IV Functions of one real variable
- V Topological vector spaces
- VI Integration
- VII Lie groups and Lie algebras
- VIII Commutative algebra
- IX Spectral theory
- X Differential and analytic manifolds
- XI Algebraic topology

Big Success: Structures

Perhaps the most important accomplishment of Bourbaki was to organize all of mathematics around the concept of a structure and to describe and analyze these structures in a rigorous, axiomatic way.

In essence, a structure is a carrier set, together with a collection of operations and relations on that set. The basic properties of the operations and relations are described by appropriate axioms and lead to several types of structures: algebraic, topological, order, . . .

There is no question that this approach has since become the de facto standard in mathematics, there is no substantial alternative.

Big Success: Unified Terminology

$\emptyset$

$A \subseteq B$

$\mathbb{N}$ $\mathbb{Z}$ $\mathbb{Q}$ $\mathbb{R}$ $\mathbb{C}$

$A \times B$ $f : A \to B$ $\langle A \rangle \subseteq G$

injective surjective bijective

filter ultrafilter

$\mathsf{Hom}(U, V)$ $\mathsf{End}(V)$ $\mathsf{GL}(n, K)$

Strangeness

- Logic is treated minimally, and exceedingly poorly.
- Problem solving is secondary to axiomatics.
- Combinatorial structure is non-essential.
- Algorithmic content is off-topic.
- Applications nowhere in sight.
- Presentation strictly linear, no external references.
- And, cela va sans dire, no pictures.

So What?

Why should anyone care? Just some math guys having innocent fun.

Because Bourbaki is currently the Holy Grail of Mathematics.

From Kuhn's perspective, this is the paradigm shift caused by the Grundlagenkrise. Not many mathematicians have actually read Bourbaki, but almost all would refer to the Elements if pressed on issues of rigor and foundations.

> It is completely clear to me which conditions caused the gradual decadence of mathematics, from its high level some 100 years ago, down to the present hopeless nadir. Degeneration of mathematics begins with the ideas of Riemann, Dedekind and Cantor which progressively repressed the reliable genius of Euler, Lagrange and Gauss. Through the influence of textbooks like those of Hasse, Schreier and van der Waerden, the new generation was seriously harmed, and the work of Bourbaki finally dealt the fatal blow.
>
> C. L. Siegel, letter to A. Weil, 1959

More Opposition

> Meanwhile, I was in a mathematics department, and this style of mathematics was not at all in fashion. Bourbaki was king: The more abstract you could be, expressing everything in terms of morphisms and categories, the better. Highly abstract methods were in favor in all the best mathematical schools. In more and more of the lectures that I was hearing at Caltech, I would find myself sitting in the audience saying to myself, "So what? So what?"
>
> Eventually I switched fields and became a professor of computer science.
>
> D. E. Knuth, 2014

Math Education

For math research, Bourbaki has a few issues, but overall the efforts was a roaring success.

For math education, though, things went sideways (the "new math" is basically a fiasco). It is hard to imagine a framework that is less suitable in a psychological and pedagogical sense. And yet, Bourbaki has deeply influenced mathematics education at all levels.

In other words:
All the math classes you have ever taken are Bourbaki's fault.

And the Rest?

For computer science, the Russell/Hilbert/Brouwer body of ideas turned out to be hugely important. It is impossible to overstate this point.

But for classical mathematics, the impact is quite small.

So far, that is.

One admittedly imperfect way to gauge the influence of logic on mathematics is to study the list of Fields medalists (starting in 1936).

There is not a whole lot. Paul Cohen (1966) worked directly in logic (independence of CH and AC), and Vladimir Voevodsky (2002) become highly engaged after the award (HoTT). Others have used methods like non-standard analysis, but that's about it.

The development of the theory of computation and of actual digital computers is another paradigm shift in mathematics and the sciences, arguably the most important one in the last century.

Are You Insane?

What about Bourbaki, category theory, quantum physics, relativity theory, modern genetics?

All true, these are major breakthroughs.

The claim is that computation is going to turn out to create even more upheaval and will ultimately change science and math in profound ways.

The Math Toolchain

- numerical computation
- symbolic computation
- visualization
- knowledge management
- communication and cooperation
- examples/counterexamples

- proof checking/searching
- generative AI

Disclaimer

We will not deal with theorem provers and proof checkers.

Those would be perfect topics for CDM, but past experience shows that there simply is not enough time.

Just two comments by late giants.

> I can't see how else it will go. I think the process will be first accepted by some small subset, then it will grow, and eventually it will become a really standard thing. The next step is when it will start to be taught at math grad schools, and then the next step is when it will be taught at the undergraduate level. That may take tens of years, I don't know, but I don't see what else could happen.
>
> Vladimir Voevodsky

In response to question about computer verified/generated proofs.

Fields Medal 2002. Astonishing connections between Martin-Löf type theory and classical homotopy theory.

> The standard of correctness and completeness necessary to get a computer program to work at all is a couple of orders of magnitude higher than the mathematical community's standard of valid proofs.
>
> Bill Thurston

Fields Medal 1982, used computers extensively in his seminal work on low-dimensional topology. Supposedly taught himself to visualize 4-dim objects.

So What's The Big Problem?

As already mentioned, Bourbaki had huge influence on math education.

> **Challenge:**
> Take a math-for-cs textbook and mark all the parts that could easily have been written 50 years ago.

That means: before the computer revolution really took off.
Make sure you have lots of markers at hand, you'll need them.

This is terrible. In effect, all you get is a rehash of the old Bourbaki style, with some combinatorics and graph theory thrown in for theatrical effect; plus a bit of bad pseudo-code.

CDM is simply an attempt to fix this:

- Focus on computation and logic.
- Select the most relevant material.
- Use the math toolchain often and systematically.

Just to be Clear

We are emphatically not proposing to ditch all the old material, quite the contrary. We just don't want to blindly copy an old syllabus.

We are not going to cancel generality and abstraction in favor of running lots of cute computational experiments. We need the big theoretical guns, avoiding them is just a waste of time.

The whole point is a shift in perspective, and the systematic use of the whole modern math toolchain.

> We shape our tools, and thereafter our tools shape us.
>
> Marshall McLuhan