CDM

Primitive Recursion

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History Angle

- Classical Recursion theory (computability theory) started in the 1930s, well before the arrival of digital computers. It arose as a critical ingredient in any attempt to handle the Grundlagenkrise (foundational crisis) in mathematics early in the 20th century. CRT is not concerned with practical computation, implementation issues or algorithms.

- Complexity theory started in the 1950/60s, in response to the increasing relevance of actual digital computation, and the need to understand resource allocation issues (analysis of algorithms). This area is in part concerned with practical, realizable computation, but the theory part of complexity theory can also be far, far removed from real computation.

Defining Computability

We need a rigorous definition of computability that is easy to understand and apply, and that matches our intuitive, pre-theoretic notion of computability.

Roughly speaking, there are two types of definitions that can be used:

- **Machine Models**
  Abstract, mathematical machines that capture the notion of a "computer" in a more or less physical sense.
- **Programms**
  A sequence of primitive instructions that can be executed in a simple, mechanical manner.

Why Bother?

Why would we even need a rigorous definition of a concept that is so utterly intuitive? Say, it's computable if there is an algorithm for it. Done.

Yes and No, but mostly: No.

A warning: in the 1930s there was tension between Church and Gödel about the proper notion of computability, the issue was finally resolved only with Turing's seminal paper.

Anything that is not obvious to these super-stars, is indeed not obvious.

Non-Computability

An informal approach is often good enough for positive results: everyone agress that the Euclidean algorithm computes gcds.

Negative results absolutely depend on real foundations: in order to show that a particular problem (say, solving Diophantine equations) fails to be computable, we need to have an airtight definition of computability.

Things get much worse when we try to show that solving a particular computable problem (say, satisfiability testing for Boolen formulae) requires such and such resources, the key concern in complexity theory.

Models of Computation

- K. Gödel: primitive recursive
- A. Church: λ-calculus
- J. Herbrand, K. Gödel: general recursiveness
- A. Turing: Turing machines
- S. C. Kleene: μ-recursive functions
- E. Post: production systems
- H. Wang: Wang machines
- A. A. Markov: Markov algorithms
- M. Minsky; J. C. Shepherdson, H. E. Sturgis: register machines
- A. Meyer, D. Ritchie; U. Schöning: loop, while programs (1967, 1992)

Comments

The models are listed roughly in historical order. Except for primitive recursive functions, they are all equivalent in a strict technical sense.

This does **not** mean that they are equally intuitive or compelling. For example, unless you have the theory-gene, you will find the λ-calculus pretty daunting.

Bad news: the second most daunting model is Turing machines. They have a beautiful motivation and are very natural in a way, but when it comes to technical details they are a nightmare.

Two Worlds

In classical computability, one works with arithmetic functions

$$f : \mathbb{N}^k \to \mathbb{N}$$

In complexity theory, one has to deal with strings instead

$$f : (\Sigma^\star)^k \to \Sigma^\star$$

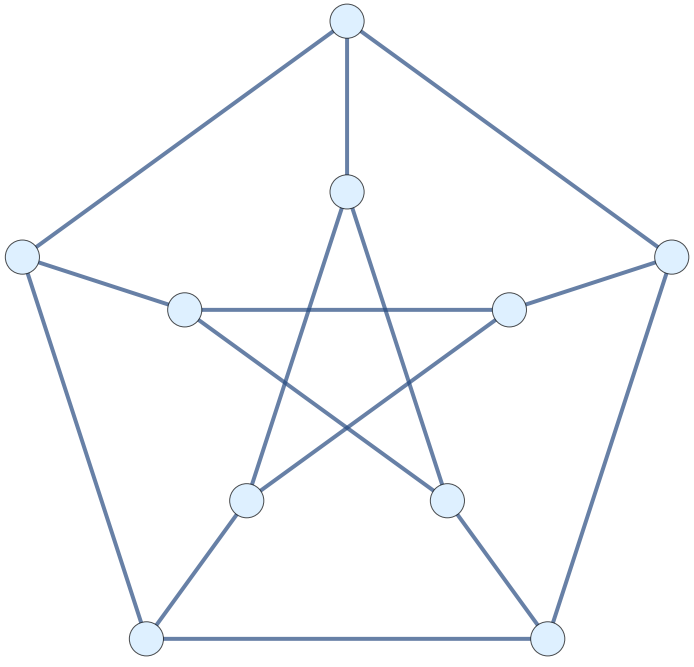
Why?

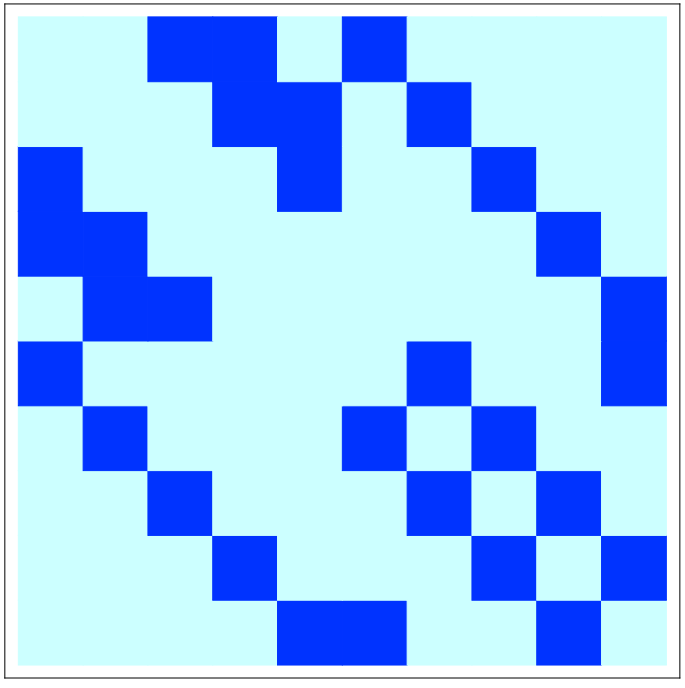
Clearly, strings can easily express natural numbers. E.g., we can us the ordinary binary expansion over the alphabet $\mathbf{2} = \{01, \}$.

We will see in a moment, it is not hard to encode strings as natural numbers. E.g., we could think of them as base k numbers where $k = |\Sigma|$.

In general, this is fine. But in low complexity classes the cost of decoding can be too high, it overwhelms the rest of the computation.

Example: Petersen Graph





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Primitive Recursion

The main idea behind our first model will be very familiar to anyone familiar with a modern programming language: we will define a function $f : \mathbb{N} \times \mathbb{N}^n \to \mathbb{N}$ by

- defining $f(0, \boldsymbol{y})$ explicitly, and
- defining $f(x+1, \boldsymbol{y})$ in terms of x, $f(x, \boldsymbol{y})$ and $\boldsymbol{y}$.

You have all seen the standard examples: addition, multiplication, exponentiation, factorials and so on. As we will see shortly, it is quite difficult to come up with an arithmetic function that is intuitively computable but not primitive recursive.

Later we will see more complicated forms of recursion.

Details

If one cares about actual implementations of primitive recursive functions, there are two basic choices.

Top-Down Implement a recursion stack so that a call to $f(n, \boldsymbol{y})$ automatically produces calls to $f(n-1, \boldsymbol{y})$, $f(n-2, \boldsymbol{y})$ ... and handles the returns properly.

Bottom-Up Compute $f(0, \boldsymbol{y})$, $f(1, \boldsymbol{y})$, $f(2, \boldsymbol{y})$ ... by iteration, which requires only a simple loop.

Again, in math this distinction does not matter much, in the early days of CS it produced some acrimonious debates[†].

[†] There were fierce fights about whether Algol 60 should include recursion. I studied with one of the people on the wrong side of the argument.

Interestingly, Gödel encountered the problem of defining computable functions working on his seminal incompleteness theorem. He introduced a class of what he called "recursive functions," that are now called primitive recursive functions.

The notion of "recursive function" today refers to an arbitrary computable function. The key difference is that primitive recursive functions can only use recursion on one variable, whereas full computability requires recursion on multiple variables as in the Herbrand-Gödel model of computation.

For primitive recursive functions it will always be crystal clear that they are intuitively computable.

Composition

Given functions $g_i : \mathbb{N}^m \to \mathbb{N}$ for $i = 1, \ldots, n$, $h : \mathbb{N}^n \to \mathbb{N}$, we define a new function $f : \mathbb{N}^m \to \mathbb{N}$ by composition as follows:

$$f(\boldsymbol{x}) = h(g_1(\boldsymbol{x}), \ldots, g_n(\boldsymbol{x}))$$

Notation: We write $h \circ (g_1, \ldots, g_n)$ inspired by the the well-known special case $m = 1$:

$$(h \circ g)(x) = h(g(x)).$$

So this is just ordinary sequential composition of functions. Clearly, computable functions are closed wrto composition: output can be re-used as input.

Primitive Recursion

We need one more operation beyond composition to get interesting functions, a form of recursion. Given $h : \mathbb{N}^{n+2} \to \mathbb{N}$ and $g : \mathbb{N}^n \to \mathbb{N}$ we define a new function $f : \mathbb{N}^{n+1} \to \mathbb{N}$ by

$$f(0, \boldsymbol{y}) = g(\boldsymbol{y})$$

$$f(x{+}1, \boldsymbol{y}) = h(x, f(x, \boldsymbol{y}), \boldsymbol{y})$$

The definition is agnostic about how to organize the computation, it just pins down the values of f.

A simple proof by induction shows that the given equations have exactly one solution, so our definition is sound.

Definition, Sort Of

Since we have now all the pieces in hand, we can concoct out first definition.

Definition (Semi-formal)

A function is primitive recursive (p.r.) if it can be generated from the basic functions zero and successor, using only composition and primitive recursion.

There, that's it.

We can now check that standard arithmetic functions like addition, multiplication, exponentiation and so on are primitive recursive.

Arithmetic Functions

All the basic functions of arithmetic are primitive recursive.

$$\begin{aligned} \mathsf{add}(0, y) &= y \\ \mathsf{add}(x{+}1, y) &= S(\mathsf{add}(x, y)) \end{aligned}$$

$$\begin{aligned} \mathsf{mult}(0, y) &= 0 \\ \mathsf{mult}(x{+}1, y) &= \mathsf{add}(\mathsf{mult}(x, y), y) \end{aligned}$$

$$\begin{aligned} \mathsf{exp}(0, y) &= 1 \\ \mathsf{exp}(x{+}1, y) &= \mathsf{mult}(\mathsf{exp}(x, y), y) \end{aligned}$$

Summation

Suppose ℓ is a primitive recursive function and we want to sum its values. No problem:

$$f(0) = 0$$
$$f(x{+}1) = \mathsf{add}(f(x), \ell(x))$$

So $f(x) = \sum_{z<x} \ell(z)$.

Similarly, if ℓ has an additional parameter, we can adjust the definition as follows:

$$f(0, y) = 0$$
$$f(x{+}1, y) = \mathsf{add}(f(x, y), \ell(x, y))$$

Summation is naturally a primitive recursive operation.

Recall: Thurston

> The standard of correctness and completeness necessary to get a computer program to work at all is a couple of orders of magnitude higher than the mathematical community's standard of valid proofs.
>
> Bill Thurston, 1994, Notices AMS

For a human reader, this is indeed all perfectly clear.

But there are a few minor issues. First off, in the summation example: there are two different zeros.

$$f(0) = 0$$

$$f(0, y) = 0$$

The first three have arity 0, but the last has arity 1. This is forced by our definition of primitive recursion.

A little more precision is needed if we wanted to, say, check proofs involving primitive recursive functions.

Arity

Sometimes it is convenient to have arity as part of the notation.

We will use a superscript (n) for this purpose:

$f^{(n)}$ a function of arity n

In particular write $\mathsf{C}_{\mathsf{a}}^{(\mathsf{n})}$ for the n-ary constant map $\boldsymbol{x} \mapsto a$.

We will call $\mathsf{C}_{\mathsf{a}}^{(0)}$ a hard constant: a function that takes no arguments.

Bureaucracy: Projections

Another problem is that composition as we defined it is not quite enough. Suppose we have a binary version add of addition, and want to define a ternary version. No problem:

$$\mathsf{add}^{(3)}(x, y, z) = \mathsf{add}^{(2)}(x, \mathsf{add}^{(2)}(y, z))$$

But, this is **not** allowed according to our definition of composition: try to find the right binding for h and the g_i.

We need a simple auxiliary tool, so-called projections:

$$\mathsf{P}^{\mathsf{n}}_{\mathsf{i}} : \mathbb{N}^n \to \mathbb{N} \qquad \mathsf{P}^{\mathsf{n}}_{\mathsf{i}}(x_1, \ldots, x_n) = x_i$$

where $1 \leq i \leq n$.

Killing Variables

Now we can write

$$\mathsf{add}^{(3)} = \mathsf{add}^{(2)} \circ (\mathsf{P}^3_1, \mathsf{add}^{(2)} \circ (\mathsf{P}^3_2, \mathsf{P}^3_3))$$

Note that no variables are needed in this notation system.

Needless to say, most humans prefer the informal notation by a long shot. But then again, the last term is very easier to parse and evaluate.

Clones

A clone or function algebra is a collection of functions that contains all projections and is closed under composition, over some carrier set.

More generally, define the collection of all finitary functions over $\mathbb{N}$ as

$$\mathfrak{F}_{\mathbb{N}} = \bigcup_{n \geq 0} (\mathbb{N}^n \to \mathbb{N})$$

Definition

A clone (over $\mathbb{N}$) is a subset $\mathcal{C} \subseteq \mathfrak{F}_{\mathbb{N}}$ that contains all projections and is closed under composition.

For example, all projections form a clone, as do all arithmetic functions.

Quoi?

Informally and intuitively, a primitive recursive function is obtained from zero, successor, composition and primitive recursion. Basta.

Projections and composition have nothing to do with the natural numbers, these concepts are perfectly general and apply to any domain.

On the other hand, 0, successor and primitive recursion are directly dependent on the naturals.

So it makes sense to separate out these two components of the definition of a primitive recursive function.

Nullary Functions

Note that we allow hard constants, nullary functions in $\mathbb{N}^0 \to \mathbb{N}$ where we think of $\mathbb{N}^0$ as a one-point set $\{*\}$.

We will write $f()$ or $f(*)$ when we evaluate such functions.

In the literature, you will also find clones without nullary functions

$$\mathcal{C} \subseteq \mathfrak{F}_{\mathbb{N}}^{(+)} = \bigcup_{n>0} (\mathbb{N}^n \to \mathbb{N})$$

This is mostly a technical detail, but one should be aware of the issue.

Actually, this is exactly the kind of pesky detail that makes programming quite so difficult.

Nullary???

Algebraists usually prefer the non-nullary approach. Most operations there are binary and unary: e.g., $(x, y) \mapsto x \cdot y$ and $x \mapsto x^{-1}$ in a group. Constants are just elements of the algebraic structure and are not considered to have anything to do with an operation.

But for those working in logic, type theory or category theory, nullary operations are not an issue at all. And, truth be told, any really solid implementation of primitive recursive functions also needs to keep track of all these gory details, otherwise things won't typecheck.

After all, a computer will not apply any algebraic common sense whatsoever, it will just follow the rules precisely as stated.

Nullary Composition

Recall composition: $h^{(n)}$, $g_i^{(m)}$, $i \in [n]$, produces $f = h \circ (g_1, \ldots, g_n) \in \mathfrak{F}_{\mathbb{N}}^{(m)}$.

It is worthwhile to consider the special case where the g_i are nullary.

Case: $m = 0$

We get the nullary constant

$$\mathsf{C}_{\mathsf{a}}^{(0)} \in \mathcal{C}$$

where

$$a = h(g_1(*), \ldots, g_n(*))$$

Another Angle

We could introduce constants $\mathsf{C}_{\mathsf{a}}^{(\mathsf{k})}$ for all a and k. Alas, that contradicts the basic principle of parsimony in axiomatization: use as few basic assumptions as possible. For example, if we have the successor function S, we can define $\mathsf{C}_{\mathsf{a}+1}^{(\mathsf{k})} = S \circ \mathsf{C}_{\mathsf{a}}^{(\mathsf{k})}$, so we only need $\mathsf{C}_{0}^{(\mathsf{k})}$.

We can use primitive recursion to deal with arity:

$$f(0, \boldsymbol{y}) = \mathsf{C}_{0}^{(\mathsf{k})}(\boldsymbol{y})$$
$$f(x{+}1, \boldsymbol{y}) = f(x, \boldsymbol{y})$$

This defines $\mathsf{C}_{0}^{(\mathsf{k}+1)}$ in terms of $\mathsf{C}_{0}^{(\mathsf{k})}$.

So all we really need is $\mathsf{C}_{0}^{(0)}$.

Generating Clones

To get something more interesting, we need to consider clones that are generated by

- certain basic functions $\mathcal{F}$, and/or
- closed under additional operations Op.

We write

$$\mathrm{clone}(\mathcal{F}; \mathrm{Op})$$

for the least clone containing $\mathcal{F}$ and closed under Op.

For example, clone(;) consists just of all projections.

This is a perfect example of a recursive datatype (rectype), one of the fundamental concepts in TCS. We have

- a collection of atoms (indecomposable items), and
- a collection of constructors that can be applied to build more complicated, decomposable objects.

Because of this inductive structure we can perform inductive arguments, both to establish properties and to define operations.

Basic Arithmetic Functions

When dealing with natural numbers, it is natural (duh) to have

- Constant zero $0 : \mathbb{N}$
- Successor function $S : \mathbb{N} \to \mathbb{N}\,,\ S(x) = x + 1$

Here constant 0 is meant to be the hard constant $\mathsf{C}_0^{(0)}$ (but recall the comment on nullary composition from above).

This is a rather spartan set of built-in functions, but as we will see it's all we need. Needless to say, these functions are trivially computable.

In fact, it is hard to give a reasonable description of the natural numbers without them (unless you are a set theorist).

Closure Operations: Primitive Recursion

We write Prec$[h, g]$ for primitive recursion: recall $h : \mathbb{N}^{n+2} \to \mathbb{N}$ and $g : \mathbb{N}^n \to \mathbb{N}$ can be used to define $f : \mathbb{N}^{n+1} \to \mathbb{N}$ by

$$f(0, \boldsymbol{y}) = g(\boldsymbol{y})$$
$$f(x+1, \boldsymbol{y}) = h(x, f(x, \boldsymbol{y}), \boldsymbol{y})$$

Definition

A function is primitive recursive (p.r.) if it lies in the clone generated by zero, successor; and closed under primitive recursion: clone$(0, S; \mathsf{Prec})$.

Two Views

	bureaucracy	basic	operator
atom	projections	zero, successor	-
constructors	composition	-	prim. rec.

Example: Factorials

The standard definition of the factorial function uses recursion like so:

$$f(0) = 1$$
$$f(x+1) = (x+1) \cdot f(x)$$

To write the factorial function in the form $f = \mathsf{Prec}[h, g]$ we need

$$g : \mathbb{N}^0 \to \mathbb{N}, \quad g() = 1$$
$$h : \mathbb{N}^2 \to \mathbb{N}, \quad h(u, v) = (u+1) \cdot v$$

More precisely, g is $\mathsf{C}_1^{(0)}$ and h is multiplication combined with the successor function:

$$f = \mathsf{Prec}[\mathsf{mult} \circ (S \circ \mathsf{P}_1^2, \mathsf{P}_2^2), \mathsf{C}_1^{(0)}]$$

Unfolding

By unfolding the definition of mult we can write a single term in our language that defines the factorial function.

$$\mathsf{Prec}[\mathsf{Prec}[\mathsf{Prec}[\mathsf{S} \circ \mathsf{P}_2^3, \mathsf{P}_1^1] \circ (\mathsf{P}_2^3, \mathsf{P}_3^3), \mathsf{C}_0^{(1)}] \circ (\mathsf{S} \circ \mathsf{P}_1^2, \mathsf{P}_2^2), \mathsf{C}_1^{(0)}]$$

The innermost Prec yields addition, the next multiplication and the last, factorial.

Again, hard to read for a human, but perfectly suited for a parser. Given the right environment, it is not hard to build an interpreter for these terms.

Arithmetic

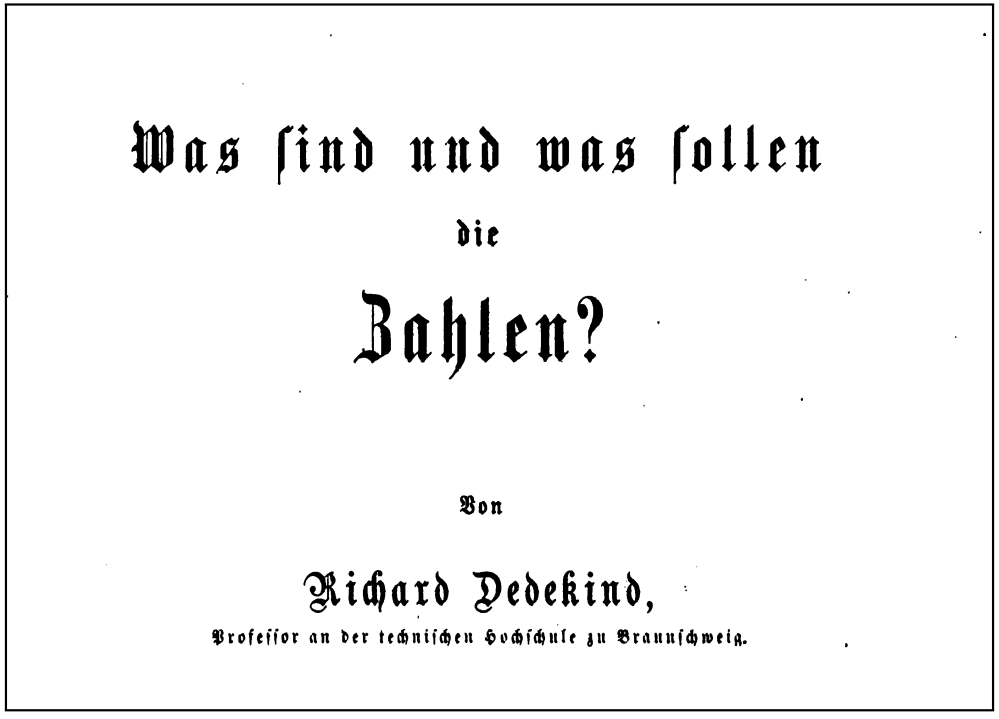
It is a good idea to go through the definitions of all the standard basic arithmetic functions from the p.r. point of view.

$$\begin{aligned}
\mathsf{add} &= \mathsf{Prec}[\mathsf{S} \circ \mathsf{P}^3_2, \mathsf{P}^1_1] \\
\mathsf{mult} &= \mathsf{Prec}[\mathsf{add} \circ (\mathsf{P}^3_2, \mathsf{P}^3_3), \mathsf{C}^{(1)}_0] \\
\mathsf{pred} &= \mathsf{Prec}[\mathsf{P}^2_1, \mathsf{C}^{(0)}_0] \\
\mathsf{sub}' &= \mathsf{Prec}[\mathsf{pred} \circ \mathsf{P}^3_2, \mathsf{P}^1_1] \\
\mathsf{sub} &= \mathsf{sub}' \circ (\mathsf{P}^2_2, \mathsf{P}^2_1)
\end{aligned}$$

Since we are dealing with $\mathbb{N}$ rather than $\mathbb{Z}$, sub here is proper subtraction: $x \dot{-} y = x - y$ whenever $x \geq y$, and 0 otherwise.

Exercise

Show that all these functions behave as expected.



Was ſind und was ſollen
die
Zahlen?

Von

Richard Dedekind,
Profeſſor an der techniſchen Hochſchule zu Braunſchweig.

These equational, inductive definitions of basic arithmetic functions date back to Dedekind's 1888 booklet "What are numbers and what is their purpose?" It is remarkable that he produced this description about 30 years before anyone started to think carefully about computability.

Admissibility

Here is an example of a closure property that is not obvious from the definitions. Apparently, we lack a mechanism for definition-by-cases:

$$f(x) = \begin{cases} 3 & \text{if } x < 5, \\ x^2 & \text{otherwise.} \end{cases}$$

We know that $x \mapsto 3$ and $x \mapsto x^2$ are p.r., but is f also p.r.?

We want to show that definition by cases is admissible in the sense that when applied to primitive recursive functions/relations we obtain another primitive recursive function. Having a solid collection of admissible operations around makes it much easier to show that some particular functions are primitive recursive.

Definition by Cases

Definition

Let $g, h : \mathbb{N}^n \to \mathbb{N}$ and $R \subseteq \mathbb{N}^n$. Define $f = \mathsf{DC}[g, h, R]$ by

$$f(\boldsymbol{x}) = \begin{cases} g(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in R, \\ h(\boldsymbol{x}) & \text{otherwise.} \end{cases}$$

We need to explain what it means for the relation R to be primitive recursive, we'll do that in a minute.

Sign and Inverted Sign

The first step towards implementing definition-by-cases is a bit strange, but we will see that the next function is actually quite useful.

The sign function is defined by

$$\mathsf{sign}(x) = \min(1, x)$$

so that $\mathsf{sign}(0) = 0$ and $\mathsf{sign}(x) = 1$ for all $x \geq 1$. Sign is primitive recursive: $\mathsf{Prec}[\mathsf{S} \circ 0, 0]$ in sloppy notation.

Similarly the inverted sign function is primitive recursive:

$$\overline{\mathsf{sign}}(x) = 1 \dot{-} \mathsf{sign}(x)$$

As usual, define the characteristic function of a relation R

$$\text{char}_R(\boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x} \in R \\ 0 & \text{otherwise.} \end{cases}$$

to translate relations into functions.

Definition

A relation is primitive recursive if its characteristic function is primitive recursive.

We will use analogous definitions later for all kinds of other types of computable functions: Turing, polynomial time, polynomial space, whatever.

Equality and Order

Define $E : \mathbb{N}^2 \to \mathbb{N}$ by

$$E = \overline{\mathsf{sign}} \circ \mathsf{add} \circ (\mathsf{sub} \circ (\mathsf{P}_1^2, \mathsf{P}_2^2), \mathsf{sub} \circ (\mathsf{P}_2^2, \mathsf{P}_1^2))$$

Or, less formally, but more intelligible:

$$E(x, y) = \overline{\mathsf{sign}}((x \stackrel{\bullet}{-} y) + (y \stackrel{\bullet}{-} x))$$

Then $E(x, y) = 1$ iff $x = y$, and 0 otherwise. Hence equality is primitive recursive. Even better, all standard order relations such as

$$\neq,\ \leq,\ <,\ \geq, \ldots$$

are primitive recursive (so we can use them e.g. in definitions by cases).

Proposition

The primitive recursive relations are closed under intersection, union and complement.

Proof.

$$\begin{aligned}
\mathsf{char}_{R\cap S} &= \mathsf{mult} \circ (\mathsf{char}_R, \mathsf{char}_S) \\
\mathsf{char}_{R\cup S} &= \mathsf{sign} \circ \mathsf{add} \circ (\mathsf{char}_R, \mathsf{char}_S) \\
\mathsf{char}_{\mathbb{N}-R} &= \mathsf{sub} \circ (1, \mathsf{char}_R)
\end{aligned}$$

□

In other words, primitive recursive relations form a Boolean algebra, and even an effective one: we can compute the Boolean operations.

Note what is really going on here: we are using arithmetic to express logical concepts such as disjunction.

The fact that this translation is possible, and requires very little on the side of arithmetic, is a central reason for the algorithmic difficulty of many arithmetic problems: logic is hard, by implication arithmetic is also difficult.

For example, finding solutions of Diophantine equations is hard.

Exercise

Show that every finite set is primitive recursive.
Show that the even numbers are primitive recursive.

DC is Admissible

Proposition

If g, h, R are primitive recursive, then $f = \mathsf{DC}[g, h, R]$ is also primitive recursive.

Proof.

$$f = \mathsf{add} \circ (\mathsf{mult} \circ (\mathsf{char}_R, g), \mathsf{mult} \circ (\overline{\mathsf{char}}_R, h))$$

Less cryptically

$$f(\boldsymbol{x}) = \mathsf{char}_R(\boldsymbol{x}) \cdot g(\boldsymbol{x}) + \overline{\mathsf{char}}_R(\boldsymbol{x}) \cdot h(\boldsymbol{x})$$

Since either $\mathsf{char}_R(\boldsymbol{x}) = 0$ and $\overline{\mathsf{char}}_R(\boldsymbol{x}) = 1$, or the other way around, we get the desired behavior. □

Bounded Sum

Proposition

Let $g : \mathbb{N}^{n+1} \to \mathbb{N}$ be primitive recursive, and define

$$f(x, \boldsymbol{y}) = \Sigma_{z<x} g(z, \boldsymbol{y})$$

Then $f : \mathbb{N}^{n+1} \to \mathbb{N}$ is again primitive recursive. The same holds for products.

Proof.

$$f = \mathsf{Prec}[\mathsf{add} \circ (g \circ (P_1^{n+2}, P_3^{n+2}, \ldots, P_{n+2}^{n+2}), P_2^{n+2}), 0^n]$$

Less formally,

$$f(0, \boldsymbol{y}) = 0$$
$$f(x{+}1, \boldsymbol{y}) = f(x, \boldsymbol{y}) + g(x, \boldsymbol{y})$$

The argument for product is similar. □

Exercise

Repeat the proof for products.

Exercise

Show that $f(x, \boldsymbol{y}) = \sum\big(g(z, \boldsymbol{y}) \mid z < x \wedge R(z)\big)$ *is primitive recursive when* g *and* R *are primitive recursive.*

Exercise

Show that $f(x, \boldsymbol{y}) = \sum_{z<h(x)} g(z, \boldsymbol{y})$ *is primitive recursive when* h *is primitive recursive.*

Bounded Search

A particularly important algorithmic technique is search over some finite domain.

For example, in brute-force factoring n we are searching over an interval $[2, n-1]$ for a number that divides n. Or in a chess program we search for the optimal next move over a space of possible next moves.

We can model search in the realm of p.r. functions as follows.

Definition (Bounded Search)

Let $g : \mathbb{N}^{n+1} \to \mathbb{N}$. Then $f = \mathsf{BS}[g] : \mathbb{N}^{n+1} \to \mathbb{N}$ is the function defined by

$$f(x, \boldsymbol{y}) = \begin{cases} \min\big(z < x \mid g(z, \boldsymbol{y}) = 0 \big) & \text{if } z \text{ exists,} \\ x & \text{otherwise.} \end{cases}$$

Keeping Things Simple

Note that $f(x, \boldsymbol{y}) = x$ simply means that the search failed. In a more luxurious environment we might set a flag, throw an exception or some such.

Here we want everything to be a simple as possible, and in particular constrained to pure arithmetic. So we code failure as a numerical value, basta.

BS is Admissible

One can show that bounded search is also admissible, it adds nothing to the class of p.r. functions.

Proposition

If g is primitive recursive, then so is BS$[g]$.

Exercise

Show that bounded search is indeed admissible ("primitive recursive functions are closed under bounded search").

Bounded Search II

This can be pushed a little further: the search does not have to end at x. Instead, we can search up to a primitive recursive function of x and $\boldsymbol{y}$.

$$f(x,\boldsymbol{y}) = \begin{cases} \min\big(\, z < h(x,\boldsymbol{y}) \mid g(z,\boldsymbol{y}) = 0 \,\big) & \text{if } z \text{ exists,} \\ h(x,\boldsymbol{y}) & \text{otherwise.} \end{cases}$$

Dire Warning:

But we have to have a p.r. bound, unbounded search as in

$$f(\boldsymbol{y}) := \min\big(\, z \mid g(z,\boldsymbol{y}) = 0 \,\big)$$

is not an admissible operation; not even when there is a suitable witness z for each $\boldsymbol{y}$. See Kleene's μ-recursive functions.

Example: Primality

Claim (1)

The divisibility relation $\mathrm{div}(x, y)$ *is primitive recursive.*

Note that

$$\mathrm{div}(x, y) \iff \exists\, z \leq y\, (x * z = y)$$

so that bounded search intuitively should suffice to obtain divisibility. Formally, we have already seen that the characteristic function $M(x, z, y)$ of $x * z = y$ is p.r. But then

$$\text{sign}\left(\sum\nolimits_{z \leq y} M(x, z, y)\right)$$

is the p.r. characteristic function of div.

Claim (2)

The primality relation is primitive recursive.

To see why, note that x is prime iff

$$1 < x \wedge \forall\, z < x\, (\mathrm{div}(z, x) \Rightarrow z = 1).$$

The building blocks $1 < x$, div and $z = 1$ are all p.r., and we can combine things by $\wedge$ and $\Rightarrow$. The only potential problem is the (bounded) universal quantifier.

But this is quite similar to the situation with div from the last slide. Time for a general solution.

Definition (Bounded Quantifiers)

$P_\forall(x, \boldsymbol{y}) \Leftrightarrow \forall\, z < x\, P(z, x, \boldsymbol{y})$

$P_\exists(x, \boldsymbol{y}) \Leftrightarrow \exists\, z < x\, P(z, x, \boldsymbol{y})$.

Note that $P_\forall(0, \boldsymbol{y}) =$ true and $P_\exists(0, \boldsymbol{y}) =$ false.

Informally, and using the dreaded ellipsis,

$$P_\forall(x, \boldsymbol{y}) \iff P(0, x, \boldsymbol{y}) \wedge P(1, x, \boldsymbol{y}) \wedge \ldots \wedge P(x-1, x, \boldsymbol{y})$$

and likewise for $P_\exists$.

Bounded Quantification

Bounded quantification is really just a special case of bounded search: for $P_\exists(x, \boldsymbol{y})$ we search for a witness $z < x$ such that $P(z, x, \boldsymbol{y})$ holds. Generalizes to $\exists\, z < h(x, \boldsymbol{y})\, P(z, x, \boldsymbol{y})$ and $\forall\, z < h(x, \boldsymbol{y})\, P(z, x, \boldsymbol{y})$.

Proposition

Primitive recursive relations are closed under bounded quantification.

Proof.

$$\mathsf{char}_{P_\forall}(x, \boldsymbol{y}) = \prod_{z<x} \mathsf{char}_P(z, x, \boldsymbol{y})$$

$$\mathsf{char}_{P_\exists}(x, \boldsymbol{y}) = \mathsf{sign}\left(\sum_{z<x} \mathsf{char}_P(z, x, \boldsymbol{y})\right)$$

□

Claim (3)

The next prime function $f(x) = \min\big(\, z > x \mid z \text{ prime} \big)$ *is p.r.*

This follows from the fact that we can bound the search for the next prime by a p.r. function:

$$f(x) \leq 2x \qquad \text{for } x \geq 1.$$

This bounding argument requires a little number theory. In general, the theory of gaps between consecutive primes is quite difficult (consider prime twins), but this result is not too bad.

Enumerating Primes

Claim (4)

The function $n \mapsto p_n$, where p_n is the nth prime, is primitive recursive.

To see this we can iterate the "next prime" function from the last claim:

$$p(0) = 2$$
$$p(n+1) = f(p(n))$$

Exercise

Show in detail that the function $n \mapsto p_n$ where p_n is the nth prime is primitive recursive. How large is the p.r. expression defining the function?

Exercise

Show that the prime counting function is primitive recursive.
Exploit it to give another p.r. construction for the nth prime function.

Exercise

Find some other closure properties of primitive recursive functions.