CS 598 (CRN 62819) Topics in Algorithms, Spring 2015

Homework # 5 **Due: April 27, 2015**

Homework is due in class (on paper) on the due date.

1. **Piazza.** Make a Piazza comment related to Chapter 5 if you have not done so yet.

2. **Cover time.**

   (a) Prove that the cover time of the n-node clique is $O(n \log n)$.

   (b) Prove that in fact a walk of length $2n \ln n$ on the n-node clique will have probability at least $1 - 1/n$ of visiting every vertex.

   (c) It is clear that adding an edge to an n-node graph can decrease its cover time (since the cover-time of an n-node clique is smaller than the cover-time of an n-node line). Can adding an edge to an n-node graph ever *increase* the cover time? (The edge must be between existing nodes.) Given a proof and/or example.

3. **Hitting time.**

   (a) What is the hitting time h_{uv} for two adjacent nodes u, v on an n-node cycle?

   (b) What is the hitting time h_{uv} for two (adjacent) nodes u, v in an n-node clique?

   (c) Consider a random walk on an n-node clique starting at vertex 1. So $\mathbf{p}^{(0)} = (1, 0, 0, \ldots, 0)$. What is $\mathbf{p}^{(1)}$ and how far is it in L_1 distance from the stationary distribution π? What is $\mathbf{p}^{(2)}$ and how far is it in L_1 distance from the stationary distribution π? Note that this shows the mixing time is much less than the hitting time for the clique.

4. **Mixing on a Line.** Consider a random walk on an N-node line graph. That is, node 1 is connected to node 2, node 2 is connected to node 3, etc., all the way up to node N. Let's add self-loops to both ends, so if you are at node 1, with 50% probability you stay at node 1 and with 50% probability you move to node 2 (similarly with node N).

   (a) What is the stationary distribution π of this walk? Give a proof.

   (b) What is the (normalized) conductance $\Phi = \min_{|S| \leq N/2} \Phi(S)$ of this Markov chain? You can assume N is even.

   (c) What $O()$ bound on ϵ-mixing time do you get if you plug this into our rapid-mixing theorem? Ignore dependence on ϵ (view it as a constant).

   (d) Now, give a proof that the ϵ-mixing time, for $\epsilon = 1/10$, is $\Omega(n^2)$, showing your bound above to be nearly tight up to logarithmic factors. Be clear in your proof. You may want to argue by contradiction, using facts we know about hitting time.

5. **Stationary distributions.** Let G be an n-node graph. Suppose we want to pick a random independent set S in G with probability proportional to $e^{|S|}$. Describe a Markov Chain whose stationary distribution has the correct probabilities (this walk will *not* be rapidly mixing: doing this efficiently would imply $RP = NP$). In particular:

   - What are the states? What are the neighbors?
   - What is the probability p_{ij} of moving to state j given that you are currently in state i?
   - Why is the stationary distribution correct? Also argue why your Markov Chain is connected.