Linear Programming & Mechanism Design

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Mechanism Design

'Optimizing' the allocation of resources.

Parameters (called type) needed to determine an optimal allocation are privately held by agents who will consume the resources to be allocated.

Those parameters determine the utility an agent will enjoy from a particular allocation.

Agents report of her type will influence the resulting allocation which in turn will affect the agents utility.

How can planner simultaneously elicit the information that is privately held and choose the optimal allocation?

Via allocation and monetary transfers.

Examples

Many mechanism design problems are optimization problems.

- Auctions
- Matching (school choice, randomized rules)
- Models of Persuasion
- Team formation

Today & Tomorrow

1. Use of polymatroids in mechanism design.
2. Use of shortest path problems to analyze rationalizability and incentive compatibility.
3. Use of iterative rounding.

Let $E = \{1, 2, \ldots, n\}$ be a ground set. Real valued function g defined on

subsets of E is

- **non-decreasing** if $S \subseteq T \Rightarrow g(S) \leq g(T)$, and
- g is **submodular** if $\forall S, T \subset E$

$$g(S) + g(T) \geq g(S \cup T) + g(S \cap T).$$

Equivalent: Suppose $S \subset T$ and $j \notin T$. Then,

$$g(T \cup j) - g(T) \leq g(S \cup j) - g(S)$$

Examples of Submodular Functions

1. E a finite set of vectors in $\Re^m$ and $g(S)$ is the rank of the subset S.
2. E a finite set of vectors in $\Re^m$ and $g(S)$ is log volume of set S.
3. E set of columns of a non-negative determinant matrix and $g(S)$ is log of determinant of principal minor associated with S (also M matrices).
4. E the edge set of a graph and $g(S)$ size of largest acyclic subset of S.
5. E the vertex set of an edge capacitated network with a distinguished source vertex and $g(S)$ the maximum flow into S.
6. E the vertex set of a graph and $g(S)$ the cardinality of the set of neighbors of S.
7. E a set of events and $-g(S)$ is the probability that all events in S are realized.
8. Entropy of joint distribution.

Polymatroid Optimization

Polymatroid:

$$P(g) = \{x \in \Re^n_+ : \sum_{j \in S} x_j \leq g(S) \quad \forall S \subseteq E\}$$

$$\max\{cx : x \in P(g)\}$$

$$c_1 \geq c_2 \ldots \geq c_k \geq 0 > c_{k+1} \ldots \geq c_n.$$

1. $S^0 = \emptyset$
2. $S^j = \{1, 2, \ldots, j\}$ for all $j \in E$.
3. $x_j = g(S^j) - g(S^{j-1})$ for $1 \leq j \leq k$
4. $x_j = 0$ for $j \geq k + 1$.

So What?

- For economic applications goal is not merely to solve the underlying optimization problem but identify qualitative properties of optimal solution and how it varies with changes in the parameters of the problem.
- Polymatroid optimizations problems are valuable because they admit a simple greedy solution.
- Reduction allows one to handle certain kinds of additional constraints like budget and quota constraints.

Allocation with Inspection

Inspired by Dekel, Lipman and Ben-Porath (2011).

Good to be allocated to agent with the highest value.

Transfers not permitted.

- n risk neutral agents
- Value each agent assigns to the good is called their **type**
- Types are independent draws from $T = \{1, \dots, m\}$
- $f_t > 0$ is probability that buyer is of type t

For a cost K, planner can verify an agents report of his type.

Direct Mechanism

Planner announces three functions whose argument is the profile of types reported.

Allocation rule: specifies what 'fraction' of the object goes to each agent as a function of profile of reported types.

Payment rule: specifies payment of each agent as a function of profile of reported types.

Inspection rule: specifies probability that an agent will be inspected as a function of profile of reported types.

Allocation Rules

For simplicity assume 2 agents.

a is an **allocation rule**

$a_i(t,s)$ is probability good is allocated to agent i when agent 1 reports t and agent 2 reports s.

Feasibility:

$$a_1(t,s) + a_2(t,s) \leq 1 \ \forall t, s$$

$$a_i(t,s) \geq 0 \ \forall i, \ \forall t, s$$

Interim Allocations

$$\mathcal{A}_t^i = \sum_{s \in T} a_i(t, s) f_s$$

$\mathcal{A}_t^i$ is the **interim allocation probability** to agent i when she reports t.

An interim allocation probability is **implementable** if there exists an allocation rule that corresponds to it.

Characterize the implementable $\mathcal{A}$'s.

Border-Maskin-Matthews-Riley

Suppose allocation rule is anonymous, i.e., does not depend on names.

$$\mathcal{A}_t^i = \mathcal{A}_t^j = \mathcal{A}_t$$

$\mathcal{A}_t$ is **implementable** iff.

$$n \sum_{t \in S} f_t \mathcal{A}_t \leq 1 - (\sum_{i \notin S} f_t)^n \ \forall S \subseteq T.$$

$g(S) = 1 - (\sum_{i \notin S} f_t)^n$ is monotone and submodular.

Assume $T = \{t, t'\}$ and $S = \{s, s'\}$. Here are all the inequalities:

$$a_1(t,s) + a_2(t,s) \leq 1$$

$$a_1(t',s) + a_2(t',s) \leq 1$$

$$a_1(t',s') + a_2(t',s') \leq 1$$

$$a_1(t,s') + a_2(t,s') \leq 1$$

$$a_1(t,s)f_s + a_1(t,s')f_{s'} = \mathcal{A}^1_t$$

$$a_1(t',s)f_s + a_1(t',s')f_{s'} = \mathcal{A}^1_{t'}$$

$$a_2(t,s)f_t + a_2(t',s)f_{t'} = \mathcal{A}^2_s$$

$$a_2(t,s')f_t + a_2(t',s')f_{t'} = \mathcal{A}^2_{s'}$$

Interim Allocations

$$f_t f_s a_1(t,s) + f_t f_s a_2(t,s) \leq f_t f_s$$
$$f_{t'} f_s a_1(t',s) + f_{t'} f_s a_2(t',s) \leq f_{t'} f_s$$
$$f_{t'} f_{s'} a_1(t',s') + f_{t'} f_{s'} a_2(t',s') \leq f_{t'} f_{s'}$$
$$f_t f_{s'} a_1(t,s') + f_t f_{s'} a_2(t,s') \leq f_t f_{s'}$$
$$f_t f_s a_1(t,s) + f_t f_{s'} a_1(t,s') = f_t \mathcal{A}^1_t$$
$$f_{t'} f_s a_1(t',s) + f_{t'} f_{s'} a_1(t',s')) = f_{t'} \mathcal{A}^1_{t'}$$
$$f_t f_s a_2(t,s) + f_{t'} f_s a_2(t',s) = f_s \mathcal{A}^2_s$$
$$f_t f_{s'} a_2(t,s') + f_{t'} f_{s'} a_2(t',s') = f_{s'} \mathcal{A}^2_{s'}$$

$x_i(u,v) = f_u f_v a_i(u,v)$.

Interim Allocations

$$x_1(t,s) + x_2(t,s) \leq f_t f_s$$
$$x_1(t',s) + x_2(t',s) \leq f_{t'} f_s$$
$$x_1(t',s') + x_2(t',s') \leq f_{t'} f_{s'}$$
$$x_1(t,s') + x_2(t,s') \leq f_t f_{s'}$$
$$x_1(t,s) + x_1(t,s') = f_t \mathcal{A}^1_t$$
$$x_1(t',s) + x_1(t',s') = f_{t'} \mathcal{A}^1_{t'}$$
$$x_2(t,s) + x_2(t',s) = f_s \mathcal{A}^2_s$$
$$x_2(t,s') + x_2(t',s') = f_{s'} \mathcal{A}^2_{s'}$$

- $\mathcal{A}_t$ is interim allocation probability to type $t \in T$.
- $1 - c(t)$ is the probability of checking a report of type t conditional on the good being allocated to a type t.
- Total value less the cost of inspection is

$$\sum_{t=1}^{m} f_t t \mathcal{A}_t - K \sum_{t=1}^{m} f_t \mathcal{A}_t (1 - c(t))$$

Allocation with Inspection

$$\max \sum_{t=1}^{m} f_t t \mathcal{A}_t - K \sum_{t=1}^{m} f_t \mathcal{A}_t (1 - c(t))$$

$$\text{s.t. } t\mathcal{A}_t \geq t\mathcal{A}_s c(s) \ \forall t, s \in T \tag{1}$$

$$0 \leq c(t) \leq 1 \ \forall t \in T \tag{2}$$

$$n \sum_{t \in S} f_t \mathcal{A}_t t \leq 1 - (\sum_{t \notin S} f_t)^n = g(S) \ \forall S \subseteq T \tag{3}$$

Allocation with Inspection: IC

Bayesian incentive compatibility constraint captured here by:

$$t\mathcal{A}_t \geq t\mathcal{A}_s c(s)\ \forall t, s \in T$$

$$\Rightarrow\ \mathcal{A}_t \geq \mathcal{A}_s c(s)\ \forall t, s \in T$$

This is dual to problem of finding a feasible flow in a **generalized** network.

Associate node with each member of T and for each ordered pair (t, s) insert a directed edge from t to s with flow multiplier $c(s)$.

System is feasible iff the associated network has no flow generating cycles.

For any subset R of types we must have $\Pi_{t \in R} c(t) \leq 1$.

Allocation with Inspection

$$\mathcal{A}_t \geq \mathcal{A}_s c(s) \;\Rightarrow \mathcal{A}_s \leq \frac{\mathcal{A}_t}{c(s)}$$

Never good to inspect $t = 1$. So, $c(1) = 1$. Therefore, $\mathcal{A}_t \geq \mathcal{A}_1$ for all $t \in T$.

$$\mathcal{A}_s \leq \frac{\mathcal{A}_1}{c(s)} \;\forall s \in T.$$

Allocation with Inspection: Relaxation

$$\max \sum_{t \in T} f_t \mathcal{A}_t [t - K + Kc(t)]$$

$$\text{s.t. } c(t) \le \frac{\mathcal{A}_1}{\mathcal{A}_t} \; \forall t \in T$$

$$\mathcal{A}_t \ge \mathcal{A}_1 \; \forall t \in T$$

$$0 \le c(t) \le 1 \; \forall t \in T$$

$$n \sum_{t \in S} f_t \mathcal{A}_t \le g(S) \; \forall S \subseteq T$$

$c(t) = \min\{\frac{\mathcal{A}_1}{\mathcal{A}_t}, 1\} = \frac{\mathcal{A}_1}{\mathcal{A}_t}$.

Allocation with Inspection: Relaxation

$$\max \sum_{t=1}^{m} f_t \mathcal{A}_t [t - K] + K\mathcal{A}_1$$

$$\text{s.t. } \mathcal{A}_1 \leq \mathcal{A}_t \ \forall t \in T$$

$$n \sum_{t \in S} f_t \mathcal{A}_t \leq g(S) \ \forall S \subseteq T$$

1. $\mathcal{A}_t = x_t + \mathcal{A}_1$ for all $t \geq 2$
2. $H(S) = g(S) - n\mathcal{A}_1 \sum_{i \in S} f_i$.
3. H is submodular.
4. For $\mathcal{A}_1 \leq \min_S \frac{g(S)}{n \sum_{t \in S} f_t}$, H is monotone.

Allocation with Inspection: Relaxation

$$(\sum_{t=1}^{m} tf_t)\mathcal{A}_1 + \max \sum_{t=2}^{m} f_t x_t[t-K]$$

$$\text{s.t. } n\sum_{t \in S} f_t x_t \leq H(S) \; \forall S \subseteq T \setminus \{1\}$$

One more change of variables: $z_t = f_t x_t$ for all $t \geq 2$.

$$\left(\sum_{t=1}^{m} t f_t\right)\mathcal{A}_1 + \max \sum_{t=2}^{m} z_t[t-K]$$

$$\text{s.t. } n\sum_{t\in S} z_t \leq H(S)\ \forall S \subseteq T \setminus \{1\}$$

1. Set $z_t = 0$ for all $t \leq K$. Therefore $\mathcal{A}_t = \mathcal{A}_1$.
2. $c(t) = 1$ for all $t \leq K$.
3. There is a cutoff, λ so that in any profile of types, award the object to the agent with the highest type provided it exceeds λ.
4. Inspect their report with positive probability. The probability of inspection rises with t.
5. If all reported types fall below the cutoff, randomize equally between all types below λ and don't inspect.

Allocation with Transfers (Myerson)

$$\max \sum_{t=1}^{m} f_t p_t$$

$$\text{s.t. } t\mathcal{A}_t - p_t \geq t\mathcal{A}_s - p_s \ \forall t, s \in T$$

$$n \sum_{t \in S} f_t \mathcal{A}_t t \leq 1 - (\sum_{t \notin S} f_t)^n = g(S) \ \forall S \subseteq T$$

Digression: Shortest Path

$\mathcal{N}$ be the node-arc incidence matrix of a network $G = (V, A)$ with a single source node s and sink node t.

c_{ij} is length of arc (i, j).

$b^{s,t}$ be the vector such that $b_i^{s,t} = 0$ for all $i \in V \setminus \{s, t\}$, $b_s^{s,t} = -1$ and $b_t^{s,t} = 1$.

Digression: Shortest Path

The shortest path problem is

$$\min\{cx : \mathcal{N}x = b^{s,t}, x \geq 0\}.$$

Dual is

$$\min y_t - y_s$$

$$\text{s.t. } y_i - y_j \leq c_{ij} \ \forall (i,j) \in E$$

Primal is bounded iff. dual is feasible.

$$p_t - p_s \leq t(\mathcal{A}_t - \mathcal{A}_s)$$

No negative cycles equivalent to $\mathcal{A}_t$ is monotone in t

p_t upper bounded by length of shortest path to t

$$p_t = t\mathcal{A}_t - \sum_{j \leq t-1} \mathcal{A}_j$$

Allocation with Transfers

$$\max \sum_{t=1}^{m} f_t \left\{ t\mathcal{A}_t - \sum_{j=0}^{t-1} \mathcal{A}_j \right\} = \sum_{y=1}^{m} f_t \left\{ t - \frac{1-F(t)}{f(t)} \right\} \mathcal{A}_t$$

$$\text{s.t. } \mathcal{A}_t \geq \mathcal{A}_{t-1} \ \forall t \in T$$

$$n \sum_{t \in S} f_t \mathcal{A}_t t \leq 1 - (\sum_{t \notin S} f_t)^n = g(S) \ \forall S \subseteq T$$

Rationalizability (Afriat)

Set of purchase decisions $\{p_i, x_i\}_{i=1}^n$ is **rationalizable** by

- locally non-satiated,
- quasi-linear,
- concave utility function $u : \mathbb{R}^m_+ \mapsto \mathbb{R}$
- for some budget B

if for all i,

$$x_i \in \arg\max\{u(x) + s : p_i \cdot x + s = B, x \in \mathbb{R}^m_+\}.$$

Rationalizability

If at price p_i, $p_i \cdot x_j \leq B$, it must be that x_j delivers less utility than x_i.

$$u(x_i) + B - p_i \cdot x_i \geq u(x_j) + B - p_i \cdot x_j$$

$$\Rightarrow u(x_j) - u(x_i) \leq p_i \cdot (x_j - x_i)$$

Given set $\{(p_i, x_i)\}_{i=1}^{n}$ we formulate the system:

$$y_j - y_i \leq p_i \cdot (x_j - x_i), \ \forall i, j \ \text{ s.t. } \ p_i \cdot x_j \leq B$$

$$y_j - y_i \leq p_i \cdot (x_j - x_i), \ \forall i,j \ \text{ s.t. } \ p_i \cdot x_j \leq B \tag{4}$$

1. One node for each i.
2. For each ordered pair (i,j) such that $p_i \cdot x_j \leq B$, an arc with length $p_i \cdot (x_j - x_i)$.
3. The system (4) is feasible iff. associated network has no negative length cycles.

Rationalizability

Use any feasible choice of $\{y_j\}_{j=1}^n$ to construct a concave utility.

Set $u(x_i) = y_i$.

For any other $x \in \mathbb{R}^n_+$ set

$$u(x) = \min_{i=1,\ldots,n} \{u(x_i) + p_i \cdot (x - x_i)\}.$$

Allocation via Rounding (Nguyen & Vohra)

Allocating Indivisible Objects

1. Physical Division
2. Hold in common
3. Compensation
4. Exchange for something divisible
5. Unbundle attributes
6. Lottery
7. Rotation
8. Removal

k-UNIT DEMAND

1. No agent wishes to consume more than k goods.
2. $u(S) = \max_{A \subseteq S:}\{u(A) : |A| \leq k\}$ for any bundle S.
3. There is a partition $P_1, \dots, P_t$ such that $|P_r| \leq k$ for all $r = 1, \dots, t$. Also
$$u(S) = \sum_{r=1}^{t} u(S \cap P_r).$$

Approximate Walrasian

1. G set of distinct goods.
2. s_j the (integral) supply of good $j \in G$
3. Assume no agent wishes to consume more than one copy of any $j \in G$.

When agents have quasi-linear preferences and k-unit demand, there exist Walrasian prices where the excess demand for any good is at most $k - 1$.

Course Assignment

1. G set of distinct classes.
2. s_j the (integral) supply of seats in class $j \in G$
3. No agent wishes to consume more than one copy of any $j \in G$.

Under the k-demand assumption there is a lottery over assignments of classes to agents that is

1. approximately efficient, and, ex-ante envy-free.
2. The lottery is asymptotically strategy-proof.
3. The allocation consumes no more than $s_j + k - 1$ seats of $j \in G$.

Iterative Rounding Mechanism

- N set of agents.
- G set of distinct goods and s_j the (integral) supply of good $j \in G$.
- No agent wishes to consume more than one copy of any $j \in G$.
- $x_i(S) = 1$ if the bundle $S \subseteq G$ is assigned to agent $i \in N$ and zero otherwise.

An assignment $\{x_i(S)\}_{i \in N, S \subseteq G}$ is feasible if:

$$\sum_{S \subseteq G} x_i(S) \leq 1 \ \forall i \in N \tag{5}$$

$$\sum_{i \in N} \sum_{S \ni j} x_i(S) \leq s_j \ \forall j \in G \tag{6}$$

Iterative Rounding

IRM takes as input an extreme point, $x^* \in \arg\max\{u \cdot x : x \in P\}$ where $u \geq 0$ and $u_i(S) = 0$ for all $i \in N$ and $S \subseteq G$ such that $|S| > k$.

Round x^* into a 0-1 vector $\bar{x}$ that satisfies

$$\sum_{S \subseteq G} x_i(S) \leq 1 \ \forall i \in N$$

and is such that

$$\sum_{i \in N} \sum_{S \ni j} \bar{x}_i(S) \leq s_j + k - 1 \ \forall j \in G. \tag{7}$$

Iterative Rounding

Beginning with x^*, remove from (5-6) all variables $x_i(S)$ for which $x_i^*(S) = 0$.

Remove from (5-6) all variables $x_i(S)$ for which $x_i^*(S) = 1$ and adjust the right hand sides of (6) accordingly.

In system that remains pick a non-negative extreme point (fractional or otherwise) that optimizes the vector c and repeat.

At some iteration, an extreme point with no variable set to 1. Call it y. There must exist a $j \in G$ such that

$$|\{i \in N : y_i(S) > 0,\ S \ni j\}| \leq s_j + k - 1.$$

For each such j, remove the corresponding constraint (6) and in relaxed system find an extreme point that optimizes c and repeat.

Iterative Rounding

1. At each iteration, inequality (5) holds. Thus, $\bar{x}$ satisfies (5).

2. At each iteration, the original program is (possibly) relaxed. Thus, $u \cdot \bar{x} \geq u \cdot x^*$.

3. Because $\bar{x}_i(S) = 1$ only if $x_i^*(S) > 0$, it follows that on the inequalities in (6) thrown away, $\sum_{i \in N} \sum_{S \ni j} \bar{x}_i(S) \leq s_j + k - 1$.

Király, Lau and Singh (2008)

Lemma

Let $u_i(S) \geq 0$ *and* $u_i(S) = 0$ *for all* $|S| > k$. *Let* x^* *be an extreme point of* P *in* $\arg\max\{u \cdot x : x \in P\}$ *such that* $x_i^*(S) < 1$ *for all* $i \in N$ *and* $S \subseteq G$. *Then, there exists a* $j \in G$ *such that*

$$|\{i \in N : x_i^*(S) > 0,\ S \ni j\}| \leq s_j + k - 1.$$

$|N| < |G|$.

An extreme point of (5-6) can have at most $|N| + |G|$ non-zero variables.

As $x_i^*(S) < 1$ for all $i \in N$ and $S \subseteq G$, it follows that for each i, $|\{S : x_i^*(S) > 0\}| \geq 2$.

Hence, the number of non-zero variables in x^* is at least $2|N|$. Therefore, $|N| \leq |G|$.

Let $z_i(S) = 1$ if $x_i^*(S) > 0$ and zero otherwise. Suppose conclusion of the lemma is false.

$$\sum_{i \in N} \max_{S \ni j} \{z_i(S)\} > s_j + k - 1 \ \forall j \in G. \tag{8}$$

Adding inequality (8) up over $j \in G$ and using the fact that $z_i(S) = 1$ implies $|S| \le k$ gives:

$$\sum_{j \in G} s_j + |G|(k-1) < \sum_{j \in G} \sum_{i \in N} \max_{S \ni j} \{z_i(S)\} \le \sum_{i \in N} k \max_{S \subseteq G} z_i(S) \le k|N|$$

As the left hand side of the above is bounded below by $k|G|$ and $|N| < |G|$, contradiction.

Let $Q_k = \{x \in P : x_i(S) = 0 \ \forall i, \ \forall |S| > k\}$ and let E_k be the set of non-negative integral solutions to

$$\sum_{S \subseteq G} x_i(S) \leq 1 \ \forall i \in N$$

$$\sum_{i \in N} \sum_{S \ni j} \bar{x}_i(S) \leq s_j + k - 1 \ \forall j \in G.$$

Theorem

Q_k is in the convex hull of E_k.

Maxi-Min Share

A choice of x^* that ensures the rounded solution $\bar{x}$ gives to each agent a bundle they prefer at least as much as their maxi-min share.

For each h and $i \in N$, set $x_i(S) = 0$ for any bundle S that i ranks below h^{th} place in her preference ordering.

Denote the corresponding restriction of P by P_h. Let h^* be the smallest index such that P_{h^*} is non-empty.

Choose u that satisfies the conditions of the lemma and let x^* be an extreme point solution of $\max\{u \cdot x : x \in P_{h^*}\}$.

The corresponding $\bar{x}_i(S) = 1$ only if $x_i^*(S) > 0$, it follows that each agent receives a bundle that she ranks in place h^* or higher.

Envy-Free

$u_i(S)$ is von-Neumann Morgenstern utility that agent i assigns to bundle S (not necessarily quasi-linear).

A fractional allocation $x \in P$ is ex-ante envy-free if

$$\sum_{S \subseteq G} u_i(S)x_i(S) \geq \sum_{S \subseteq G} u_i(S)x_j(S) \ \forall i \ \forall j \neq i$$

Choose an x^* in Q_k that is envy-free and maximizes $\sum_i \sum_S u_i(S)x_i(S)$. It can be expressed as a lottery over the elements in E_k.